Chapter 4

Partitions of Integers

4.1 The Basic Partition Problem

Shortly after De Moivre used generating functions to find a formula for the n-th Fibonacci number, Euler used them to solve problems about partitions of integers.

Let us suppose that we are given an integer n and we wish to find the number of ways of expressing n as the sum of positive integers. If we regard the order of the summands as important, so that two sums differing only in the order of their summands are nonetheless counted as different sums, then the answer is 2^{n-1}. To see this, imagine n ones arranged in a line, with a space between each of the ones. For each space we have the choice either of leaving it blank or inserting a comma. Since there are $n-1$ spaces and we have two choices for the disposition of each space, we conclude there are 2^{n-1} ways of arranging our blank spaces and commas. Since the number of such arrangements is in one-one correspondence with the number of ordered partitions, we are finished.

Things become considerably more complicated when we consider unordered partitions. By this we mean that two partitions of n that differ only in the order of their summands are to be considered identical. Let us denote by $p(n)$ the number of unordered partitions of the integer n. By the basic partition problem we mean the determination of $p(n)$. From now on we will drop the term "unordered" from our description of "partition".

For example, we find that $p(n) = 1$ while $p(2) = 2$, namely 2 and $1+1$. Moving on, we find $p(3) = 3$ (3, $2+1$ and $1+1+1$). We also have $p(5) = 7$

(5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$). This grows tedious.

As a beginning we may define a generating function

$$G(x) = \sum_{n=0}^{\infty} p(n)x^n.$$

Partitions of integers arise naturally upon multiplication of power series. For example, we note that

$$\left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{k=0}^{\infty} x^k\right) = \sum_{j=0}^{\infty} c_j x^j,$$

where c_j equals the number of ways of partitioning j into the sum of two integers, where the order of the summands is taken to be important.

That is not quite what we want, but it suggests that we might be able to realize $G(n)$ as the product of several other generating functions. Define the notation $p_k(n)$ to denote the number of partitions of n into summands drawn from the set $\{1, 2, 3, \cdots, k\}$. Suppose we had chosen slightly different geometric series for our product. For example, observe that

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right) = \sum_{n=0}^{\infty} p_2(n)x^n.$$

By the same logic,

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right) = \sum_{n=0}^{\infty} p_3(n)x^n.$$

We could continue this process for any number of summands. If we desire to have no restrictions on the number of summands we may utilize, we can write

$$G(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}.$$

This formula assumes that the summands we are interested in are drawn from the set of all positive integers. If instead we had a set of positive integers $\{a_1, a_2, \cdots, a_k\}$ and wanted to find the number of ways of expressing n as a sum of integers drawn from that set, the appropriate generating function would be

$$\left(\frac{1}{1-x^{a-1}}\right)\left(\frac{1}{1-x^{a_2}}\right)\cdots\left(\frac{1}{1-x^{a_k}}\right).$$

As an application of this last formula, consider the problem of making change. To keep things simple, we will assume we have pennies, nickles, dimes and quarters. Let n denote the quantity for which we desire change. To avoid having to deal with decimals, we will assume that the monetary values of everything have been multiplied by one hundred. Then accoridng to our formula, the number of ways of making change for n is

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{10}}\right)\left(\frac{1}{1-x^{25}}\right).$$

4.2 The Size of $p(n)$

The problem of counting the number of unordered partitions of a number n has intrigued mathematicians for over a century, and a substantial body of work has developed to address it. Fundamental in this regard is a formula obtained by Hardy and Ramanujan in the early twentieth century. They showed that

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

This result was later improved by Rademacher, who obtained a formula which, when rounded to the nearest integer, would produce $p(n)$. Rademacher's formula is one of the crowning achievments of analytic number theory, but its proof is far too complicated to be included here. The Hardy-Ramanujan bound is likewise difficult to prove, so we will content ourselves with a more accessible result.

Specifically, we will show that for all n, we have that $p(n) < e^{\left(\frac{\pi^2}{6}+1\right)\sqrt{n}}$. To do that we set $G(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$ and take logarithms of both sides. When we do, we obtain

$$\log G(x) = -(\log(1-x) + \log(1-x^2) + \log(1-x^3) + \cdots).$$

We can now use the Taylor expansion

$$-\log(1-y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \cdots$$

to obtain

$$\begin{aligned}\log G(x) &= \sum_{i=1}^{\infty} \frac{x^i}{i} + \sum_{j=1}^{\infty} \frac{x^{2j}}{j} + \sum_{k=1}^{\infty} \frac{x^{3k}}{k} + \cdots \\ &= \sum_{i=1}^{\infty} x^i + \sum_{j=1}^{\infty} \frac{x^{2j}}{2} + \sum_{k=1}^{\infty} \frac{x^{3j}}{3} + \cdots \\ &= \left(\frac{x}{1-x}\right) + \frac{1}{2}\left(\frac{x^2}{1-x^2}\right) + \frac{1}{3}\left(\frac{x^3}{1-x^3}\right) + \cdots\end{aligned}$$

Further progress will only be made once we have better understood the function $\frac{x^n}{1-x^n}$. We will restrict our attention to values of x strictly between zero and one. For any such x we have that

$$x^{n-1} < x^{n-2} < x^{n-3} < \cdots < x^2 < x < 1.$$

Since the average of a set of numbers is larger than the smallest of those numbers, we can write

$$\frac{x^{n-1}}{1 + x + x^2 + \cdots + x^{n-1}} < \frac{1}{n}.$$

It follows that

$$\frac{x^n}{1-x^n} = \frac{x}{1-x} \cdot \frac{x^{n-1}}{1 + x + x^2 + \cdots + x^{n-1}} < \frac{x}{n(1-x)}.$$

If we assemble the pieces we have constructed thus far, we see that we have

$$\log G(x) < \left(\frac{x}{1-x}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{x}{1-x}\zeta(2),$$

where ζ denotes the Riemann zeta function defined in the previous chapter. Since we know that $\zeta(2) = \frac{\pi^2}{6}$, we see that

$$\log G(x) < \frac{\pi^2}{6}\left(\frac{x}{1-x}\right).$$

Since the sum defining $G(x)$ consists entirely of positive terms, we know that $G(x)$ is larger than any one of its terms. Therefore, $G(x) > p(n)x^n$ for any n. Taking logarithms of both sides leads to the discovery that

$$\log p(n) < \log G(x) - n \log x < \frac{\pi^2}{6}\left(\frac{x}{1-x}\right) - n \log x.$$

But we also know that for any real number $y > 1$, we have $\log y < y - 1$. It follows that

$$-\log(x) = \log \frac{1}{x} < \frac{1}{x} - 1 = \frac{1-x}{x}.$$

Our inequality now becomes

$$\log p(n) < \frac{\pi^2}{6}\left(\frac{x}{1-x}\right) + n\left(\frac{1-x}{x}\right).$$

If we now take $x = \dfrac{\sqrt{n}}{\sqrt{n}+1}$, then we obtain

$$\log p(n) < \left(\frac{\pi^2}{6} + 1\right)\sqrt{n},$$

from which the result follows.

4.3 Restricted Partitions

Our change-making example from section one suggests a new investigative direction for us to pursue. Suppose that instead of partitioning an integer into summands drawn from the full set of positive integers, we require instead that our summands be drawn our summands from various other sets.

In light of the result we established in section one, we can write down the following generating functions:

- The generating function for $p_k(n)$, the number of partitions of n using summands not exceeding k is

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^k)}.$$

- The generating function for $p_o(n)$, the number of partitions of n using odd summands, is

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}.$$

- The generating function for $p_e(n)$, the number of partitions of n using even summands, is

$$\frac{1}{(1-x^2)(1-x^4)(1-x^6)\cdots}.$$

In everything we have considered so far we have allowed the possibility of repeated summands in our partition. Let us suppose that we desire to contemplate partitions using distinct summands only. We will denote the number of such partitions of n by the notation $p_d(n)$.

The answer comes from the binomial theorem. There we saw that in expanding a binomial such as $(1 + x)$ to the power n, we effectively had to choose either an x or a 1 out of each term in the expansion. Thus, the coefficient of the term x^k was the number of ways of choosing k x's out of our n terms.

But what if instead of expanding $(1 + x)$, we considered a sequence of binomials of the form $(1 + x^{k_i})$, where $\{k_i\}$ is a sequence of distinct positive integers? In this case, every power of x in the resulting product would appear precisely as many times as it could be expressed as a sum of elements in $\{k_i\}$. Since the (k_i)'s are distinct, our coefficients will record the number of ways of expressing an integer n as the sum of distinct elements drawn from the (k_i)'s.

Therefore, if we took our sequence to be the set of all postive integers, we would obtain

$$p_d(n) = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots = \prod_{k=1}^{\infty}(1+x^k).$$

If instead we had sought the number of ways of expressing n as a sum of distinct powers of two, which we will denote by $p_t(n)$, we would have found that

$$p_t(n) = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = \prod_{k=1}^{\infty}(1+x^{(2^k)}).$$

In glancing over all of these generating functions, you might have noticed that some of them bear striking similarities to others. This suggests that there are intriguing relationships among them to be found.

For example, notice that for any integer k we have that $(1+x^k) = \dfrac{1-x^{2k}}{1-x^k}$. If we multiply over all possible values of k, we obtain

$$\prod_{k=1}^{\infty}(1+x^k) = \prod_{k=1}^{\infty}\left(\frac{1-x^{2k}}{1-x^k}\right) = \prod_{k=1}^{\infty}\frac{1}{1-x^{2k-1}}.$$

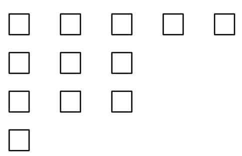
But the expression on the left is just $p_d(n)$ and the expression on the right is $p_o(n)$. It follows that these quantities are equal, and the number of ways of partitioning an integer into the sum of distinct integers is equal to the number of ways of expressing it as the sum of odd integers. Had we chosen the number six, for example, we would find that $p_o(6) = 4\ \{5+1,\ 3+3,\ 1+1+1+3,\ 1+1+1+1+1+1\}$. Similarly, $p_d(6) = 4\ \{6,\ 5+1,\ 4+2,\ 1+2+3\}$.

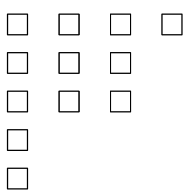
We also know that for any integer n there is precisely one base two expansion of n. Since a base two expansion involves expressing an integer as the sum of distinct powers of two, we see that $p_t(n)$ must be the generating function for the sequence consisting entirely of ones. We already have a name for this function, leading us to conclude that

$$\prod_{k=1}^{\infty}(1+x^k) = \sum_{k=1}^{\infty}x^n = \frac{1}{1-x}.$$

4.4 Ferrers Diagrams

Sometimes a geometric approach can shed light on partition problems, and we now consider a useful heuristic that has been developed for treating them. Let us consider the partition $5+3+3+1$ of 12. We can depict this partition geometrically via the first of the two diagrams below:





The second diagram corresponds to the partition $4+3+3+1+1$. It was obtained from the first by reflecting it across the line at forty-five degrees. The partition on the left is said to be conjugate to the one on the right. More precisely, let $(p_1, p_2, \cdots, p_k)$ be a partition of some integer n into k parts. Then the partition given by $(r_1, r_2, \cdots, r_s)$, where r_i denotes the number of the (p_j)'s larger than i, is the conjugate of $(p_1, p_2, \cdots, p_k)$.

Now suppose that we are given a partition of n into k parts. Then its conjugate consists entirely of summands no greater than k. We also note that every partition is conjugate to precisely one other. Therefore, we have established a one-one correspondence between partitions of n into k parts and partitions of n with summands not exceeding k. We conclude that the numbers of such partitions are equal.

As an especially clever application of a Ferrers diagram, we address the problem of finding the reciprocal of the generating function for the partition numbers. Earlier we saw that

$$G(x) = \sum_{k=1}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)}.$$

We then set

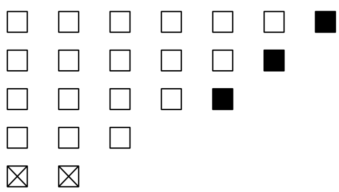
$$\frac{1}{G(x)} = \prod_{k=1}^{\infty}(1-x^k) = \sum_{k=1}^{\infty} c_n x^n,$$

and seek a formula for the coefficients c_n. When we carry out the above multiplication, we find that the coefficient of x^n will be related to the number of ways of expressing n as the sum of distinct integers. In fact, we know from our previous work that $\prod_k(1+x^k)$ is the generating function for $p_d(n)$. To take the minus sign in to consideration, we make a distinction between partitions of n into an even number of distinct summands and those possessing an odd number of distinct summands. We will denote the former by $Q_e(n)$ and the latter by $Q_o(n)$. Then we have that

$$c_n = Q_e(n) - Q_o(n).$$

How are we to evaluate this difference?

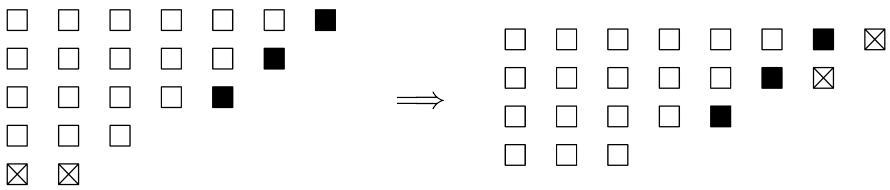
Since our interest is in the difference between $Q_e(n)$ and $Q_o(n)$, we will attempt to establish a one-one correspondence between these two sets. Towards that end, consider the Ferrers diagram below:



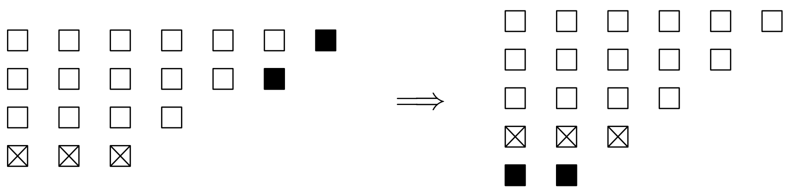
This corresponds to the partition $7+6+5+3+2$ of 23. In general, given a Ferrers diagram, we will define its slope, denoted by S, to be the number of squares appearing in a forty-five degree line beginning with the northeastern-most square in the diagram. In this case the slope is three, as indicated by the three black squares in the diagram. We will also refer call the three squares themselves the slope; we trust this will cause no confusion.

We will also define the base of the partition, denoted by B, to be the size of its smallest summand. Here that quantity is two, indicated by the boxes with $\times$'s in them. The squares making up the smallest summand will also be called the base.

We now define two operations that might be carried out on a Ferrers diagram. If, as in the present case, we have $B \leq S$ then we construct a new Ferrers diagram by removing the base, and placing its squares at the far eastern side of the diagram. This creates a new slope. The effect of this change on the above diagram is given below:

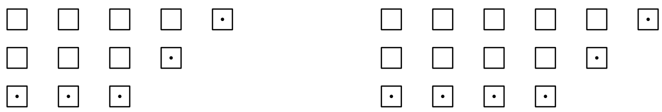


On the other hand, if we have a partition for which $B > S$, we produce a new Ferrer diagram by moving the squares that form the slope to the extreme south end of the diagram. An example of this operation is shown in the diagram below:



As long as the set of squares forming the slope is disjoint from the set of squares forming the base, it is always possible to carry out one of these two operations. The result of either operation is to transform a given partition of n into distinct summands into a different such partition. What changes is the parity of the number of parts in the partition. We note further that for any such diagram only one of the two operations can be carried out. Furthermore, applying the first operation to a given diagram produces a new diagram to which the second operation can be applied. Finally, we note that if we apply the first operation to a diagram and then apply the second operation to this new diagram, we arrive back at our original diagram. In that sense we can say the two operations are inverses of each other.

For what sorts of diagrams is it impossible to carry out either of our operations? We need two conditions. The first is that the set of squares forming the slope must intersect the set of squares forming the base. If this happens, the second condition is that $B = S$ or $B = S + 1$. These cases are illustrated respectively in the two diagrams below:



We next determine the values of n for which diagrams such as the above are possible. Let us set $\epsilon = 0$ or 1. Then we have $S = B + \epsilon$. Since we are assuming that the slope intersects the base, we know the partition must comprise consecutive numbers. We also know that the total number of terms in the partition is equal to S. With that in mind we compute

$$\begin{aligned} n &= (k+\epsilon) + (k+\epsilon+1) + (k+\epsilon+2) + \cdots + (2k+\epsilon-1) \\ &= \frac{k}{2}(3k + 2\epsilon - 1) = \frac{k}{2}(3k \pm 1). \end{aligned}$$

It follows that if k is not of the form $\dfrac{3k^2 \pm k}{2}$ then $c_k = 0$.

When k is of the form $\dfrac{3k^2 \pm k}{2}$, then we have precisely one partition of k for which it is impossible to carry out either of the two operations defined above. Whether that partition belongs to $Q_e(k)$ or $Q_o(k)$ will depend on the parity of k. Regardless, its contribution to the difference $Q_e(k) - Q_o(k)$ will be $(-1)^k$. It follows that $c_k = (-1)^k$ in these cases.

We can summarize all of our new found wisdom in the following identity, first proved by Euler:

$$\prod_{i=1}^{\infty}(1 - x^i) = 1 + \sum_{k=1}^{\infty}(-1)^k\left(x^{\frac{3k^2-k}{2}} + x^{\frac{3k^2+k}{2}}\right).$$

4.5 Summary

In this chapter we defined the following terms:

Definition 12. *By an **ordered partition** of an integer n we mean any way of expressing n as the sum of positive integers, where two partitions differing only in the number of summands are nonetheless to be considered equal. Repeated summands are allowed. If we do not regard two such partitions as different, then we talk about **unordered partitions**. The number of unordered partitions of an integer n is denoted by $p(n)$.*

Definition 13. *We also defined the following sorts of partition functions:*

- *By $p_k(n)$ we mean all the partitions of n using summands not exceeding k.*
- *By $p_o(n)$ we mean all partitions of n using odd summands only.*
- *By $p_e(n)$ we mean all partitions of n using even summands only.*
- *By $p_d(n)$ we mean all partitions of n using distinct summands.*
- *By $p_t(n)$ we mean all partitions of n using distinct powers of two.*

Definition 14. *A **Ferrers diagram** is a way of presenting a geometric picture of a particular partition. The summands in the partition are presented in horizontal rows, in decreasing order starting from the top, with each row comprising a number of squares equal to the particular summand. Thus, the partition $6+4+3+2+1$ of 16 would have the Ferrers diagram:*

□ □ □ □ □ □
□ □ □ □
□ □ □
□ □
□

Definition 15. *Given a partition $(p_1, p_2, \cdots, p_k)$ of an integer n, its **conjugate partition** is the partition $(r_1, r_2, \cdots, r_s)$ where r_i denotes the number of the (p_j)'s larger than i. Thus, the partition conjugate to $6+4+3+2+1$ is $5+4+3+2+1+1$. The Ferrers diagrams of conjugate partitions are related by a reflection across the forty-five degree line.*

We also proved the following theorems.

Theorem 4.5.1. *If $S = \{a_1, a_2, \cdots, a_k\}$ is a set of positive integers, then the generating function $G(x)$ for the number of ways of expressing an integer n as the sum of elements of S is*

$$G(x) = \prod_{i=1}^{k} \frac{1}{1 - x^{a_i}}.$$

Theorem 4.5.2. *We also have the following generating functions:*

- $\sum_{n=1}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$.
- $\sum_{n=1}^{\infty} p_k(n)x^n = \prod_{i=1}^{k} \frac{1}{1-x^i}$.
- $\sum_{n=1}^{\infty} p_o(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}}$.
- $\sum_{n=1}^{\infty} p_d(n)x^n = \prod_{i=1}^{\infty}(1 + x^k)$.

Theorem 4.5.3. *The partition numbers $p(n)$ satisfy the bound*

$$p(n) < e^{\left(\frac{\pi^2}{6}+1\right)\sqrt{n}}.$$

Theorem 4.5.4. *The number of partitions of n into distinct summands is equal to the number of partitions of n into odd summands. Thus, for all n, we have $p_o(n) = p_d(n)$.*

Theorem 4.5.5. *The following equation is true:*

$$\prod_{i=1}^{\infty}(1 - x^i) = 1 + \sum_{k=1}^{\infty}(-1)^k(x^{\frac{3k^2-k}{2}} + x^{\frac{3k^2+k}{2}}).$$

4.6 Problems

1. Prove that the number of partitions of an integer n is equal to the number of partitions of $n + 1$ whose smallest part is 1.

2. Let $p^*(n)$ denote the number of partitions of n whose summands all exceed one. Prove that $p^*(n) = p(n) - p(n - 1)$.

3. Recall that we defined the quantity $p_k(n)$ to be the number of partitions of n with summands no larger than k. Let k and n be integers with $1 < k < n$. Prove that

$$p_k(n) = p_{k-1}(n) + p_k(n-k).$$

4. Prove that for any integer n, we have $p(n+2) + p(n) \geq 2p(n+1)$.

5. In how many ways can 100 identical apples be placed into 20 identical bags if we require that each bag has at least 3 apples? A formula is fine; there is no need to come up with the exact number.

6. A given partition of an integer k is said to be self-conjugate if it is equal to its own conjugate. Prove that the number of self-conjugate partitions of k is equal to the number of partitions of k into distinct, odd summands.

7. Prove that the number of partitions of m is equal to the number of partitions of $2m$ into m parts.

8. Prove that the number of partitions of n into exactly r parts where partitions differing in the order of their summands are to be considered different is $\binom{n-1}{r-1}$.

9. Find the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers such that $x_1 + x_2 + x_3 + x_4 = 98$.

10. Prove that the number of partitions of $(a-c)$ into exactly $(b-1)$ parts, none of which is larger than c is equal to the number of partitions of $(a-b)$ into $(c-1)$ parts none of which is larger than b. (HINT: Begin by taking the Ferrers diagram of any partition of $(a-c)$ into exactly $(b-1)$ parts, none of which is larger than c. Add one more part of size c to create a partition of a into b parts, the largest of which is c. Go on from there.)

11. Prove that the number of partitions of n with no summand greater than k is equal to the number of partitions of $n+k$ with exactly k parts.

12. Given a partition P of a positive integer n, we say P is perfect if it contains precisely one partition of every number less than n. For example, the perfect partitions of 7 are $(1,1,1,1,1,1,1)$, $(4,1,1,1)$, $(4,2,1)$ and $(2,2,2,1)$. Prove that the number of perfect partitions of an integer n is equal to the number of ordered factorizations of $n+1$. Factors of one are not counted. By an ordered factorization, we mean that two factorizations of $n+1$ differing only in the order of the factors are nonetheless to be considered as different factorizations. When $n = 7$ we consider the ordered factorizations of 8 and find that there are four of them: 8, 4(2), 2(4) and 2(2)(2).