# Border Correlation of Binary Words

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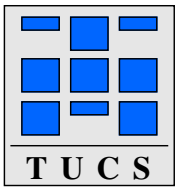


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**Abstract**

The border correlation function $\beta\colon A^* \to A^*$, for $A = \{a, b\}$, specifies which conjugates (cyclic shifts) of a given word $w$ of length $n$ are bordered, in other words, $\beta(w) = c_0 c_1 \dots c_{n-1}$, where $c_i = a$ or $b$ according to whether the $i$-th cyclic shift $\sigma^i(w)$ of $w$ is unbordered or bordered. Except for some special cases, no binary word $w$ has two consecutive unbordered conjugates ($\sigma^i(w)$ and $\sigma^{i+1}(w)$). We show that this is optimal: in every cyclically overlap-free word every other conjugate is unbordered. We also study the relationship between unbordered conjugates and critical points, as well as, the dynamic system given by iterating the function $\beta$. We prove that, for each word $w$ of length $n$, the sequence $w, \beta(w), \beta^2(w), \dots$ terminates either in $b^n$ or in the cycle of conjugates of the word $ab^k ab^{k+1}$ for $n = 2k + 3$.

**Keywords:** combinatorics on words, border correlation, binary words

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# 1 Introduction

A word $w$ is said to be *unbordered* (or *self-uncorrelated* [13]), if the only border of $w$ is the word itself, that is, if $w = uv = vu'$ for a nonempty word $v$, then $v = w$ and, consequently, $u = u' = \varepsilon$, the empty word. A word $u$ is a *factor* of a word $w$, if $w = w_1uw_2$ for some (possibly empty) words $w_1$ and $w_2$. Unbordered words and factors of words play a significant role in some proofs concerning combinatorial properties of words. The questions involving periodicity of finite and infinite words are naturally related to the border structure of words see, e.g., [3, 4, 5, 6, 7, 11]. As another example, we mention that the existence of borders in words appear in the study of coding properties of sets of words as well as in unavoidability studies of words; see, e.g., [1, 13].

In this paper we study the border structure of words with respect to conjugation. We shall consider solely binary words. To this end, we fix our alphabet to be $A = \{a, b\}$. Let $A^*$ denote the monoid of all finite words over $A$ including the empty word, denoted by $\varepsilon$. Let $\sigma\colon A^* \to A^*$ be the (cyclic) *shift function* of words, where $\sigma(\varepsilon) = \varepsilon$ and $\sigma(cw) = wc$ for all $w \in A^*$ and $c \in A$. The *border correlation function* $\beta\colon A^* \to A^*$ is defined such that $\beta(w)$ specifies which conjugates of $w$ are unbordered: Let $w \in A^*$ be a word of length $n$. Then $\beta(w) = c_0c_1 \dots c_{n-1}$, where

$$c_i = \begin{cases} a & \text{if } \sigma^i(w) \text{ is unbordered,} \\ b & \text{if } \sigma^i(w) \text{ is bordered.} \end{cases}$$

Let $\beta(\varepsilon) = \varepsilon$. For example, let $w = aabab$. Then

$$\sigma^0(w) = w = aabab,\ \sigma^1(w) = ababa,\ \sigma^2(w) = babaa,$$
$$\sigma^3(w) = abaab,\ \sigma^4(w) = baaba,$$

and hence $\beta(w) = ababb$, since only $\sigma^0(w)$ and $\sigma^2(w)$ are unbordered. While we consider only binary words in this paper, note that $\beta$ can be applied to words over any alphabet and yields always a binary word.

It is rather easy to show (see Lemma 1) that the image $\beta(w)$ of a binary word $w$ cannot have two consecutive $a$'s (except for some trivial words), that is, for no $i$ are both $\sigma^i(w)$ and $\sigma^{i+1}(w)$ unbordered. In Section 2 we show that the bound given by this fact is optimal. Indeed, we prove that in every cyclically overlap-free word every other conjugate (that is, either $\sigma^i(w)$ or $\sigma^{i+1}(w)$ for each $i$) is unbordered.

A word $w \in A^*$ is *overlap-free*, if it does not have self-overlapping factors, that is, $w$ does not have a factor of the form $cxcxc$ where $c \in A$ and $x \in A^*$.

Moreover, $w$ is *cyclically overlap-free*, if all its conjugates are overlap-free. The cyclically overlap-free binary words were characterized by Thue [15]; see Section 2.

There is a close relationship between unbordered conjugates of a word and its critical points, when critical points are considered independent of cyclic shifts. This relation is elaborated on in Section 3.

In Section 4 we shall study the dynamic system given by the border correlation function $\beta$. We prove that, for each word $w$ of length $n$, the sequence $w, \beta(w), \beta^2(w), \ldots$ terminates either in the word $b^n$ or in the cycle of the conjugates of the word $ab^kab^{k+1}$ for $k = (n-3)/2$.

The border correlation function provides a similarity function among the strings. Related functions of similarity are the *auto-correlation* function of Guibas and Odlyzko [8], and the *border-array* function of Moore, Smyth, and Miller [12].

We end this section with some definitions and notation needed in the rest of the paper. We refer to Lothaire's book [11] for more basic and general definitions of combinatorics on words.

We denote the length of a word $w$ by $|w|$. Also, if $w \in A^*$ and $c \in A$, then $|w|_c$ denotes the number of occurrences of letter $c$ in $w$. For instance, we have for $w = abaab$ that $|w|_a = 3$ and $|w|_b = 2$. Suppose $w = uv$. Then $u$ is called a *prefix* of $w$, denoted by $u \leq w$, and $v$ is called a *suffix* of $w$. A nonempty word $u \in A^*$ is a *border* of a word $w \in A^*$, if $w = uv = v'u$ for some suitable nonempty words $v$ and $v'$ in $A^*$.

We call two words $u$ and $v$ *conjugates*, denoted by $u \sim v$, if $u = \sigma^k(v)$ for some $k \geq 0$. Two conjugates $u$ and $v$ are called *adjacent*, if $\sigma(u) = v$. Clearly, $\sim$ is an equivalence relation. Let $[u] = \{v \mid u \sim v\}$ denote the *conjugate class* of $u$. A word $w$ is *primitive* if it is not a proper power of another word, that is, $w = u^k$ implies $u = w$ and $k = 1$. A word $w$ is called a *Lyndon word* if it is primitive and minimal among all its conjugates with respect to some lexicographic order. In the binary case $A = \{a, b\}$, there are two orders given by $a \lhd b$ and its inverse $b \lhd^{-1} a$. It is well known (see, e.g., Lothaire [11]), that each primitive word $w$ has a unique Lyndon conjugate with respect to a given order. For example, consider $w = abaabb$. Then $aabbab$ and $bbabaa$ are conjugates of $w$ and they are minimal with respect to the order $\lhd$ and $\lhd^{-1}$, respectively. These words are thus Lyndon words.

# 2 Optimal words for border correlation

Let $w$ be a nonempty word of length $n$ in $A^*$. If it is not primitive, that is, $w = u^k$ for some $u$ and $k \geq 2$, then it is immediate that all conjugates of $w$

are nonprimitive, and thus bordered. Therefore, $\beta(w) = b^n$ in this case. It is also clear that $\beta$ is invariant under renaming. That is, if $w'$ is obtained from $w$ by exchanging the letters $a$ and $b$, then $\beta(w') = \beta(w)$. Therefore $\beta$ is not injective, and thus not surjective, that is, there are at most $2^{n-1}$ words of length $n$ that are $\beta$-images. In fact, this number is much lower as we will show later with Corollary 7.

The following lemma gives some useful properties of the images $\beta(w)$. By the second case of the lemma, $\beta(w)$ does not contain two adjacent letters $a$ unless $w$ is a conjugate of the special words $ab^{n-1}$ or $ba^{n-1}$. Notice that $\beta(ab^{n-1}) = aab^{n-2} = \beta(ba^{n-1})$.

**Lemma 1.** *Let $w \in A^*$ of length $n \geq 4$.*

*(i) If $w$ is primitive, then $|\beta(w)|_a \geq 2$.*

*(ii) For each $i = 0, 1, \ldots, n-1$, $\sigma^i(w)$ or $\sigma^{i+1}(w)$ is bordered, or $w \in [ab^{n-1}]$ or $w \in [ba^{n-1}]$.*

*(iii) The word $w$ can have at most $\lfloor |w|/2 \rfloor$ unbordered conjugates.*

*Proof.* For (i), we notice, as mentioned in the introduction, that each primitive word $w$ has two Lyndon conjugates. Since Lyndon words are unbordered (see Lothaire [11]), the claim follows.

For (ii), assume that $w$ is not a conjugate of $ab^{n-1}$ nor of $ba^{n-1}$, and hence, it has at least two occurrences of $a$ and of $b$. Let $w' = \sigma^i(w)$ be any unbordered conjugate of $w$. Without loss of generality, we assume that $w'$ begins with $a$, and, consequently, $w' = ab^k x a b^j$, where $j > k \geq 0$ and the word $xa$ begins with $a$, since $w'$ is unbordered. (We may have $x = \varepsilon$.) Now, $\sigma(w') = b^k x a b^j a$ has a border $b^k a$, and hence, $\sigma^{i+1}(w)$ is bordered, as required.

The claim (iii) is clear from (ii). □

In particular, if the length of $w$ is an odd number $\geq 5$, then $w$ has two adjacent conjugates that are both bordered.

**Example 2.** Consider $w = abbabaa$. Although the image $\beta(w) = bababab$ does not contain $b^2$ as a factor, it has a conjugate that does so. Indeed, the adjacent conjugates $\sigma^6(w) = aabbaba$ and $\sigma^7(w) = w$ are both bordered.

Lemma 1 (iii) states that a word of length 4 or more has at most $\lfloor |w|/2 \rfloor$ many conjugates. The next example shows such words.

**Example 3.** There are words for which the maximum number $\lfloor |w|/2 \rfloor$ of unbordered conjugates is obtained. Every second conjugate of $w$ is unbordered, for instance, in the following cases $w = aabb$ and $w = abaabbaabab$. In these examples, $\beta(w) = (ab)^{|w|/2}$. However, there is no word of length 10 that has 5 unbordered conjugates (see Theorem 6). Also, e.g., for $w = aabbbab$ of odd length, we have $\beta(w) = ababbab$, and hence, $|\beta(w)|_a = 3 = \lfloor |w|/2 \rfloor$ in this case.

There is a close relationship between overlap-free binary words and the maximum number of unbordered conjugates. Theorems 5 and 6 clarify this relation. Before we prove these theorems, let us recall that the *Thue-Morse* morphism [14, 15] $\tau\colon A^* \to A^*$ is defined by $\tau(a) = ab$ and $\tau(b) = ba$.

The following result is due to Thue [15] (see also [9]).

**Lemma 4.** *Let $w \in A^*$ be a cyclically overlap-free word.*

*(i)* *$\tau(w)$ is cyclically overlap-free.*

*(ii)* *$\tau^{-1}(w)$ is cyclically overlap-free if $w \in \{ab, ba\}^*$.*

*(iii)* *Either $w$ or $\sigma(w)$ has a factorization in terms of $ab$ and $ba$, that is, $w \in \{ab, ba\}^*$.*

*(iv)* *For some $u \in \{a, b, aab, abb\}$ and $n \geq 0$, $w \in [\tau^n(u)]$. In particular, $|w| = 2^n$ or $3 \cdot 2^n$ for some $n \geq 0$.*

Note, that cyclically overlap-free words longer than 3 are of even length. Theorem 5 shows that cyclically overlap-free binary words have a maximum number of unbordered conjugates. In the theorem, "every other conjugate of $w$ is unbordered" means, by Lemma 1(iii), that $\beta(w)$ is $(ab)^{n/2}$ or $(ba)^{n/2}$ for some even $n$.

**Theorem 5.** *Let $w \in A^*$ and $|w| > 3$. Every other conjugate of $w$ is unbordered, if and only if $w$ is a cyclically overlap-free word.*

*Proof.* Let $w$ be a word of length $n$ that contains an overlapping factor, i.e., $w = ucxcxcv$, where $c \in A$ and $u, v, x \in A^*$. Let $i = |ucx|$. Then the conjugates $\sigma^i(w) = cxcvucx$ and $\sigma^{i+1}(w) = xcvucxc$ are both bordered, with borders $cx$ and $xc$, respectively.

In the other direction, suppose that $w$ is cyclically overlap-free word such that both $\sigma(w)$ and $\sigma^2(w)$ are bordered. Clearly, $|w| \geq 4$. We derive a contradiction which proves the claim. Let $u$ be the shortest border of $\sigma(w)$ and $v$ be the shortest border of $\sigma^2(w)$. Note that in general the shortest border of a word $g$ is not longer than $\lfloor |x|/2 \rfloor$

We shall assume that $a \leq w$. The case $b \leq w$ is symmetric, and it can be thus omitted.

**Case 1:** Assume first that $aa \leq w$. Then $u = a$, and $\sigma(w) \in \{ab, ba\}^*$ by Lemma 4(iii). It follows that $aab \leq w$, and hence $w = aabw_0b$ where $w_0 \in \{ab, ba\}^*$ and the $\tau$-factorization of $\sigma(w)$ is given by $\sigma(w) = (ab)w_0(ba)$. Now, $\sigma^2(w) = bw_0baa$. Note that $v \neq baa$ for the border $v$ of $\sigma^2(w)$, because $w_0 \in \{ab, ba\}^*$. Consequently, $v = bv'baa$ for some $v' \in A^*$. Since we have $\sigma^2(w) = vzv$ for some nonempty $z$, and $\sigma(w) \in \{ab, ba\}^*$, $w$ has a conjugate $vvby$ (where $z = by$). This is a contradiction, since $v$ begins with $b$ and so $vvb$ is not overlap-free.

**Case 2:** Assume that $ab \leq w$. We have that $bb$ is a suffix of $w$, since $w$ is unbordered. Therefore again $\sigma(w) \in \{ab, ba\}^*$ which implies that $u = ba$, and also $aba \leq w$, say $w = abaw_0b$. We have $w = abaw_1bb$, since $w$ is unbordered. Moreover, $w = abaw_2abb$, since $\sigma(w) \in \{ab, ba\}^*$. Actually, $w = abaabw_3abb$, since $\tau^{-1}(\sigma(w))$ is cyclically overlap-free by Lemma 4(ii) and thus it is also in $\{ab, ba\}^*$. We have the following $\tau$-factorization $\sigma(w) = (ba)(ab)w_3(ab)(ba)$, where $w_3 \in \{ab, ba\}^*$. Now, the shortest border $v$ of $\sigma^2(w)$ is either (2a) $v = aabbab$ or (2b) $v = aabv'abbab$ for some word $v'$. In Case (2a), we have $\sigma^2(w) = aabbabw_4aabbab$, where $w_4 \neq \varepsilon$ (otherwise, $\tau^{-1}(\sigma(w)) \notin \{ab, ba\}^*$). Hence, $\sigma(w) = (ba)(ab)(ba)(bw_4a)(ab)(ba)$ and so $w_4 = aw_5b$, that is,

$$\sigma(w) = (ba)(ab)(ba)(ba)w_5(ba)(ab)(ba)\,,$$

and thus $\tau^{-1}(\sigma(w) = babb\tau^{-1}(w_5)bab$, and therefore $babbabb$ is a factor in a conjugate of the preimage $\tau^{-1}(\sigma(w))$ contradicting the overlap-freeness requirement. In Case (2b), we have that $vvay$ occurs in a conjugate of $w$. This is a contradiction, since $v$ begins with $a$, and thus $vva$ is an overlapping factor. This completes the proof of the theorem. □

The next theorem shows that words (of even length) with a maximum number of unbordered conjugates are cyclically overlap-free with two exceptions.

**Theorem 6.** *Let $n \geq 1$. Every word of length $2n$ that has $n$ unbordered conjugates is either cyclically overlap-free or a conjugate of abbb or aaab.*

*Proof.* Note that $\beta(abbb) = aabb$ and $\beta(aaab) = abba$. The claim follows easily now from Lemma 1 and Theorem 5. □

Theorems 5 and 6 show that every word with a maximum number of unbordered conjugates is cyclically overlap-free, except for the conjugates of $abbb$ and $aaab$. By Lemma 4(iv), each such word has length either $2^n$ or $3 \cdot 2^n$ for some $n \geq 1$.

Lemma 6 and Theorem 5 give an upper bound on the number of $\beta$-images. Let $A^n$ denote all words over $A$ of length $n$, and let $B_n$ denote the number of all $\beta$-images of length $n$. Let $F_n$ be the $n$-th Fibonacci number so that $F_0 = 1$ and $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 3$.

**Corollary 7.** *Let* $\mathcal{M} = \{2i \mid i \geq 0\} \setminus \{2^j, 3 \cdot 2^j \mid j \geq 0\}$. *Then for all* $n \geq 3$

$$\beta(A^n) \subseteq \left[aab^{n-2}\right] \cup \left\{w \,\middle|\, |w|_a \geq 2,\ a^2 \text{ not in } ww\right\} \setminus \left\{(ab)^k, (ba)^k \,\middle|\, k \in \mathcal{M}\right\} \tag{1}$$

*and*

$$B_n \leq F_n + F_{n-2} - m \tag{2}$$

*where* $m = 2$, *if* $n \in \mathcal{M}$, *and* $m = 0$ *otherwise.*

*Proof.* Clearly, (1) follows from Lemma 6 and Theorem 5. We show how (2) follows from (1).

Let $A_n$ denote the set of words of length $n$ that have no factors $a^2$. Now,

(i) each $w \in A_{n-1}$ yields an element $wb \in A_n$, and all elements of $A_n$ ending in $b$ can be so obtained;

(ii) each $w \in A_{n-1}$ ending with $b$ yields $wa \in A_n$, and all elements of $A_n$ ending in $a$ can be so obtained.

By case (i), the number of required words $w$ in case (ii) is equal to $|A_{n-2}|$. Therefore, $|A_n| = |A_{n-1}| + |A_{n-2}|$. Since $|A_1| = 2$, we have that $|A_n| = F_{n+1}$ for all $n \geq 1$.

Moreover, for $n \geq 5$, the words $w \in A_n$ that begin and end in $a$ are of the form $w = abvba$, where $v \in A_{n-4}$. Hence the number of these words is $F_{n-3}$. We conclude that there are $F_{n+1} - F_{n-3} = F_n + F_{n-2}$ words of length $n$ with $n \geq 5$ whose conjugates do not have the factor $a^2$.

We do not consider the $n$ different words of length $n$ with exactly one $a$. Therefore, $\{w \mid |w|_a \geq 2,\ a^2 \text{ not in } ww\}$ has $F_n + F_{n-2} - n$ elements. Clearly, $[aab^{n-2}]$ has $n$ elements. The claim then follows for $n \geq 5$ from Lemma 1. By inspection, we see that (2) holds for $n = 3$ and 4, and thus the claim follows for all $n \geq 3$. □

*Remark* 8. We have calculated $B_n$ for all $n \leq 30$ using a computer; see Table 1.

It is remarkable that the bound (2) given in Corollary 7 is tight for all $n \leq 30$ except if $n = 12$. That is $B_n = F_n + F_{n-2} - m$ for all $3 \leq n \leq 30$ except if $n = 12$ where $m = 2$, if $n \in \mathcal{M}$, and $m = 0$ otherwise. Actually, there exists no word $w$ such that $\beta(w) \in [ababbbabab]$. We have that $B_{12} = F_{12} + F_{10} - 12$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 1 | 2 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 121 | 199 | 310 | 521 | 841 | 1364 |

| $n$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| $m$ | 2207 | 3571 | 5776 | 9349 | 15125 | 24476 | 39601 | 64079 | 103682 |

| $n$ | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|
| $m$ | 167761 | 271441 | 439204 | 710645 | 1149851 | 1860496 |

Table 1: The number $m$ of $\beta$-images for lengths $1 \leq n \leq 30$

# 3 Unbordered Conjugates and Critical Factorizations

In this section we investigate the relation between the border correlation function and critical factorizations. First we introduce the critical points of words.

Let $w = a_0 a_1 \dots a_{n-1} \in A^*$, where $a_i \in A$ for each $i$. An integer $1 \leq q \leq n$ is a *period* of $w$, if $a_i = a_{i+q}$ for all $0 \leq i < n - q$. The smallest period of $w$ is denoted by $\partial(w)$. For instance, $\partial(w) = |w|$, if and only if $w$ is unbordered. It is easy to see that $q$, with $1 \leq q \leq |w|$, is a period of $w$, if and only if there is a word $v$ of length $q$ such that $w$ is a factor of $v^n$ for some $n \geq 1$. Let for example $w = abaababa$. Then the periods of $w$ are $5, 7$, and $8 = |w|$. In this example, $\partial(w) = 5$.

An integer $p$ with $1 \leq p < |w|$ is called a *point* in $w$. Intuitively, a point $p$ denotes the place between $a_{p-1}$ and $a_p$ in $w$ above. A nonempty word $u$ is called a *repetition word* at point $p$ if $w = xy$ with $|x| = p$ and there exist $x'$ and $y'$ such that $u$ is a suffix of $x'x$ and a prefix of $yy'$. For a point $p$ in $w$, let

$$\partial(w, p) = \min\{|u| \mid u \text{ is a repetition word at } p\}$$

denote the *local period* at point $p$ in $w$. Let for example $w = abaabab$. Now, for instance, $\partial(w, 2) = 3$, since the shortest repetition word at $p = 3$ is $aab$. Indeed, $aw = (aab)(aab)ab$. The shortest repetition words of $w$ for the points $p = 1, 2, \dots, 6$ are, respectively, $ba, aab, aba, babaa, ab$, and $ba$. We notice that $\partial(w) = 5 = \partial(w, 4)$.

Note that the repetition word of length $\partial(w, p)$ at point $p$ is necessarily unbordered and $\partial(w, p) \leq \partial(w)$. A factorization $w = uv$, with $u, v \neq \varepsilon$ and $|u| = p$, is called *critical*, if $\partial(w, p) = \partial(w)$, and, if this holds, then $p$ is called *critical point*.

We recall the critical factorization theorem next [11] (see also [10]).

**Theorem 9.** *Every word $w$, with $|w| \geq 2$, has at least one critical factorization $w = uv$, with $u, v \neq \varepsilon$ and $|u| < \partial(w)$, i.e., $\partial(w, |u|) = \partial(w)$.*

The following lemma is a consequence of the critical factorization theorem. It is proven in [2].

**Lemma 10.** *Let $w = uv$ be unbordered and $|u|$ be a critical point. Then $vu$ is unbordered.*

There is no direct relationship between critical points and unbordered conjugates in general, since, for instance, the number of critical points is not invariant under cyclic shifts whereas the border correlation function is; see Remark 15 in the next section. Moreover, if $w = uv$ such that $vu$ is unbordered, then $|u|$ is not a critical point in general.

**Example 11.** Consider the conjugate class of $w = ababa$

$$[w] = \{ababa,\ babaa,\ abaab,\ baaba,\ aabab\}$$

with 4, 1, 2, 2, and 1 critical points, respectively. However, the word $w$ has exactly two unbordered conjugates *babaa* and *aabab*.

In general, it is not so that there is a word $w'$ in the conjugate class of some word $w$ such that the critical points of $w'$ mark the unbordered conjugates of $w$ like *babaa* and *aabab* in the above example.

**Example 12.** Consider the conjugate class of $w = abbabaab$. We have exactly two critical points for every $w' \in [w]$ but four unbordered conjugates in $[w]$.

However, if critical points are considered modulo cyclic shifts, the situation changes. Let $w$ be a word of length $n$. We call an integer $p$, with $0 \leq p < n$, an *internal critical point* of $w$, if $p + n$ is a critical point of $www$. The following lemma shows that internal critical points are invariant under cyclic shifts.

**Lemma 13.** *Let $w$ be a word of length $n$. The point $p$ is internal critical of $w$, if and only if the point*

$$q = \begin{cases} p - i & \text{if } p \geq i\,, \\ p + n - i & \text{otherwise}\,, \end{cases}$$

*with $0 \leq i < n$, is an internal critical point of $u = \sigma^i(w)$.*

*Proof.* Clearly, $www$ contains all conjugates of $ww$. Moreover, it follows from $\sigma(ww) = \sigma(w)\sigma(w)$ that $uuu$ also contains all conjugates of $ww$. In fact, let $v \in [w]$ such that $v = \sigma^j(w)$ with $0 \leq j < n$, then $vv = \sigma^j(ww)$ and $www = xvvz$ where $|x| = j$. In particular, $uuu = x'vvz'$, where $|x'| = j - i$, if $j \geq i$, and $|x'| = j + n - i$ otherwise.

Surely, the implication directions of the claim are symmetric to each other. Assume $p$ is an internal critical point of $w$. Let $v$ be the shortest repetition word at point $p + n$ in $www$. We have that $v$ is a conjugate of $w$, since $p + n$ is critical. So, $www = xvvz$ where $|x| = p$. Now, $uuu = x'vvz'$ where $|x'| = p - i$, if $p \geq i$, and $|x'| = p + n - i$ otherwise, and hence, the point $q + n$ is critical, and this proves the claim. □

**Theorem 14.** *Let $w$ be a primitive word of length $n$, and let $0 \leq p < n$. Then the following statements are equivalent:*

- *$p$ is an internal critical point of $w$.*
- *the conjugate $\sigma^p(w)$ is unbordered.*

*Proof.* Assume $p$ is an internal critical point of $w$. Then $www = xvvz$ where $|x| = p$ and $v$ is an unbordered factor of length $n$ in $ww$. Hence, $\sigma^p(w) = v$.

Assume $v = \sigma^p(w)$ is an unbordered conjugate of $w$. Then $www = xvvz$ with $|x| = p$, and $p + n$ is a critical point of $www$. Hence, $p$ is an internal critical point of $w$. □

# 4 Iterations of the Border Correlation Function

In this section we investigate iterations of the border correlation function. We start by considering the *$\beta$-graph* $G_\beta(n)$ for each $n \geq 1$. It is the directed graph with the set $A^n = \{w \mid |w| = n, w \in A^*\}$ as vertices, and with edges determined by the border correlation function $\beta$, that is, there is a (directed) edge $u \to v$, if and only if $\beta(u) = v$. Note that every vertex has exactly one outgoing edge. In order to avoid trivial exceptions, we assume in this section that $n \geq 3$.

*Remark* 15. It is straightforward to see that $\beta(\sigma(w)) = \sigma(\beta(w))$, that is, the following diagram commutes.

$$\begin{array}{ccc} w & \xrightarrow{\beta} & u \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ w' & \xrightarrow{\beta} & u' \end{array}$$

So, the $\beta$-graph $G_\beta(n)$ consists of components where each component contains exactly one cycle, since for all members of one conjugate class $[w]$ the images are mapped to the conjugate class $[\beta(w)]$ and every vertex has not more than one outgoing edge.

In the following we show that any cycle in the graph $G_\beta(n)$ consists of exactly one conjugate class. Moreover, we describe all conjugate classes that form a cycle.

Let $\kappa\colon A^* \to \mathbb{N}$ where $\kappa(w)$ denotes the minimum $k$ such that $ab^ka$ occurs in any conjugate of $w$, or $w$ is a conjugate of $ab^k$, or $w = b^k$. Note that $k = 0$, if, and only if, $a^2$ occurs in $w$ or $\sigma(w)$. Let $\mu\colon A^* \to \mathbb{N} \times \mathbb{N}$ be defined such that $\mu(w) = (|w|_a, |w| - \kappa(w))$. Note that $\mu(w) = \mu(\sigma(w))$. Let $<$ denote the extension of the ordering of natural numbers to the lexicographic order on $\mathbb{N} \times \mathbb{N}$; in other words, $(p, q) < (r, s)$ if $p < r$, or $p = r$ and $q < s$.

**Theorem 16.** *Let $w$ be a word not in $b^*$ and not in $[ab^k] \cup [ab^kab^{k+1}]$, for all $k \geq 0$. Then $\mu(\beta(w)) < \mu(w)$.*

*Proof.* Let $w$ be a word of length $n$ that is not in $b^* \cup [ab^{n-1}]$ and not in $[ab^kab^{k+1}]$, for $k = (n-3)/2$. Note that $a$ occurs at least twice in $w$. If $w$ is not primitive, then $\beta(w) = b^n$ and, in this case, it is clear that $\mu(\beta(w)) < \mu(w)$. Assume then that $w$ is primitive. Because $\mu(w) = \mu(\sigma(w))$, we can choose any conjugate of $w$ without changing its $\mu$ image. Therefore, we can assume that $w$ begins with $a$ and that it is unbordered. For example, we may take the Lyndon word in the conjugate class $[w]$ with respect to the order $a \lhd b$. We have now a unique factorization in the form $w = B_1B_2 \cdots B_r$, where each $B_i = ab^{k_i}$ with $r \geq 2$ and $k_i \geq 0$ for all $1 \leq i \leq r$. Let $m$ be the minimum of all $k_i$.

Note that $|\beta(w)|_a \leq |w|_a$ by Lemma 1. So, every occurrence of the letter $a$ in $w$ implies at most one $a$ in $\beta(w)$, because if the $i$-th letter of $w$ be $a$, then not both $\sigma^{i-1}(w)$ and $\sigma^i(w)$ can be unbordered conjugates of $w$ by Lemma 1(ii). If an occurrence of $a$ in $w$ does not imply an $a$ in $\beta(w)$, we say that this occurrence of $a$ is *dropped*.

The claim follows, if $|\beta(w)|_a < |w|_a$, and therefore, we can assume that $|\beta(w)|_a = |w|_a$, that is, no occurrence of $a$ is dropped: for every $i \geq 1$, if the $i$-th letter in $w$ is an $a$, then either $\sigma^{i-1}(w)$ or $\sigma^i(w)$ is unbordered. Since $w$ begins with $a$ and is unbordered, we have that $\beta(w) = B'_1B'_2 \cdots B'_r$, where $B'_i = ab^{k'_i}$ and $k'_i > 0$ for all $1 \leq i \leq r$. Note that the $a$ in $B'_i$ corresponds to the unbordered conjugate of $w$, if $w$ is factored either before or after the occurrence of $a$ in $B_i$. We show that $\kappa(w) < \kappa(\beta(w))$ in this case.

Let $i + 1$ be modulo $r$ in the following, and let $j = |B_1B_2 \cdots B_i|$. If $k_i = k_{i+1}$ then the $a$ in $B_{i+1}$ is dropped, that is, neither $\sigma^j(w)$ nor $\sigma^{j+1}(w)$ is bordered; a contradiction. So, assume that $k_i \neq k_{i+1}$.

Note that if $k_i > k_{i+1}$ then $\sigma^{j+1}(w)$ is bordered and $\sigma^j(w)$ is unbordered by assumption, and if $k_i < k_{i+1}$ then $\sigma^j(w)$ is bordered and $\sigma^{j+1}(w)$ is unbordered by assumption.

If $k_i > k_{i+1}$ then $k'_i = k_i$, in case $k_{i-1} > k_i$, and $k'_i = k_i - 1$, in case $k_{i-1} < k_i$.

If $k_i < k_{i+1}$ then $k'_i = k_i + 1$.

Now, we have that $|k_i - k'_i| \leq 1$. If $k_i = m$ then $k'_i = k + 1$. However, we get $k'_i = m$, if and only if $k_{i-1} = m$ and $k_i = k + 1$ and $k_{i+1} = m$, and $r \geq 4$, since $w \notin [ab^k ab^{k+1}]$ and, by assumption, $|\beta(w)|_a = |w|_a$. Therefore, we also have $k_{i-2} > m$ and $b^{m+1}ab^m ab^{m+1}ab^m a$ occurs in a conjugate of $w$, and both $\sigma^j(w)$ and $\sigma^{j+1}(w)$ are bordered; a contradiction.

So, $k'_\ell > m$, for all $1 \leq \ell \leq r$, if $|\beta(w)|_a = |w|_a$, and therefore we have $\mu(\beta(w)) < \mu(w)$. □

**Lemma 17.** *Let $w \in [ab^k ab^{k+1}]$ with $k \geq 0$. Then*

$$[ab^k ab^{k+1}] = \{\beta^i(w) \mid 0 \leq i < |w|\} \ .$$

*Proof.* We have that $w = b^r ab^s ab^t$, where either $r + t = k$ and $s = k + 1$, or $r + t = k + 1$ and $s = k$. Now $\beta(w) = b^{r+1}ab^{s-1}ab^t = \sigma^s(w)$ in the former case and $\beta(w) = b^r ab^{s+1}ab^{t-1} = \sigma^{s+1}(w)$ in the latter case. That is, $\beta(w) = \sigma^{k+1}(w)$, and the claim follows, since $2k + 3$ and $k + 1$ are relatively prime. □

Note that the proof of Lemma 17 gives $\beta^j(\sigma^i(ab^k ab^{k+1})) = \sigma^m(ab^k ab^{k+1})$ where $m$ is $i + j(k + 1)$ modulo $(2k + 3)$.

We are now ready to show that iterations of $\beta$ on any binary word result in a word of a certain shape.

**Theorem 18.** *For every word $w$, there exists an $i \geq 0$ such that $\beta^i(w) \in b^*$ or $\beta^i(w) \in [ab^k ab^{k+1}]$.*

*Proof.* Let $w$ be a word of length $n$. Note that $\beta(w) = b^n$, if $w$ is not primitive. Assume thus that $w$ is primitive. Note that if $\mu(w) \neq \mu(u)$ then $[w] \neq [u]$, and that $\beta(w) \notin [ab^{n-1}]$, since $w$ has at least two unbordered conjugates. If $w \in [ab^{n-1}]$ then $\beta(w) \in [aab^{n-2}]$. If $w \in [ab^k ab^{k+1}]$ then $\beta(w) \in [ab^k ab^{k+1}]$ by Lemma 17.

Suppose now that $w$ is different from $b^n$ and $w$ is not in $[ab^{n-1}] \cup [ab^k ab^{k+1}]$ for $k = (n - 3)/2$. Since the values of $\mu$ strictly decrease after an application of $\beta$, by Theorem 16, we conclude that there exists an $i \geq 1$ such that $\beta^i(w) = b^n$ or $\beta^i(w) \in [ab^k ab^{k+1}]$. □

Observe that by Theorem 18 for every word $w$ of even length there exists an $i \geq 0$ such that $\beta(w) = b^{|w|}$.

Consider then the graph $G_\beta^\sim(n)$, which consists of the conjugate classes $[w]$, for $|w| = n$, as its vertices and there is an edge $[u] \to [v]$ if $\beta(u) = v$. By the above results, this graph is well defined, and it consists of trees when disregarding reflexive loops $[u] \to [u]$. (See Figure 1 for the graph $G_\beta^\sim(7)$.)

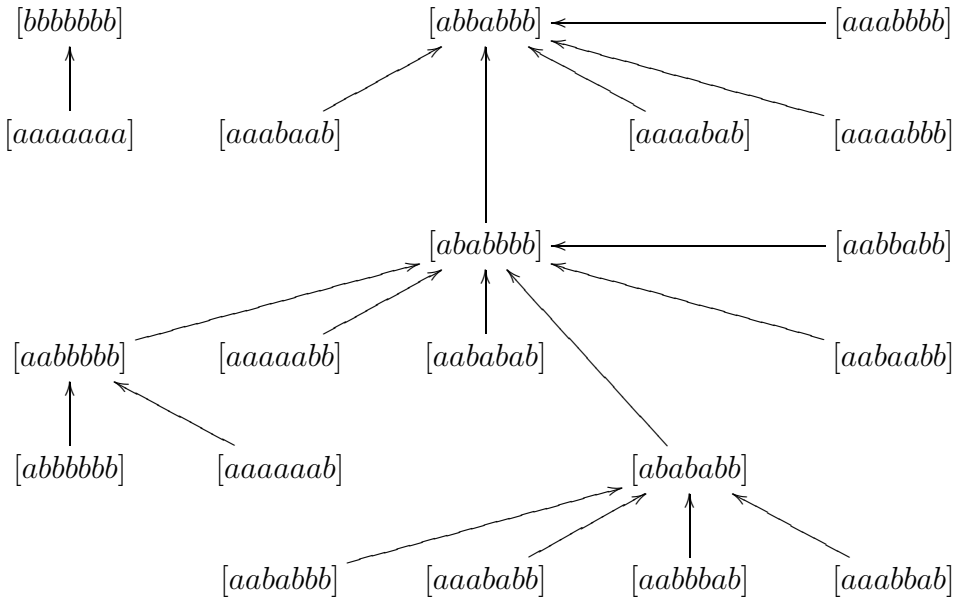


Figure 1: The graph $G_\beta^\sim(7)$. We have omitted the loops of the vertices $[b^7]$ and $[abbabbb]$.

# 5 Discussion

We have investigated the border correlation function $\beta$ of binary words. The shape of $\beta$ images for words with a minimal and maximal number of unbordered conjugates has been clarified. Nevertheless, the set $\beta(A^*)$ has not been completely described. Corollary 7 seems to give a very good estimation. All $\beta$-images up to length 30 have been checked and only words of length 12 seem to be exceptional.

Apart from the border correlation function $\beta$ one could investigate an extension $\beta' \colon A^* \to \mathbb{N}^*$ of that function such that a word $w$ of length $n$ is mapped to $m_0 m_1 \cdots m_{n-1}$ where $m_i$ is the length of the shortest border of

$\sigma^i(w)$ for all $0 \leq i < n$. We just notice here that $\beta'$ is injective, since, if $u = wau'$ and $v = wbv'$, then clearly the shortest borders of the $|w|$-th conjugates $au'w$ and $bv'w$ are different, because one of them is equal to 1, and the other is not.


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# References


[1] J. Berstel and D. Perrin. *Theory of codes*, volume 117 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1985.

[2] D. Breslauer, T. Jiang, and Z. Jiang. Rotations of periodic strings and short superstrings. *J. Algorithms*, 24(2), 1997.

[3] C. Choffrut and J. Karhumäki. Combinatorics of words. In A. Salomaa and G. Rozenberg, editors, *Handbook of Formal Languages*, volume 1, pages 329–438. Springer-Verlag, Berlin, 1997.

[4] W.-F. Chuan. Unbordered factors of the characteristic sequences of irrational numbers. *Theoret. Comput. Sci.*, 205(2):337–344, 1998.

[5] J. C. Costa. Biinfinite words with maximal recurrent unbordered factors. *Theoret. Comput. Sci.*, 290(3):2053–2061, 2003.

[6] J.-P. Duval. Relationship between the period of a finite word and the length of its unbordered segments. *Discrete Math.*, 40(1):31–44, 1982.

[7] A. Ehrenfeucht and D. M. Silberger. Periodicity and unbordered segments of words. *Discrete Math.*, 26(2):101–109, 1979.

[8] L. J. Guibas and A. Odlyzko. String overlaps, pattern matching, and nontransitive games. *J. Combin. Theory Ser. A*, 30(2):183–203, 1981.

[9] T. Harju. On cyclically overlap-free words in binary alphabets. In G. Rozenberg and A. Salomaa, editors, *The Book of L*, pages 125–130. Springer, Berlin, 1985.

[10] T. Harju and D. Nowotka. Density of critical factorizations. *Theor. Inform. Appl.*, 36(3):315–327, 2002.

[11] M. Lothaire. *Algebraic Combinatorics on Words*, volume 90 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, United Kingdom, 2002.

[12] D. Moore, W. F. Smyth, and D. Miller. Counting distinct strings. *Algorithmica*, 23(1):1–13, 1999.

[13] H. Morita, A. J. van Wijngaarden, and A. J. Han Vinck. On the construction of maximal prefix-synchronized codes. *IEEE Trans. Inform. Theory*, 42:2158–2166, 1996.

[14] M. Morse. Recurrent geodesics on a surface of negative curvature. *Trans. Amer. Math. Soc.*, 22(1):84–100, 1921.

[15] A. Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Det Kongelige Norske Videnskabersselskabs Skrifter, I Mat.-nat. Kl. Christiania*, 1:1–67, 1912.

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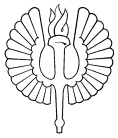

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