“Random Walks, Trees and Extensions of Riordan Group Techniques”

by
Naiomi T. Cameron
Howard University
Department of Mathematics

Advisor: Dr. Louis Shapiro
Department of Mathematics

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OUTLINE

1. Introduction

   - The Catalan Numbers and Their Interpretations/Functions Related to the Catalan Function

   - The Ternary Numbers and Generalized Dyck Paths

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2. Main Results

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   - The Expected Number of Returns

   - Area Under Ternary Paths

3. Future Work

- Determinant Sequences of the Ternary Numbers

4. Conclusion

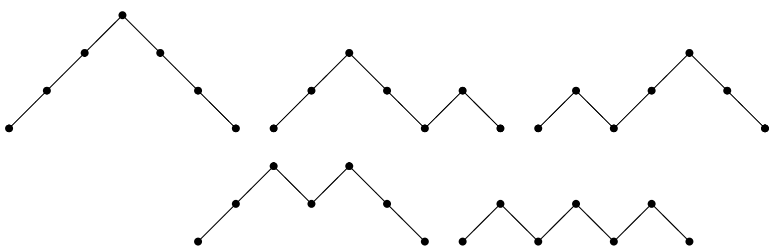
A **path**, P, of length n from (x_0, y_0) to (x, y) with step set S is a sequence of points in the plane,

$$(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) = (x, y)$$

such that all $(x_{i+1} - x_i, y_{i+1} - y_i) \in S$.

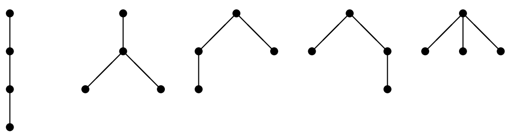
The points are called **vertices**. The **height** of a vertex, v, is the ordinate of that point. A path, P, is **positive** if each of its vertices has nonnegative height.

Dyck paths are positive paths from $(0, 0)$ to $(2n, 0)$ with $S = \{(1, 1), (1, -1)\}$. Below are the five Dyck paths of length 6.



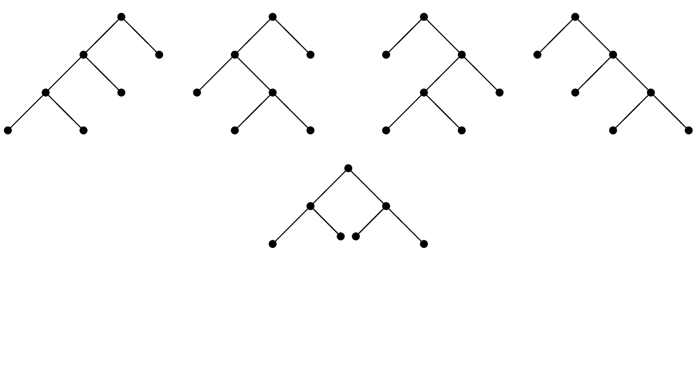
A (rooted) **plane tree** or **tree**, T, is a connected graph with no cycles, where one vertex is designated as the root.

The five plane trees on 4 vertices:



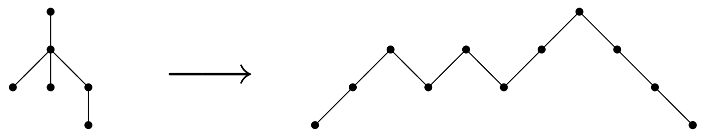
A plane m-ary tree is a plane tree in which every vertex (including the root) has degree 0 or m.

The five binary (or 2-ary) trees with 6 edges:



Let D_n denote the set of all Dyck paths of length $2n$. Let A_n denote the set of all plane trees with n edges.

$$|D_n| = |A_n| = \frac{1}{n+1}\binom{2n}{n} \qquad (1)$$



The nth Catalan number, $\frac{1}{n+1}\binom{2n}{n}$, is denoted c_n, and

$$C(z) = \sum_{n=0}^{\infty} c_n z^n$$

is called the Catalan function. We have that

$$C(z) = 1 + zC^2(z)$$

and

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}$$

The nth Catalan number, $c_n = \frac{1}{n+1}\binom{2n}{n}$, is the number of:

- triangulations of a convex $(n+2)$-gon into n triangles with $n-1$ nonintersecting (except at a vertex) diagonals,

- planted (i.e., degree of the root is 1) plane trees with $2n+2$ vertices where every non-root vertex has degree 0 or 2,

- Dyck paths from $(0,0)$ to $(2n+2,0)$ with no peak at height 2,

- noncrossing partitions of $[n]$.

Definition 1. *The Central Binomial function, $B(z)$, is the generating function for the central binomial coefficients, $\binom{2n}{n}$. That is,*

$$B(z) = \sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}}$$

Definition 2. *The Fine function, $F(z)$, is the generating function for the Fine numbers,*

$$1, 0, 1, 2, 6, 18, 57, 186, ...$$

$$F(z) = \sum_{n=0}^{\infty} f_n z^n = \frac{1-\sqrt{1-4z}}{z(3-\sqrt{1-4z})}$$

Definition 3. *The Motzkin function, $M(z)$, is the generating function for the Motzkin numbers* $1, 1, 2, 4, 9, 21, 51, 127, 323, ...$

$$M(z) = \sum_{n=0}^{\infty} m_n z^n = \frac{1-z-\sqrt{1-2z-3z^2}}{2z}$$

Identity 1. $B(z) = 1 + 2zC(z)B(z)$

Identity 2. $F(z) = \frac{C(z)}{1+zC(z)}$

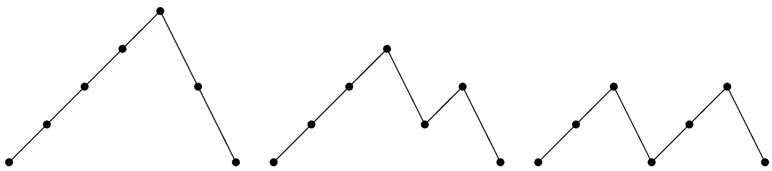
Identity 3. $M(z) = \frac{1}{1-z}C\left(\frac{z^2}{(1-z)^2}\right)$

$B(z)$ counts paths from $(0,0)$ to $(2n,0)$ with $S = \{(1,1),(1,-1)\}$.

$F(z)$ counts Dyck paths with no hills and plane trees with no leaf at height 1.

$M(z)$ counts the number of positive paths from $(0,0)$ to $(n,0)$ using $S = \{(1,1),(1,0),(1,-1)\}$, and the number of plane trees with n edges where every vertex has degree ≤ 2.

Let $\mathcal{T}_n$ denote the set of all positive paths from $(0,0)$ to some $(3n,0)$ with $S=\{(1,1),(1,-2)\}$. We call these paths **ternary paths**.



Let $t_n = |\mathcal{T}_n|$.

Theorem 1. *Let* $T(z)=\sum_{n=0}^{\infty} t_n z^n$. *Then:*

(i.) $T(z) = 1 + zT^3(z)$

(ii.) $t_n = \frac{1}{2n+1}\binom{3n}{n}$

A **generalized t-Dyck path** is a positive path from $(0,0)$ to $((t+1)n,0)$ with $S=\{(1,1),(1,-t)\}$.

The ternary numbers, $\frac{1}{2n+1}\binom{3n}{n}$, also count:

- positive paths starting at $(0,0)$ and ending at $(2n,0)$ using steps in

  $\{(1,1),(1,-1),(1,-3),(1,-5),(1,-7),\ldots,\}$;

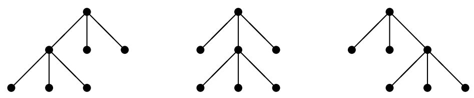
- rooted plane trees with $2n$ edges where every vertex (including the root) has even degree (referred to as "even trees");

- dissections of a convex $2(n+1)$-gon into n quadrangles by drawing $n-1$ diagonals;

- noncrossing trees on $n+1$ points;

- rooted plane trees with $3n$ edges where every node (including the root) has degree zero or three (called **ternary trees**)

  The three ternary trees with 6 edges:



Definition 4. *An infinite lower triangular matrix,* $L = \left(l_{n,k}\right)_{n,k\geq 0}$ *is a* **Riordan matrix** *if there exist generating functions* $g(z) = \sum g_n z^n$, $f(z) = \sum f_n z^n$, $f_0 = 0$, $f_1 \neq 0$ *such that* $l_{n,0} = g_n$ *and* $\sum_{n\geq k} l_{n,k} z^n = g(z)\left(f(z)\right)^k$.

L is called **Riordan** and we write $L = (g(z), f(z))$, or simply $L = (g, f)$.

Example 1.

$$(C(z), zC(z)) = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 2 & 2 & 1 & & & & \\ 5 & 5 & 3 & 1 & & & \\ 14 & 14 & 9 & 4 & 1 & & \\ 42 & 42 & 28 & 14 & 5 & 1 & \\ 132 & 132 & 90 & 48 & 20 & 6 & 1 \\ & & & \cdots & & & \end{bmatrix}$$

Theorem 2. *(Chung, Feller) Let* $c_{n,k}$ *denote the number of paths from* $(0,0)$ *to* $(2n,0)$ *with steps in* $\{(1,1),(1,-1)\}$ *such that* k *up steps lie above the* x*-axis,* $k=0,1,\ldots,n$*. Then* $c_{n,k} = c_n = \frac{1}{n+1}\binom{2n}{n}$*, independently of* k*.*

Theorem 3. *Let* $u_{n,k}$ *be the number of paths from* $(0,0)$ *to* $(3n,0)$ *with steps in* $\{(1,1),(1,-2)\}$ *such that* k *up steps lie above the* x*-axis,* $k=0,1,\ldots,2n$*. Then* $u_{n,k} = t_n = \frac{1}{2n+1}\binom{3n}{n}$*.*

Proof:

$z \leftrightarrow$ an up step

$y \leftrightarrow$ an up step above the x-axis

$$U(y,z) = \sum_{0\le k\le n} u_{n,k} y^k z^n$$

$$U(y,z) = \frac{T(z^2)}{1-\Psi(y,z)}$$

where

$$\Psi(y,z) = y^2z^2T^2(y^2z^2)T(z^2) + yz^2T(y^2z^2)T^2(z^2)$$

Theorem 4. *The function* $N(z) = \sum \binom{3n}{n} z^n$ *counts*

(i.) paths in $Z \times Z$ *starting at* $(0,0)$ *and ending at* $(3n, 0)$ *using steps in* $\{(1,1),(1,-2)\}$*, and*

(ii.) even trees with $2n$ *edges and one distinguished vertex.*

Theorem 5.

(i.) $N(z) = 1 + 3zT^2(z)N(z)$

(ii.) $N(z)T^s(z) = \sum_{n=0}^{\infty} \binom{3n+s}{n} z^n$

Proof: (i) Note $\frac{d}{dz}\left(zT(z^2)\right) = N(z^2)$

Theorem 6. *(i.) The "Motzkin analogue", $M_T(z)$, satisfies*

$$M_T(z) = 1+z+2z^2+5z^3+13z^4+36z^5+104z^6+...$$

$$= 1 + zM_T(z) + z^2M_T^2(z) + z^3M_T^3(z)$$

(ii.) $M_T(z)$ counts positive paths from $(0,0)$ to $(n,0)$ using $\{(1,1),(1,-1),(1,-2),(1,0)\}$.

Theorem 7. *(i.) The "Fine Analogue," $F_T(z)$ satisfies*

$$F_T(z) = \frac{1}{1+z-zT^2(z)}$$

$$T(z) = \frac{F_T(z)}{1-zF_T(z)}$$

(ii.) $F_T(z)$ counts ternary paths with no bumps.

Theorem 8. *The expected number of returns for generalized Dyck paths from* $(0, 0)$ *to* $((t + 1)n, 0)$ *is*

$$\frac{(t + 2)n}{tn + 2}$$

with variance

$$\frac{2tn(n - 1)\,((t + 1)n + 1)}{(tn + 2)^2(tn + 3)}$$

.

Proof: Let $Y(n)$ denote the number of returns. Let $p_m(n)$ denote the probability that a randomly chosen path of length $(t+1)n$ has m returns. Then

$$P_{Y(n)}(z) := \sum_{m=0}^{\infty} p_m(n) z^n$$

$$= \frac{t(tn + 1)}{(t + 1)((t + 1)n - 1)} zF(2, 1-n; 2-(t+1)n; z)$$

The limiting distribution of $Y(n)$ is $negbin(2, \frac{t}{t+1})$.

Theorem 9. *(Woan, Rogers, Shapiro) The sum of the areas of all strict Dyck paths of length* $2n$ *is* 4^{n-1}*.*

Theorem 10. *(Kreweras/Merlini et al./Chapman) The sum of the areas of all Dyck paths of length* $2n$ *is* $4^n - \frac{1}{2}\binom{2n+2}{n+1}$*.*

Let $\mathcal{D}_n$ denote the set of Dyck paths of length $2n$. Define the **area** of $D \in \mathcal{D}_n$, $a(D)$, to be the area of the region bounded by D and the x-axis.

Let $\mathcal{B}_n$ denote the set of all plane binary trees with $2n$ edges. Let $B \in \mathcal{B}_n$. Define the **total weight** of B, $\omega(B)$ by

$$\omega(B) := \sum_{v \in B} hgt(v)$$

Theorem 11. *For all* $n \in \mathbb{N}$*,*

$$\sum_{B \in \mathcal{B}_n} \omega(B) = 2 \sum_{D \in \mathcal{D}_n} a(D)$$

Let $\mathcal{S}_n$ denote the set of all ternary paths in $\mathcal{T}_n$ with exactly 1 return (**strict** ternary paths).

Let a_n^s (resp. a_n^t) denote the total area of the region bounded by the paths of $\mathcal{S}_n$ (resp. $\mathcal{T}_n$) and the x-axis.

Let $h_{n,k}^s$ (resp. $h_{n,k}^t$) denote the number of points in $\mathcal{S}_n \cap \mathbb{Z} \times \mathbb{Z}$ (resp. $\mathcal{T}_n \cap \mathbb{Z} \times \mathbb{Z}$) which have height k.

Theorem 12. *Let* $H_k^s(z) = \sum_{n=0}^{\infty} h_{n,k}^s z^n$ *and* $H_k^t(z) = \sum_{n=0}^{\infty} h_{n,k}^t z^n$.

(i.)

$$H_k^s(z) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} (zT^3(z))^{k-j}$$

$$= (zT^3(z))^k \frac{\frac{z^2}{1-z} T^3(\frac{z^2}{1-z})}{z^2 T^3(\frac{z^2}{1-z}) - 1}$$

(ii.)

$$H_k^t(z) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} (zT^3(z))^{k-j} T^2(z)$$

$$= (zT^3(z))^{k+2} \frac{\frac{z^2}{1-z} T^3(\frac{z^2}{1-z})}{z^2 T^3(\frac{z^2}{1-z}) - 1}$$

$$
\begin{bmatrix}
1 & & & & & & \\
0 & 1 & 1 & & & & \\
0 & 3 & 4 & 2 & 1 & & \\
0 & 12 & 18 & 13 & 9 & 3 & 1 \\
 & & & \cdots & & &
\end{bmatrix}
\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \cdots \end{bmatrix}
=
\begin{bmatrix} 0 \\ 3 \\ 21 \\ 144 \\ 981 \\ 6663 \\ \vdots \\ a_n^s \\ \vdots \end{bmatrix}
$$

$$
\begin{bmatrix}
1 & & & & & & \\
2 & 1 & 1 & & & & \\
7 & 5 & 6 & 2 & 1 & & \\
30 & 25 & 33 & 17 & 11 & 3 & 1 \\
 & & & \cdots & & &
\end{bmatrix}
\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \cdots \end{bmatrix}
=
\begin{bmatrix} 0 \\ 3 \\ 27 \\ 207 \\ 1506 \\ 10692 \\ \vdots \\ a_n^t \\ \vdots \end{bmatrix}
$$

Let $\mathcal{W}_n$ denote the set of all ternary trees with $3n$ edges.

Conjecture 1. *For all $n \in \mathbb{N}$,*

$$\sum_{W \in \mathcal{W}_n} \omega(W) = \sum_{T \in \mathcal{T}_n} a(T)$$

Let $t(k,n)$ denote the total number of vertices at height k in $\mathcal{W}_n$ and $\tau(y,z) = \sum t(k,n) y^k z^n$. Then

$$\frac{d}{dy}(\tau(y,z))|_{y=1} = \frac{3zT^3(z)}{(1-3zT^2(z))^2}$$

$$= 3z + 27z^2 + 207z^3 + 1506z^4 + 10692z^5 + \ldots$$

Given a sequence $\{a_n\}_{n=0}^{\infty}$, define

$$A_n^k := \begin{vmatrix} a_k & a_{k+1} & a_{k+2} & \cdots & a_{n+k-1} \\ a_{k+1} & a_{k+2} & a_{k+3} & \cdots & a_{n+k} \\ a_{k+2} & a_{k+3} & a_{k+4} & \cdots & a_{n+k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n+k-1} & a_{n+k} & a_{n+k+1} & \cdots & a_{2n+k-2} \end{vmatrix}$$

Theorem 13. *(Gronau et al.) Let p_{ij} be the number of paths leading from a_i to b_j in G, let p^+ be the number of disjoint path systems W in (G, A, B) for which $\sigma(W)$ is an even permutation, and let p^- be the number of such systems with for which $\sigma(W)$ is odd. Then $det(p_{ij}) = p^+ - p^-$.*

Let Q_n^k be the family of all sets of pairwise vertex disjoint paths in $\mathcal{G}$, ξ_0, ξ_1, ..., ξ_{n-1}, such that ξ_i joins $(-3i, 0)$ with $(3(i+k), 0)$, $i = 0, 1, \ldots, n-1$. The theorem implies that

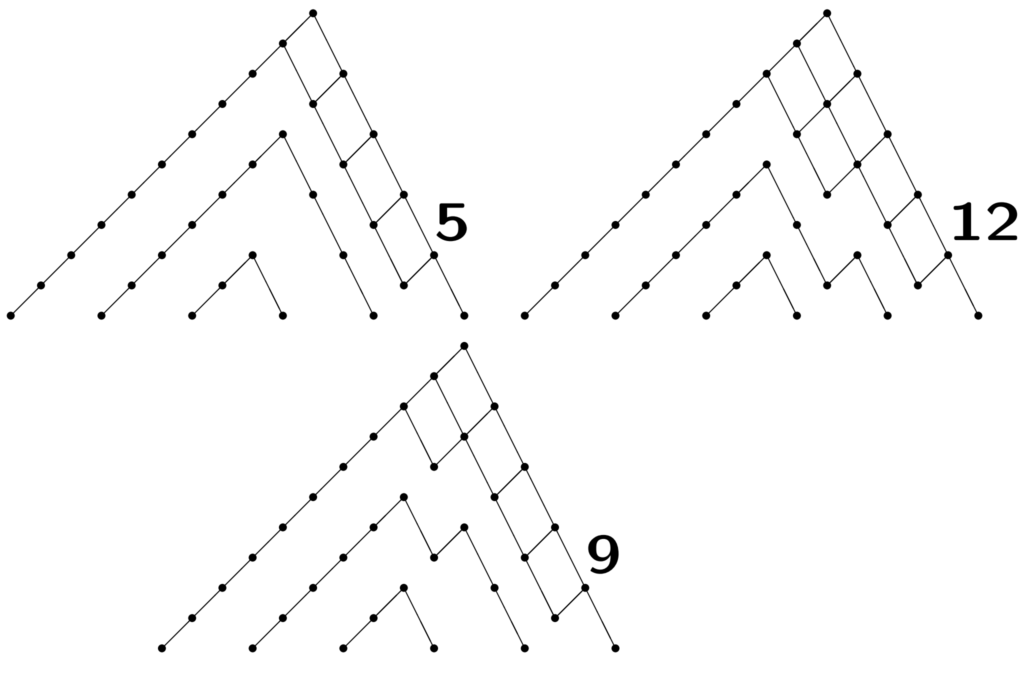
$$A_n^k = \left|Q_n^k\right|$$

When $k = 1$, $n = 3$, we have

$$\left|Q_3^1\right| = A_3^1 = \begin{vmatrix} 1 & 3 & 12 \\ 3 & 12 & 55 \\ 12 & 55 & 273 \end{vmatrix} = 26$$

The corresponding vertex disjoint path systems

are



n	A_n^0	A_n^1
0	1	1
1	2	3
2	11	26
3	170	646
4	7429	45,885
5	920,460	9,304,650
6	323,801,820	5,382,618,658

Theorem 14. *(U. Tamm)*

$$A_n^0 = \prod_{j=0}^{n-1} \frac{(3j+1)(6j)!(2j)!}{(4j+1)!(4j)!}$$

$$A_n^1 = \prod_{j=1}^{n} \frac{\binom{6j-2}{2j}}{2\binom{4j-1}{2j}}$$

Question 1. *What about A_n^k for $k \geq 2$?*

Definition 5. *An* **alternating sign matrix** *is a square matrix of 0s, 1s, and -1s for which*

- *the entries of each row and each column sum to 1,*

- *the non-zero entries of each row and each column alternate in sign.*

Example 2. $$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Conjecture 2. *(Stanley/Mills, Robbins, Rumsey)*

Let $F(2n+1)$ *denote the number of* $(2n+1)\times(2n+1)$ *alternating sign matrices which are invariant under a reflection about the vertical axis.*

$$F(2n+1) = \prod_{j=1}^{n} \frac{\binom{6j-2}{2j}}{2\binom{4j-1}{2j}}$$

Question 2. *Is there a bijection between vertex disjoint ternary path systems and alternating sign matrices invariant under vertical reflection?*