The bridge between the continuous and the discrete via original sources

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Dedicated to the memory of my friend and inspiration John Fauvel.

We describe a chapter in a forthcoming book based on original historical sources, emerging from an upper level capstone undergraduate mathematics course. Each chapter tells an important mathematical story spanning several centuries, examined in-depth through original sources. The chapter featured here follows the development of the connection between the continuous and the discrete, in the context of two interlocked themes: the search for formulas for sums of numerical powers in relation to integration, and Euler's summation formula in relation to infinite series. Further material is at `math.nmsu.edu/~history.`

For more than a decade several faculty at New Mexico State University have been teaching courses based on student study of original historical sources to learn great mathematics via the rich insights and motivation such sources can provide. Two of our courses have evolved to follow entire themes through original sources from many centuries (Laubenbacher & Pengelley, 1992, 1996), and a book based on the original sources for our lower division course already exists (Laubenbacher & Pengelley, 1998). We are completing a second book of annotated original sources, emerging from a capstone course for college juniors and seniors with substantial mathematics background (Knoebel, Laubenbacher, Lodder, & Pengelley, to appear). Here we describe one of its five independent chapter themes in the context of the pedagogy of teaching based on original sources. A detailed description of the topics for the entire book is available at our web site (Laubenbacher & Pengelley, 1999).

Following a mathematical theme over a long period via original sources is a motivating and unifying experience for students, since one sees how a breadth of mathematical areas and techniques all come together (de Guzmán, 1993). Original sources also expose one to the thrill of exploring the unknown that motivates most mathematicians, and shows that mathematics is a living, breathing, human subject, often raising more questions than it answers. In all these respects, teaching with original sources creates a mathematical experience not

as narrow as traditional courses (Laubenbacher, Pengelley, & Siddoway, 1994). More information on our work and that of others teaching with original historical sources is at (Laubenbacher & Pengelley, 1999).

Each chapter in our forthcoming book begins with a comprehensive introduction that tells an important mathematical story extending over several centuries. The chapter introduction refers to the individual sections of the chapter for in-depth study of the high points of that story through studying original sources. Each subsequent section features original source material surrounded by connecting mathematical and historical annotation, and extensive exercises based on the original sources. A supplementary web site devoted to the book will contain sample materials, and additional source translations we have made, along with more exercises and topics for further exploration.

In this paper we summarize the story told through original sources from our chapter on the relationship between the continuous and the discrete, hinging historically on two interlocking themes: the search for formulas for sums of numerical powers, and Euler's development of his summation formula in relation to sums of infinite series.

Sums of powers and Euler's summation formula: two interlocked themes

Historically our story begins in ancient times with the Greek discrete approximations used to obtain continuous areas and volumes by the method of exhaustion; in particular, Archimedes determined sums of squares to find the area inside a spiral. This was followed by medieval approaches in India and the Arab world producing individual formulas for sums of cubes and fourth powers. By the mid-seventeenth century Fermat and Pascal realized the general connection between the figurate and binomial coefficient numbers and sums of numerical powers, with application to computing areas under higher parabolas. Jakob Bernoulli, during his work on probability with combination numbers, was the first to conjecture a general pattern for the polynomial formulas for sums of powers, introducing the Bernoulli numbers into mathematics. Motivated by the Basel Problem of determining the infinite sum of the reciprocal squares, Euler then discovered the general connection between integration and discrete sums of either finite or infinite series, embodied in the Euler-Maclaurin summation formula. This achievement allowed him to solve the Basel Problem, prove Bernoulli's conjectured general formula for sums of powers, and obtain spectacular numerical approximations for sums of infinite series. It also inaugurated the study of the zeta function, at the heart of the development of modern mathematics. Our book chapter follows these two interlocked themes comprehensively; here we briefly sketch the progression via original sources, along with some pedagogical comments.

The Basel Problem

In the 1670s James Gregory and Gottfried Leibniz discovered (as essentially

had the mathematicians of Kerala in southern India two centuries before) that (Katz, 1998, pp. 493ff,527)

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Since aside from geometric series, very few infinite series then had a known sum, this beautiful and astonishing result spurred Leibniz and the brothers Jakob and Johann Bernoulli to seek the sums of other series, and particularly the reciprocal squares

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \ ?$$

The latter came to be known as the Basel Problem, and Jakob expressed his eventual frustration with its elusive nature in the comment "*If someone should succeed in finding what till now withstood our efforts and communicate it to us, we shall be much obliged to him.*" (Young, 1992, p. 345) In the early 1730s Leonhard Euler solved the problem, proving that its sum is exactly $\pi^2/6$, in part by first broadening the context to produce a general "summation formula" for $\sum_{i=1}^{n} f(i)$, with n possibly infinite. His new setting thus encompassed both the Basel Problem, $\sum_{i=1}^{\infty} \frac{1}{i^2}$, and the problem of finding precise formulas for sums of powers, $\sum_{i=1}^{n} i^k \approx \int_0^n x^k \, dx$, which had been studied from antiquity for solving area and volume problems. The summation formula Euler developed helped him resolve both questions. This is a spectacular pedagogical illustration of how generalization and abstraction can lead to the combined solution of seemingly independent problems.

Archimedes: The area inside a spiral and sums of squares

The Pythagoreans, in the sixth century B.C.E., knew that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

As a first original source, we can read Archimedes (third century B.C.E.), expressing in his book *On spirals* (Archimedes, 1952) a "formula" for a sum of squares by

If a series of any number of lines be given, which exceed one another by an equal amount, and the difference be equal to the least, and if other lines be given equal in number to these and in quantity to the greatest, the squares on the lines equal to the greatest, plus the square on the greatest and the rectangle contained by the least and the sum of all those exceeding one another by an equal amount will be the triplicate of all the squares on the lines exceeding one another by an equal amount.

Students can interpret and transform this into the equivalent modern formulation

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Archimedes applied this to deduce the area inside what we call an Archimedean spiral by the classical Greek method of exhaustion:

The area bounded by the first turn of the spiral and the initial line is equal to one-third of the first circle.

The ability to sum yet higher powers was key to finding areas and volumes of other geometric objects, and we find sums of cubes understood in work of Nicomachus of Gerasa (first century B.C.E.), Āryabhaṭa in India (499 C.E.), and al-Karajī in the Arab world (c. 1000) (Boyer, 1943)(Heath, 1963, p. 68f)(Katz, 1998, p. 212f,251ff):

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

The first evidence of a general relationship between various exponents is in the work of Abū 'Alī al-Ḥasan ibn al-Haytham (965–1039), who needed a formula for a sum of fourth powers in order to find the volume of a general paraboloid of revolution (Katz, 1998, p. 255f). Although not stated in full generality, his discovery was essentially the recursive relationship

$$(n+1) \sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i^{k+1} + \sum_{p=1}^{n} \left(\sum_{i=1}^{p} i^k \right).$$

Pierre de Fermat and Blaise Pascal: Figurate numbers, the arithmetical triangle, and sums of powers

In his correspondence of 1636, Pierre de Fermat called the problem of finding formulas for sums of powers "*what is perhaps the most beautiful problem of all arithmetic*", and claimed a recursive solution using figurate numbers, which could then be applied to integrate the "higher parabolas" x^k (Boyer, 1943)(Katz, 1998, p. 481ff). The only details Fermat gave were his claims about "figurate numbers" that

The last number multiplied by the next larger number is double the collateral triangle;
the last number multiplied by the triangle of the next larger is three times the collateral pyramid;
the last number multiplied by the pyramid of the next larger is four times the collateral triangulo-triangle;
and so on indefinitely in this same manner.

Interpreting and understanding what Fermat meant by this is a delightful mystery for students and instructor to unravel, and naturally leads to understanding how figurate numbers, which are formed from arrays of dots, are one and the same as combination numbers and binomial coefficient numbers. In brief, as an example, if $F_{n,2}$ denotes the number of dots evenly spaced in an

equilateral triangle with n dots on a side, and $F_{n,3}$ is the number of dots in an analogous triangular pyramid, and so on to higher dimensions, then Fermat claims that these figurate numbers obey equations such as

$$nF_{n+1,1} = 2F_{n,2} \text{ and } nF_{n+1,2} = 3F_{n,3}.$$

So $F_{n,3} = \dfrac{n}{3}F_{n+1,2} = \dfrac{n}{3}\dfrac{n+1}{2}F_{n+2,1} = \dfrac{n}{3}\dfrac{n+1}{2}\dfrac{n+2}{1}$
and $F_{i,2} = \dfrac{i}{2}F_{i+1,1} = \dfrac{i}{2}\dfrac{i+1}{1}.$

Moreover, since a triangular pyramid of dots consists of a sloping pile of triangles of dots, we have

$$F_{n,3} = \sum_{i=1}^{n} F_{i,2}.$$

Thus $\dfrac{n\,(n+1)\,(n+2)}{3\cdot 2\cdot 1} = \sum_{i=1}^{n} \dfrac{i(i+1)}{2} = \dfrac{1}{2}\sum_{i=1}^{n} i^2 + \dfrac{1}{2}\sum_{i=1}^{n} i,$

from which we easily deduce the formula for a sum of squares from knowing a formula for a sum of first powers.

Next we read the words of Blaise Pascal, who in 1654, also with the aim of applications to integration, wrote an entire treatise on *Sums of Numerical Powers* (Pascal, 1976, v. III, pp. 341–367), beginning with

Given, starting with the unit, some consecutive numbers, for example 1, 2, 3, 4, one knows, by the methods the Ancients made known to us, how to find the sum of their squares, and also the sum of their cubes; but these methods, applicable only to the second and third degrees, do not extend to higher degrees. In this treatise, I will teach how to calculate not only the sum of squares and of cubes, but also the sum of the fourth powers and those of higher powers up to infinity: and that, not only for a sequence of consecutive numbers beginning with the unit, but for ...

Pascal uses generalizable example, binomial expansions, and telescoping sums to obtain a recursive relationship (Boyer, 1943), which in modern formulation says

$$(k+1)\sum_{i=1}^{n} i^k = (n+1)^{k+1} - 1 - \sum_{j=0}^{k-1} \binom{k+1}{j} \sum_{i=1}^{n} i^j.$$

Clearly one can solve here, if tediously and one exponent at a time, for an explicit formula for the sum of k-th powers, by using at each stage the already known formulas for lower exponents. One can also discern some patterns in

the sums of powers formulas for the first few values of k, which students can prove by mathematical induction for general k from Pascal's equation. This leads us to hope there is a pattern to the remaining coefficients.

Jakob Bernoulli: A pattern emerges

In 1713 Jakob Bernoulli's posthumous book on the nascent field of probability appeared, *The Art of Conjecturing*, and in a section on permutations and combinations, we find him first list the formulas for *Sums of Powers* up to exponent ten (using the notation $\int$ for the discrete sum from 1 to n), and then claim a general pattern to the formulas (Bernoulli, 1975, vol. 3, pp. 164–167):

$$\int n = \frac{1}{2}nn + \frac{1}{2}n.$$

$$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n.$$

$$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn.$$

$$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 * -\frac{1}{30}n.$$

$$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 * -\frac{1}{12}nn.$$

$$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 * -\frac{1}{6}n^3 * +\frac{1}{42}n.$$

$$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 * -\frac{7}{24}n^4 * +\frac{1}{12}nn.$$

$$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 * -\frac{7}{15}n^5 * +\frac{2}{9}n^3 * -\frac{1}{30}n.$$

$$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 * -\frac{7}{10}n^6 * +\frac{1}{2}n^4 * -\frac{3}{20}nn.$$

$$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 * -1n^7 * +1n^5 * -\frac{1}{2}n^3 * +\frac{5}{66}n.$$

Indeed, a pattern can be seen in the progressions herein which can be continued by means of this rule: Suppose that c is the value of any power; then the sum of all n^c or

$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c \cdot c - 1 \cdot c - 2}{2 \cdot 3 \cdot 4}Bn^{c-3}$$
$$+ \frac{c \cdot c - 1 \cdot c - 2 \cdot c - 3 \cdot c - 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}Cn^{c-5}$$
$$+ \frac{c \cdot c - 1 \cdot c - 2 \cdot c - 3 \cdot c - 4 \cdot c - 5 \cdot c - 6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}Dn^{c-7} \ldots,$$

where the value of the power n continues to decrease by two until it reaches n or nn. The uppercase letters A, B, C, D, etc., in order,

denote the coefficients of the final term of $\int nn, \int n^4, \int n^6, \int n^8$, *etc.,
namely*

$$A = \frac{1}{6}, \; B = -\frac{1}{30}, C = \frac{1}{42}, \; D = -\frac{1}{30}.$$

These coefficients are such that, when arranged with the other coefficients of the same order, they add up to unity: so, for D, which we said signified $-\frac{1}{30}$, we have

$$\frac{1}{9} + \frac{1}{2} + \frac{2}{3} - \frac{7}{15} + \frac{2}{9}(+D) - \frac{1}{30} = 1.$$

These special numbers in his conjecture are the first occurrence of the "Bernoulli numbers", which today play such an important role in modern mathematics.

We could now conceivably return, in anticipation, to Euler's broader context of $\sum_{i=1}^{n} f(i)$, for which Bernoulli's claim engages the test function $f(x) = x^k$, and venture a rash generalization based on Bernoulli's conjecture:

$$\sum_{i=1}^{n} f(i) \approx C + \int^{n} f(x)dx + \frac{f(n)}{2} + A\frac{f'(n)}{2!} + B\frac{f'''(n)}{4!} + \cdots?$$

Leonhard Euler: Dances between continuous and discrete

Euler calculated without any apparent effort, just as men breathe, as eagles sustain themselves in the air.

Arago (Young, 1992, p. 354)

Around the year 1730, the 23-year old Euler, along with his frequent correspondents Christian Goldbach and Daniel Bernoulli, developed ways to find increasingly accurate fractional or decimal estimates for the sum of the series of reciprocal squares; but these estimates were challenging, since the series converges very slowly. They were likely trying to guess the exact value of the sum, hoping to recognize that their approximations hinted something familiar, perhaps involving π, like Leibniz's series which summed to $\pi/4$. Euler hit gold with the discovery of his summation formula. One of his first major uses of it was in a paper submitted to the St. Petersburg Academy of Sciences on the 13th of October, 1735, in which he approximated the sum of reciprocal squares correct to twenty decimal places! Only seven and a half weeks later Euler astonished his contemporaries when he presented another paper, solving the famous Basel Problem, demonstrating by a completely different method that the precise sum of the series is $\pi^2/6$: "*Now, however, quite unexpectedly, I have found an elegant formula for $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}+$ etc., depending upon the quadrature of the circle [i.e., upon π].*" (Weil, 1983, p. 261) And

Johann Bernoulli reacted with "*And so is satisfied the burning desire of my brother [Jakob] who, realizing that the investigation of the sum was more difficult than anyone would have thought, openly confessed that all his zeal had been mocked. If only my brother were alive now.*" (Young, 1992, p. 345)

In our course and our book, one reads Euler's mature view of the subject, presented in 1755 in part two of his book *Foundations of Differential Calculus*[1] (Euler, 1911–, vol. 10)(Euler, 1981). He devotes chapters 5 and 6 of part two to his summation formula and its applications; we have translated extensive excerpts from these chapters at (Euler, 2000b). Among other things, Euler derives his summation formula, analyzes the generating function for Bernoulli numbers in relation to power series from calculus, derives properties of the Bernoulli numbers, shows that they grow supergeometrically, proves Bernoulli's formulas for sums of powers, and finds the exact sums of all infinite series of reciprocal even powers in terms of Bernoulli numbers.

Our rash guess above was correct for Euler's summation formula, which we can write, using his notation for the Bernoulli numbers, as

$$\sum_{i=1}^{n} f(i) \approx C + \int^{n} f(x)dx + \frac{f(n)}{2} + \mathfrak{A}\frac{f'(n)}{2!} - \mathfrak{B}\frac{f'''(n)}{4!} + \cdots .$$

Let us see how Euler uses it to approximate the sum in the Basel Problem: He writes

After considering the harmonic series we wish to turn to examining the series of reciprocals of the squares, letting

$$s = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{xx}.$$

Since the general term of this series is $z = \frac{1}{xx}$, then $\int z dx = \frac{-1}{x}$, the differentials of z are

$$\frac{dz}{2dx} = -\frac{1}{x^3}, \quad \frac{ddz}{2 \cdot 3 dx^2} = \frac{1}{x^4}, \quad \frac{d^3z}{2 \cdot 3 \cdot 4 dx^3} = -\frac{1}{x^5} \quad \textit{etc.,}$$

and the sum is

$$s = C - \frac{1}{x} + \frac{1}{2xx} - \frac{\mathfrak{A}}{x^3} + \frac{\mathfrak{B}}{x^5} - \frac{\mathfrak{C}}{x^7} + \frac{\mathfrak{D}}{x^9} - \frac{\mathfrak{E}}{x^{11}} + \textit{etc.,}$$

where the added constant C is determined from one case in which the sum is known. We therefore wish to set $x = 1$. Since then $s = 1$, one has

$$C = 1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \textit{etc.,}$$

but this series alone does not give the value of C, since it diverges strongly. Above we demonstrated that the sum of the series to infinity

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[1] Part one has recently appeared in English translation (Euler, 2000a), but not part two.

is $= \frac{\pi\pi}{6}$, and therefore setting $x = \infty$, and $s = \frac{\pi\pi}{6}$, we have $C = \frac{\pi\pi}{6}$, because then all other terms vanish. Thus it follows that

$$1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \textit{etc.} = \frac{\pi\pi}{6}.$$

If the sum of this series were not known, then one would need to determine the value of the constant C from another case, in which the sum were actually found. To this aim we set $x = 10$ and actually add up ten terms, obtaining

	s	$=$	$1,549767731166540690$	.
Further, add	$\frac{1}{x}$	$=$	$0,1$	
subtr.	$\frac{1}{2xx}$	$=$	$0,005$	
			$1,644767731166540690$	
add	$\frac{\mathfrak{A}}{x^3}$	$=$	$0,000166666666666666$	
			$1,644934397833207356$	
subtr.	$\frac{\mathfrak{B}}{x^5}$	$=$	$0,000000333333333333$	
			$1,644934064499874023$	
add	$\frac{\mathfrak{C}}{x^7}$	$=$	$0,000000002380952381$	
			$1,644934066880826404$	
subtr.	$\frac{\mathfrak{D}}{x^9}$	$=$	$0,000000000033333333$	
			$1,644934066847493071$	
add	$\frac{\mathfrak{E}}{x^{11}}$	$=$	$0,000000000000757575$	
			$1,644934066848250646$	
subtr.	$\frac{\mathfrak{F}}{x^{13}}$	$=$	$0,000000000000025311$	
			$1,644934066848225335$	
add	$\frac{\mathfrak{G}}{x^{15}}$	$=$	$0,000000000000001166$	
subtr.	$\frac{\mathfrak{H}}{x^{17}}$	$=$	71	
			$1,644934066848226430$	$= C.$

This number is likewise the value of the expression $\frac{\pi\pi}{6}$, as one can find by calculation from the known value of π. From this it is clear that, although the series A, B, C, etc. diverges, it nevertheless produces a true sum.

This source provides delightful material for stimulating a class of students. On the one hand, the summation formula diverges for every x, and yet it can be used to make spectacular approximations, in fact, arbitrarily close approximations! The resolution of this apparent contradiction belongs to the modern theory of asymptotic series. And students can explore the interplay of calculation versus accuracy achieved by different choices for x. Euler clearly also considers this a way to recheck his knowledge by other means that the sum

is $\pi^2/6$, i.e., to reaffirm and strengthen the mesh of his total knowledge, an essential tool for students to learn.

The role of divergent series, and their acceptability to mathematicians, continued as a subject of controversy for a long time, as illustrated by our final two quotations.

> *The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes. ...*

<div align="right">

Niels Henrik Abel, 1826 (Kline, 1972, p. 973f)

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> *The series is divergent; therefore we may be able to do something with it.*

<div align="right">

Oliver Heaviside (1850–1925) (Kline, 1972, p. 1096)

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