CATALAN ADDENDUM

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version of 27 May 2005

The problems below are a continuation of those appearing in Chapter 6 of *Enumerative Combinatorics*, volume 2. Combinatorial interpretations of Catalan numbers are numbered as a continuation of Exercise 6.19, while algebraic interpretations are numbered as a continuation of Exercise 6.25. Combinatorial interpretations of Motzkin numbers are numbered as a continuation of Exercise 6.38. The remaining problems are numbered 6.C1, 6.C2, etc. I am grateful to Emeric Deutsch for providing parts (ooo), (rrr), (ttt), (a^4), (e^4), (k^4), (p^4), (j^5), (k^5), (m^5) and (n^5) of Exercise 6.19, and to Roland Bacher for providing (r^4).

Note. In citing results from this Addendum it would be best not to use the problem numbers (or at the least give the version date), since I plan to insert new problems in logical rather than numerical order.

Note. At the end of this addendum is a list of all problems added on or after December 17, 2001, together with the date the problem was added. The problem numbers always refer to the version of the addendum in which the list appears.

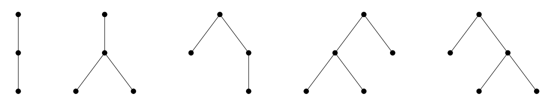
Note. Throughout this addendum we let C_n denote the Catalan number $\frac{1}{n+1}\binom{2n}{n}$ and $C(x)$ the generating function

$$C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

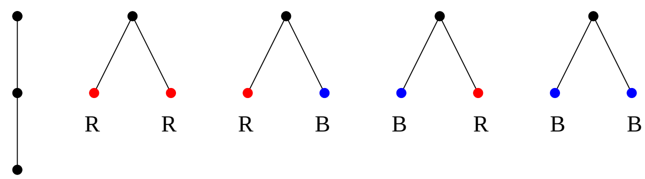
Moreover, we let

$$\begin{aligned} E(x) &= \frac{C(x)+C(-x)}{2} &= \sum_{n\geq 0} C_{2n}x^{2n} \\ O(x) &= \frac{C(x)-C(-x)}{2} &= \sum_{n\geq 0} C_{2n+1}x^{2n+1}. \end{aligned}$$

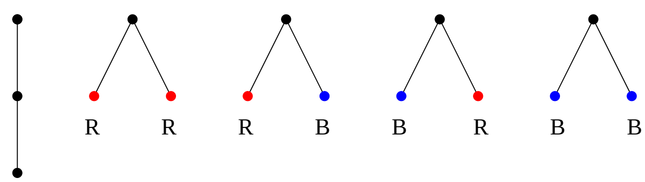
6.19(ooo) Plane trees with $n-1$ internal nodes, each having degree 1 or 2, such that nodes of degree 1 occur only on the rightmost path



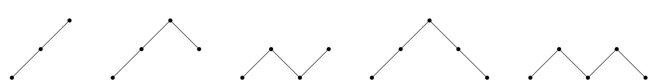
(ppp) Plane trees with n vertices, such that going from left to right all subtrees of the root first have an even number of vertices and then an odd number of vertices, with those subtrees with an odd number of vertices colored either red or blue



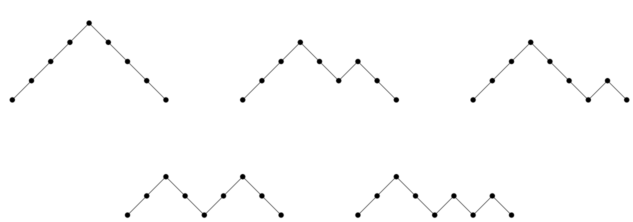
(qqq) Plane trees with n vertices whose leaves at height one are colored red or blue



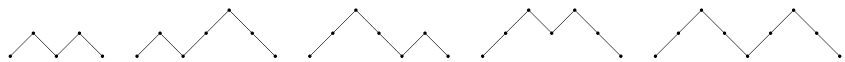
(rrr) Left factors L of Dyck paths such that L has $n-1$ up steps



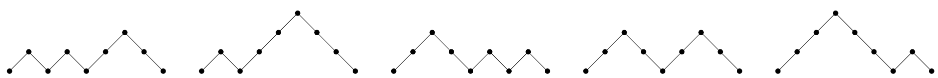
(sss) Dyck paths of length $2n+2$ whose first downstep is followed by another downstep



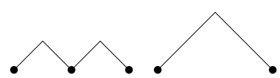
(ttt) Dyck paths with $n-1$ peaks and without three consecutive up steps or three consecutive down steps



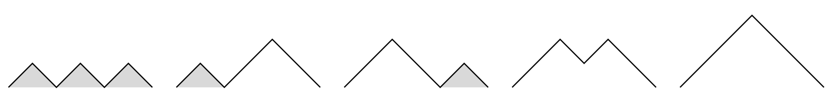
(uuu) Dyck paths D from $(0,0)$ to $(2n+2,0)$ such that there is no horizontal line segment L with endpoints (i,j) and $(2n+2-i,j)$, with $i>0$, such that the endpoints lie on P and no point of L lies above D



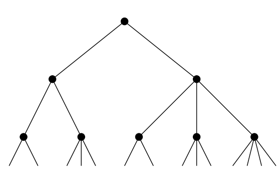
(vvv) Points of the form $(m,0)$ on all Dyck paths from $(0,0)$ to $(2n-2,0)$



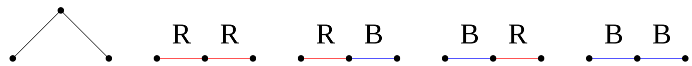
(www) Peaks of height one in all Dyck paths from $(0,0)$ to $(2n,0)$



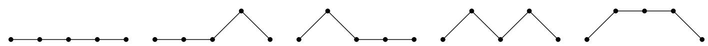
(xxx) Vertices of height $n-1$ of the tree T defined by the property that the root has degree 2, and if the vertex x has degree k, then the children of x have degrees $2,3,\ldots,k+1$



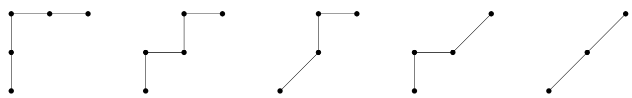
(yyy) Motzkin paths (as defined in Exercise 6.38(d), though with the typographical error (n,n) instead of $(n,0)$) from $(0,0)$ to $(n-1,0)$, with the steps $(1,0)$ colored either red or blue



(zzz) Motzkin paths $a_1, \ldots, a_{2n-2}$ from $(0,0)$ to $(2n-2,0)$ such that each odd step a_{2i+1} is either $(1,0)$ (straight) or $(1,1)$ (up), and each even step a_{2i} is either $(1,0)$ (straight) or $(1,-1)$ (down)



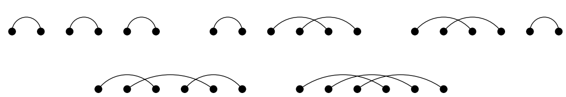
(a^4) Lattice paths from $(0,0)$ to $(n-1,n-1)$ with steps $(0,1)$, $(1,0)$, and $(1,1)$, never going below the line $y=x$, such that the steps $(1,1)$ only appear on the line $y=x$



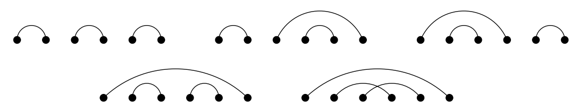
(b^4) Lattice paths of length $n-1$ from $(0,0)$ to the x-axis with steps $(\pm 1, 0)$ and $(0, \pm 1)$, never going below the x-axis

$$(-1,0)+(-1,0) \qquad (-1,0)+(1,0) \qquad (0,1)+(0,-1)$$
$$(1,0)+(-1,0) \qquad (1,0)+(1,0)$$

(c^4) *Nonnesting matchings* on $[2n]$, i.e., ways of connecting $2n$ points in the plane lying on a horizontal line by n arcs, each arc connecting two of the points and lying above the points, such that no arc is contained entirely below another



(d^4) Ways of connecting $2n$ points in the plane lying on a horizontal line by n arcs, each arc connecting two of the points and lying above the points, such that the following condition holds: for every edge e let $n(e)$ be the number of edges e' that nest e (i.e., e lies below e'), and let $c(e)$ be the number of edges e' that begin to the left of e and that cross e. Then $n(e)-c(e)=0$ or 1.



(e[4]) Ways of connecting any number of points in the plane lying on a horizontal line by nonintersecting arcs lying above the points, such that the total number of arcs and isolated points is $n-1$ and no isolated point lies below an arc

(f[4]) Ways of connecting n points in the plane lying on a horizontal line by noncrossing arcs above the line such that if two arcs share an endpoint p, then p is a left endpoint of both the arcs

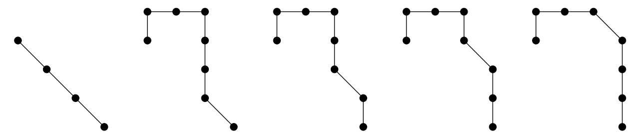
(g[4]) Ways of connecting $n+1$ points in the plane lying on a horizontal line by noncrossing arcs above the line such that no arc connects adjacent points and the right endpoints of the arcs are all distinct

(h[4]) Lattice paths in the first quadrant with n steps from $(0,0)$ to $(0,0)$, where each step is of the form $(\pm 1, \pm 1)$, or goes from $(2k,0)$ to $(2k,0)$ or $(2(k+1),0)$, or goes from $(0,2k)$ to $(0,2k)$ or $(0,2(k+1))$

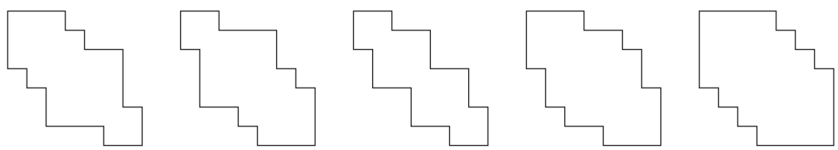
$$(0,0) \to (0,0) \to (0,0) \to (0,0)$$
$$(0,0) \to (0,0) \to (1,1) \to (0,0)$$
$$(0,0) \to (1,1) \to (0,0) \to (0,0)$$
$$(0,0) \to (2,0) \to (1,1) \to (0,0)$$
$$(0,0) \to (0,2) \to (1,1) \to (0,0)$$

(i[4]) Lattice paths from $(0,0)$ to $(n,-n)$ such that (α) from a point (x,y) with $x < 2y$ the allowed steps are $(1,0)$ and $(0,1)$, (β) from a point (x,y) with $x > 2y$ the allowed steps are $(0,-1)$ and $(1,-1)$,

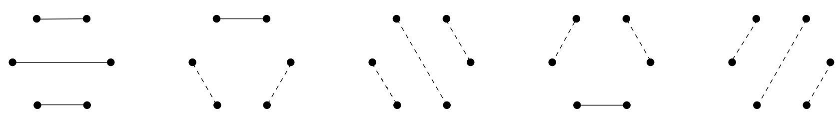
(γ) from a point $(2y, y)$ the allowed steps are $(0,1)$, $(0,-1)$, and $(1,-1)$, and (δ) it is forbidden to enter a point $(2y+1, y)$



(j^4) Symmetric parallelogram polyominos (as defined in the solution to Exercise 6.19(l)) of perimeter $4(2n+1)$ such that the horizontal (equivalently, vertical) boundary steps on each level form an unbroken line



(k^4) All horizontal chords in the nonintersecting chord diagrams of (n) (with the vertices drawn so that one of the diagrams has n horizontal chords)



(l^4) *Kepler towers with n bricks*, i.e., sets of concentric circles, with "bricks" (arcs) placed on each circle, as follows: the circles come in sets called *walls* from the center outwards. The circles (or *rings*) of the ith wall are divided into 2^i equal arcs, numbered $1, 2, \dots, 2^i$ clockwise from due north. Each brick covers an arc and extends slightly beyond the endpoints of the arc. No two consecutive arcs can be covered by bricks. The first (innermost) arc within each wall has bricks at positions $1, 3, 5, \dots, 2^i - 1$. Within each wall, each brick B not on the innermost ring must be supported by another brick B' on the next ring toward the center, i.e., some ray from the center must intersect both B and B'. Finally, if $i > 1$ and the ith wall is nonempty, then wall $i-1$ must also by nonempty.

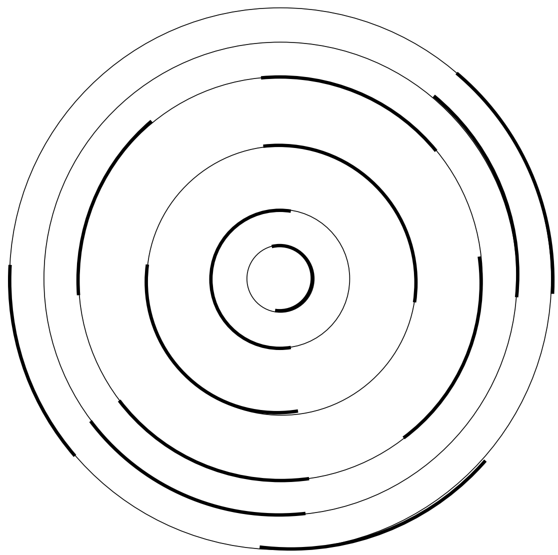
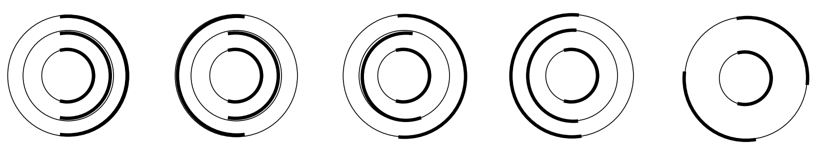


Figure 1: A Kepler tower with 3 walls, 6 rings, and 13 bricks

Figure 1 shows a Kepler tower with three walls, six rings, and 13 bricks.



(m[4]) Compositions of n whose parts equal to k are colored with one of C_{k-1} colors (colors are indicated by subscripts below)

$$1_a + 1_a + 1_a \quad 1_a + 2_a \quad 2_a + 1_a \quad 3_a \quad 3_b$$

(n[4]) Sequences $(a_1, \ldots, a_n)$ of nonnegative integers satisfying $a_1 + \cdots + a_i \geq i$ and $\sum a_j = n$

111 120 210 201 300

(o[4]) Sequences $a_1, \ldots, a_{2n}$ of nonnegative integers with $a_1 = 1$, $a_{2n} = 0$ and $a_i - a_{i-1} = \pm 1$:

123210 121210 121010 101210 101010

(p^4) Sequences of $n-1$ 1's and any number of -1's such that every partial sum is nonnegative

$$1,1 \qquad 1,1,-1 \qquad 1,-1,1 \qquad 1,1,-1,-1 \qquad 1,-1,1,-1$$

(q^4) Sequences $a_1a_2\cdots a_{2n-2}$ of $n-1$ 1's and $n-1$ -1's such that if $a_i=-1$ then either $a_{i+1}=a_{i+2}=\cdots=a_{2n-2}=-1$ or $a_{i+1}+a_{i+2}+\cdots+a_{i+j}>0$ for some $j\geq 1$

$$1,1,-1,-1 \quad 1,-1,1,-1 \quad -1,1,1,-1 \quad -1,1,-1,1 \quad -1,-1,1,1$$

(r^4) Sequences $a_1a_2\cdots a_n$ of integers such that $a_1=1$, $a_n=\pm 1$, $a_i\neq 0$, and $a_{i+1}\in\{a_i,a_i+1,a_i-1,-a_i\}$ for $2\leq i\leq n$

$$1,1,1 \qquad 1,1,-1 \qquad 1,-1,1 \qquad 1,-1,-1 \qquad 1,2,1$$

(s^4) Sequences $a_1a_2\cdots a_n$ of nonnegative integers such that $a_j=\#\{i: i<j, a_i<a_j\}$ for $1\leq j\leq n$

$$000 \qquad 002 \qquad 010 \qquad 011 \qquad 012$$

(t^4) Sequences $a_1a_2\cdots a_{n-1}$ of nonnegative integers such that each nonzero term initiates a factor (subsequence of consecutive elements) whose length is equal to its sum

$$00 \quad 01 \quad 10 \quad 11 \quad 20$$

(u^4) Sequences $a_1a_2\cdots a_{2n+1}$ of positive integers such that $a_{2n+1}=1$, some $a_i=n+1$, the first appearance of $i+1$ follows the first appearance of i, no two consecutive terms are equal, no pair ij of integers occur more than once as a factor (i.e., as two consecutive terms), and if ij is a factor then so is ji

$$1213141 \quad 1213431 \quad 1232141 \quad 1232421 \quad 1234321$$

(v^4) Sequences $(a_1,\ldots,a_n)\in\mathbb{N}^n$ for which there exists a distributive lattice of rank n with a_i join-irreducibles of rank i, $1\leq i\leq n$

$$300 \quad 210 \quad 120 \quad 201 \quad 111$$

(w[4]) Pairs of sequences $1 \leq i_1 < i_2 < \cdots < i_k \leq n-1$ and $2 \leq j_1 < \cdots < j_k \leq n$ such that $i_r < j_r$ for all r.

$$(\emptyset, \emptyset) \quad (1,2) \quad (1,3) \quad (2,3) \quad (12,23)$$

(x[4]) Ways two persons can each start with 0 and alternating add positive integers to their numbers so that they first have equal numbers when that number is n (notation such as $1,2;4,3;5,5$ means that the first person adds 1 to 0 to obtain 1, then the second person adds 2 to 0 to obtain 2, then the first person adds 3 to 1 to obtain 4, etc.)

$$3,3 \quad 2,3;3 \quad 2,1;3,3 \quad 1,2;3,3 \quad 1,3;3$$

(y[4]) Cyclic equivalence classes (or *necklaces*) of sequences of $n+1$ 1's and n 0's (one sequence from each class is shown below)

$$1111000 \quad 1110100 \quad 1110010 \quad 1101100 \quad 1101010$$

(z[4]) Partitions of an integer which are both n-cores and $(n+1)$-cores, in the terminology of Exercise 7.59(d).

$$\emptyset \quad 1 \quad 2 \quad 11 \quad 311$$

(a[5]) Equivalence classes of the equivalence relation on the set $S_n = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : \sum a_i = n\}$ generated by $(\alpha, 0, \beta) \sim (\beta, 0, \alpha)$ if β (which may be empty) contains no 0's. For instance, when $n = 7$ one equivalence class is given by $\{3010120, 0301012, 1200301, 1012003\}$.

$$\{300, 030, 003\} \quad \{210, 021\} \quad \{120, 012\} \quad \{201, 102\} \quad \{111\}$$

(b[5]) Pairs (α, β) of compositions of n with the same number of parts, such that $\alpha \geq \beta$ (dominance order, i.e., $\alpha_1 + \cdots + \alpha_i \geq \beta_1 + \cdots + \beta_i$ for all i)

$$(111,111) \quad (12,12) \quad (21,21) \quad (21,12) \quad (3,3)$$

(c[5]) weak ordered partitions (P, V, A, D) of $[n]$ into four blocks such that there exists a permutation $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ (with $a_0 =$

$a_{n+1} = 0$) satisfying

$$\begin{aligned} P &= \{i \in [n] : a_{i-1} < a_i > a_{i+1}\} \\ V &= \{i \in [n] : a_{i-1} > a_i < a_{i+1}\} \\ A &= \{i \in [n] : a_{i-1} < a_i < a_{i+1}\} \\ D &= \{i \in [n] : a_{i-1} > a_i > a_{i+1}\}. \end{aligned}$$

$(3, \emptyset, 12, \emptyset)$ $(3, \emptyset, 1, 2)$ $(23, 1, \emptyset, \emptyset)$ $(3, \emptyset, 2, 1)$ $(3, \emptyset, \emptyset, 12)$

(d^5) Permutations $w \in \mathfrak{S}_n$ satisfying the following condition: let $w = R_{s+1}R_s \cdots R_1$ be the factorization of w into maximal ascending runs (so $s = \mathrm{des}(w)$, the number of descents of w). Let m_k and M_k be the smallest and largest elements in the run R_k. Let n_k be the number of symbols in R_k for $1 \leq k \leq s+1$; otherwise set $n_k = 0$. Define $N_k = \sum_{i \leq k} n_i$ for all $k \in \mathbb{Z}$. Then $m_{s+1} > m_s > \cdots > m_1$ and $M_i \leq N_{i+1}$ for $1 \leq i \leq s+1$.

123 213 231 312 321

(e^5) Permutations $w \in \mathfrak{S}_n$ satisfying, in the notation of (d^5) above, $m_{s+1} > m_s > \cdots > m_1$ and $m_{i+1} > N_{i-1} + 1$ for $1 \leq i \leq s$

123 213 231 312 321

(f^5) 321-avoiding permutations $w \in \mathfrak{S}_{2n+1}$ such that i is an excedance of w (i.e., $w(i) > i$) if and only if $i \neq 2n+1$ and $w(i) - 1$ is not an excedance of w (so that w has exactly n excedances)

4512736 3167245 3152746 4617235 5671234

(g^5) 321-avoiding alternating permutations in $\mathfrak{S}_{2n}$

214365 215364 314265 315264 415263

(h^5) 321-avoiding fixed-point-free involutions of $[2n]$

214365 215634 341265 351624 456123

(i^5) 321-avoiding involutions of $[2n-1]$ with one fixed point

13254 14523 21354 21435 34125

(j^5) 213-avoiding fixed-point-free involutions of $[2n]$

$$456123 \quad 465132 \quad 564312 \quad 645231 \quad 654321$$

(k^5) 213-avoiding involutions of $[2n-1]$ with one fixed point

$$14523 \quad 15432 \quad 45312 \quad 52431 \quad 54321$$

(l^5) 3412-avoiding (or noncrossing) involutions of a subset of $[n-1]$

$$\emptyset \quad 1 \quad 2 \quad 12 \quad 21$$

(m^5) Standard Young tableaux with at most two rows and with first row of length $n-1$

$$\begin{array}{l}12\\ \end{array} \qquad \begin{array}{l}12\\3\end{array} \qquad \begin{array}{l}13\\2\end{array} \qquad \begin{array}{l}12\\34\end{array} \qquad \begin{array}{l}13\\24\end{array}$$

(n^5) Standard Young tableaux with at most two rows and with first row of length n, such that for all i the ith entry of row 2 is not $2i$

$$\begin{array}{l}123\\ \end{array} \qquad \begin{array}{l}123\\4\end{array} \qquad \begin{array}{l}124\\3\end{array} \qquad \begin{array}{l}123\\45\end{array} \qquad \begin{array}{l}124\\35\end{array}$$

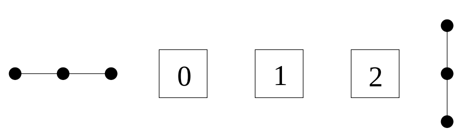
(o^5) Standard Young tableaux of shape $(2n+1, 2n+1)$ such that adjacent entries have opposite parity

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 8 & 9 \\ 6 & 7 & 10 & 11 & 12 & 13 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 10 & 11 \\ 6 & 7 & 8 & 9 & 12 & 13 & 14 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 6 & 7 & 8 & 9 \\ 4 & 5 & 10 & 11 & 12 & 13 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 6 & 7 & 10 & 11 \\ 4 & 5 & 8 & 9 & 12 & 13 & 14 \end{bmatrix}$$

(p^5) Plane partitions with largest part at most two and contained in a rectangle of perimeter $2(n-1)$ (including degenerate $0 \times (n-1)$ and $(n-1) \times 0$ rectangles)



(q^5) Triples (A, B, C) of pairwise disjoint subsets of $[n-1]$ such that $\#A = \#B$ and every element of A is less than every element of B

$$(\emptyset, \emptyset, \emptyset) \qquad (\emptyset, \emptyset, 1) \qquad (\emptyset, \emptyset, 2) \qquad (\emptyset, \emptyset, 12) \qquad (1, 2, \emptyset)$$

(r^5) Subsets S of $\mathbb{N}$ such that $0 \in S$ and such that if $i \in S$ then $i+n, i+n+1 \in S$

$$\mathbb{N}, \quad \mathbb{N} - \{1\}, \quad \mathbb{N} - \{2\}, \quad \mathbb{N} - \{1,2\}, \quad \mathbb{N} - \{1,2,5\}$$

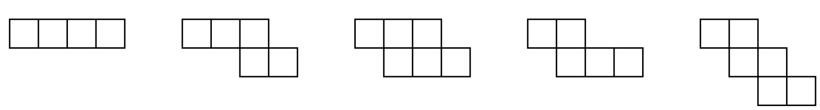
(s^5) $(n+1)$-element multisets on $\mathbb{Z}/n\mathbb{Z}$ whose elements sum to 0

$$0000 \qquad 0012 \qquad 0111 \qquad 0222 \qquad 1122$$

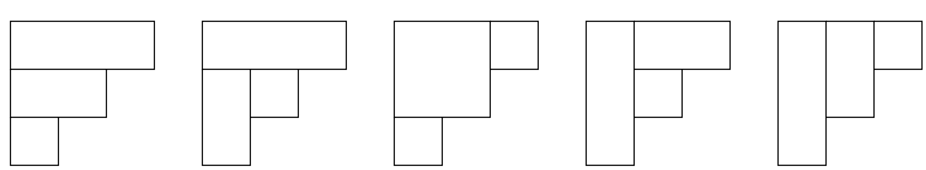
(t^5) Ways to write $(1, 1, \ldots, 1, -n) \in \mathbb{Z}^{n+1}$ as a sum of vectors $e_i - e_{i+1}$ and $e_j - e_{n+1}$, without regard to order, where e_k is the kth unit coordinate vector in $\mathbb{Z}^{n+1}$:

$$(1,-1,0,0) + 2(0,1,-1,0) + 3(0,0,1,-1)$$
$$(1,0,0,-1) + (0,1,-1,0) + 2(0,0,1,-1)$$
$$(1,-1,0,0) + (0,1,-1,0) + (0,1,0,-1) + 2(0,0,1,-1)$$
$$(1,-1,0,0) + 2(0,1,0,-1) + (0,0,1,-1)$$
$$(1,0,0,-1) + (0,1,0,-1) + (0,0,1,-1)$$

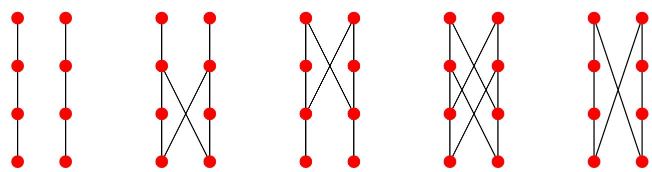
(u^5) Horizontally convex polyominoes (as defined in Example 4.7.18) of width $n+1$ such that each row begins strictly to the right of the beginning of the previous row and ends strictly to the right of the end of the previous row



(v^5) tilings of the staircase shape $(n, n-1, \ldots, 1)$ with n rectangles



(w^5) Nonisomorphic $2(n+1)$-element posets that are a union of two chains, that are not a (nontrivial) ordinal sum, and that have a nontrivial automorphism (compare (eee))



(x^5) $n \times n$ $\mathbb{N}$-matrices $M = (m_{ij})$ where $m_{ij} = 0$ unless $i = n$ or $i = j$ or $i = j - 1$, with row and column sum vector $(1, 2, \ldots, n)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

(y^5) Bounded regions into which the cone $x_1 \geq x_2 \geq \cdots \geq x_{n+1}$ in $\mathbb{R}^{n+1}$ is divided by the hyperplanes $x_i - x_j = 1$, $1 \leq i < j \leq n+1$ (compare (lll), which illustrates the case $n = 2$ of the present item)

(z^5) Extreme rays of the closed convex cone generated by all flag f-vectors (i.e., the functions $\beta(P, S)$ of Section 3.12) of graded posets of rank n with $\hat{0}$ and $\hat{1}$ (the vectors below lie on the extreme rays, with the coordinates $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$ in that order)

$$(0,0,0,1) \quad (0,0,1,1) \quad (0,1,0,0) \quad (0,1,1,1) \quad (1,1,1,1)$$

6.25 (j) Degree of the Grassmannian $G(2, n+2)$ (as a projective variety under the usual Plücker embedding) of 2-dimensional planes in $\mathbb{C}^{n+2}$

(k) Dimension (as a $\mathbb{Q}$-vector space) of the ring $\mathbb{Q}[x_1, \ldots, x_n]/Q_n$, where Q_n denotes the ideal of $\mathbb{Q}[x_1, \ldots, x_n]$ generated by all quasisymmetric functions in the variables $x_1, \ldots, x_n$ with 0 constant term

(l) Multiplicity of the point X_{w_0} in the Schubert variety Ω_w of the flag manifold $\mathrm{GL}(n, \mathbb{C})/B$, where $w_0 = n, n-1, \ldots, 1$ and $w = n, 2, 3, \ldots, n-2, n-1, 1$

(m) Conjugacy classes of elements $A \in \mathrm{SL}(n, \mathbb{C})$ such that $A^{n+1} = 1$

6.38 (n) 2143-avoiding involutions (or vexillary involutions) in $\mathfrak{S}_n$

6.C1 (a) [3] Let $a_{i,j}(n)$ (respectively, $\bar{a}_{i,j}(n)$) denote the number of walks in n steps from $(0,0)$ to (i,j), with steps $(\pm 1, 0)$ and $(0, \pm 1)$, never touching a point $(-k, 0)$ with $k \geq 0$ (respectively, $k > 0$) once leaving the starting point. Show that

$$\begin{aligned} a_{0,1}(2n+1) &= 4^n C_n \\ a_{1,0}(2n+1) &= C_{2n+1} \\ a_{-1,1}(2n) &= \frac{1}{2} C_{2n} \\ a_{1,1}(2n) &= 4^{n-1} C_n + \frac{1}{2} C_{2n} \\ \bar{a}_{0,0}(2n) &= 2 \cdot 4^n C_n - C_{2n+1}. \end{aligned} \tag{1}$$

(b) [3] Show that for $i \geq 1$ and $n \geq i$,

$$a_{-i,i}(2n) = \frac{i}{2n} \frac{\binom{2i}{i}\binom{n+i}{2i}\binom{4n}{2n}}{\binom{2n+2i}{2i}} \tag{2}$$

$$a_{i,i}(2n) = a_{-i,-i} + 4^n \frac{i}{n} \binom{2i}{i} \binom{2n}{n-i}. \tag{3}$$

(c) [3] Let $b_{i,j}(n)$ (respectively, $\bar{b}_{i,j}(n)$) denote the number of walks in n steps from $(0,0)$ to (i,j), with steps $(\pm 1, \pm 1)$, never touching a point $(-k, 0)$ with $k \geq 0$ (respectively, $k > 0$) once leaving the starting point. Show that

$$\begin{aligned} b_{1,1}(2n+1) &= C_{2n+1} \\ b_{-1,1}(2n+1) &= 2 \cdot 4^n C_n - C_{2n+1} \\ b_{0,2}(2n) &= C_{2n} \end{aligned}$$

$$b_{2i,0}(2n) = \frac{i}{n} \binom{2i}{i} \binom{2n}{n-i} 4^{n-i}, \ i \geq 1 \tag{4}$$

$$\bar{b}_{0,0}(2n) = 4^n C_n. \tag{5}$$

(d) [3–] Let

$$f(n) = \sum_P (-1)^{w(P)},$$

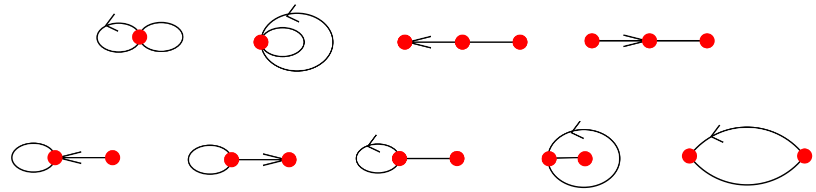


Figure 2: The rooted planar maps with two edges

where (i) P ranges over all lattice paths in the plane with $2n$ steps, from $(0,0)$ to $(0,0)$, with steps $(\pm 1, 0)$ and $(0, \pm 1)$, and (ii) $w(P)$ denotes the winding number of P with respect to the point $(\frac{1}{2}, \frac{1}{2})$. Show that $f(n) = 4^n C_n$.

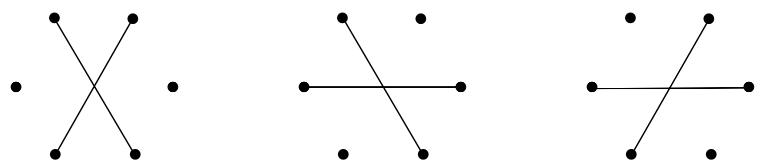
6.C2 [3] A *rooted planar map* is a planar embedding of an (unlabelled) connected planar graph rooted at a *flag*, i.e, at a triple (v, e, f) where v is a vertex, e is an edge incident to v, and f is a face incident to e. Two rooted planar maps G and H are considered the same if, regarding them as being on the 2-sphere $\mathbb{S}^2$, there is a flag-preserving homeomorphism of $\mathbb{S}^2$ that takes G to H. Equivalently, a rooted planar map may be regarded as a planar embedding of a connected planar graph in which a single edge on the outer face is directed in a counterclockwise direction. (The outer face is the root face, and the tail of the root edge is the root vertex.) Figure 2 shows the nine rooted planar maps with two edges.

(a) [3] Show that the number of rooted planar maps with n edges is equal to
$$\frac{2\,(2n)!\,3^n}{n!\,(n+1)!} = \frac{2 \cdot 3^n}{n+2} C_n.$$

(b) [2+] Show that the total number of vertices of all rooted planar maps with n edges is equal to $3^n C_n$.

6.C3 A *k-triangulation* of a convex n-gon C is a maximal collection of diagonals such that there are no $k+1$ of these diagonals for which any two intersect in their interiors. A 1-triangulation is just an ordinary triangulation, enumerated by the Catalan number C_{n-2} (Corollary 6.2.3(vi)). Note that any k-triangulation contains all diagonals between vertices at most distance k apart (where the distance between two vertices u, v

is the least number of edges of C we need to traverse in walking from u to v along the boundary of C). We call these nk edges *superfluous*. For example, there are three 2-triangulations of a hexagon, illustrated below (nonsuperfluous edges only).



(a) [3–] Show that all k-triangulations of an n-gon have $k(n-2k-1)$ nonsuperfluous edges.

(b) [3] Show that the number $T_k(n)$ of k-triangulations of an n-gon is given by

$$\begin{aligned} T_k(n) &= \det\left[C_{n-i-j}\right]_{i,j=1}^{k} \\ &= \prod_{1\leq i<j\leq n-2k} \frac{2k+i+j-1}{i+j-1}, \end{aligned}$$

the latter inequality by by Theorem 2.7.1 and Exercise 7.101(a).

(c) [5] It follows from (b) and Exercise 7.101(a) that $T_k(n)$ is equal to the number of plane partitions, allowing 0 as a part, of the staircase shape $\delta_{n-2k} = (n-2k-1, n-2k-2, \ldots, 1)$ and largest part at most k. Give a bijective proof.

6.C4 [3–] Let D be a Dyck path with $2n$ steps, and let $k_i(D)$ denote the number of up steps in D from level $i-1$ to level i. Show that

$$\sum_{D,D'} \sum_i k_i(D)k_i(D') = C_{2n} - C_n^2,$$

where the first sum ranges over all pairs (D, D') of Dyck paths with $2n$ steps.

6.C5 Let $t_0, t_1, \ldots$ be indeterminates. If S is a finite subset of $\mathbb{N}$, then set $t^S = \prod_{i\in S} t_i$. For $X = \mathbb{N}$ or $\mathbb{P}$, let $U_n(X)$ denote the set of all n-subsets of X that don't contain two consecutive integers.

(a) [2+] Show that the following three power series are equal:

(i) The continued fraction

$$\cfrac{1}{1-\cfrac{t_0x}{1-\cfrac{t_1x}{1-\cfrac{t_2x}{1-\cdots}}}}$$

(ii) $$\frac{\displaystyle\sum_{n\geq 0}(-1)^n\left(\sum_{S\in U_n(\mathbb{P})} t^S\right)x^n}{\displaystyle\sum_{n\geq 0}(-1)^n\left(\sum_{S\in U_n(\mathbb{N})} t^S\right)x^n}$$

(iii) $\displaystyle\sum_T \prod_{v\in V(T)} t_{\mathrm{ht}(v)}^{\deg(v)} x^{\#V(T)-1}$, where T ranges over all (nonempty) plane trees. Moreover, $V(T)$ denotes the vertex set of T, $\mathrm{ht}(v)$ the height of vertex v (where the root has height 0), and $\deg(v)$ the degree (number of children) of vertex v.

The three power series begin

$$1+t_0x+(t_0^2+t_0t_1)x^2+(t_0^3+2t_0^2t_1+t_0t_1^2+t_0t_1t_2)x^3+\cdots. \quad (6)$$

(b) [2] Deduce that

$$\lim_{n\to\infty}\frac{\sum_{i\geq 0}(-1)^i\binom{n-i}{i}x^i}{\sum_{i\geq 0}(-1)^i\binom{n+1-i}{i}x^i}=\sum_{k\geq 0}C_kx^k. \quad (7)$$

(c) [5] Let $t_i=(2i+1)^2$ in equation (6), yielding the power series

$$\sum_{n\geq 0}D_nx^n=1+x+10x^2+325x^3+22150x^4+\cdots.$$

Show that $\nu_2(D_n)=\nu_2(C_n)$, where $\nu_2(m)$ is the exponent of the largest power of 2 dividing m.

6.C6 (a) [2+] Start with the monomial $x_{12}x_{23}x_{34}\cdots x_{n,n+1}$, where the variables x_{ij} commute. Continually apply the "reduction rule"

$$x_{ij}x_{jk}\to x_{ik}(x_{ij}+x_{jk}) \quad (8)$$

in any order until unable to do so, resulting in a polynomial $P_n(x_{ij})$. Show that $P_n(x_{ij}=1)=C_n$. (NOTE. The polynomial $P_n(x_{ij})$ itself depends on the order in which the reductions are applied.) For instance, when $n=3$ one possible sequence of reductions (with the pair of variables being transformed shown in boldface) is given by

$$\begin{aligned}
\boldsymbol{x_{12}x_{23}}x_{34} &\rightarrow \boldsymbol{x_{13}}x_{12}\boldsymbol{x_{34}}+\boldsymbol{x_{13}}x_{23}\boldsymbol{x_{34}} \\
&\rightarrow x_{14}x_{13}x_{12}+x_{14}x_{34}x_{12} \\
&\qquad +x_{14}x_{13}x_{23}+x_{14}\boldsymbol{x_{34}x_{23}} \\
&\rightarrow x_{14}x_{13}x_{12}+x_{14}x_{34}x_{12} \\
&\qquad +x_{14}x_{13}x_{23}+x_{14}x_{24}x_{23}+x_{14}x_{24}x_{34} \\
&= P_3(x_{ij}).
\end{aligned}$$

(b) [3–] More strongly, replace the rule (8) with

$$x_{ij}x_{jk}\rightarrow x_{ik}(x_{ij}+x_{jk}-1),$$

this time ending with a polynomial $Q_n(x_{ij})$. Show that

$$Q_n\left(x_{ij}=\frac{1}{1-x}\right)=\frac{N(n,1)+N(n,2)x+\cdots+N(n,n)x^{n-1}}{(1-x)^n},$$

where $N(n,k)$ is a Narayana number (defined in Exercise 6.36).

(c) [3–] Even more generally, show that

$$Q_n(x_{ij}=t_i)=\sum(-1)^{n-k}t_{i_1}\cdots t_{i_k},$$

where the sum ranges over all pairs $((a_1,a_2,\ldots,a_k),(i_1,i_2,\ldots,i_k))\in\mathbb{P}^k\times\mathbb{P}^k$ satisfying $1=a_1<a_2<\cdots<a_k\le n$, $1\le i_1\le i_2\le\cdots\le i_n$, and $i_j\le a_j$. For instance,

$$Q_3(x_{ij}=t_i)=t_1^3+t_1^2t_2+t_1^2t_3+t_1t_2^2+t_1t_2t_3-2t_1^2-2t_1t_2-t_1t_3+t_1.$$

(d) [3+] Now start with the monomial $\prod_{1\le i<j\le n+1}x_{ij}$ and apply the reduction rule (8) until arriving at a polynomial $R_n(x_{ij})$. Show that

$$R_n(x_{ij}=1)=C_1C_2\cdots C_n.$$

(e) [5–] Generalize (d) in a manner analogous to (b) and (c).

6.C7 [3–] Let V_r be the operator on (real) polynomials defined by

$$V_r\left(\sum_{i\geq 0} a_i q^i\right) = \sum_{i\geq r} a_i q^i.$$

Define $B_1(q) = -1$, and for $n > 1$,

$$B_n(q) = (q-1)B_{n-1}(q) + \ V_{(n+1)/2}\left(q^{n-1}(1-q)B_{n-1}(1/q)\right).$$

Show that $B_{2n}(1) = B_{2n+1}(1) = (-1)^{n+1}C_n$.

6.C8 (a) [3+] Let $g(n)$ denote the number of $n\times n$ $\mathbb{N}$-matrices $M = (m_{ij})$ where $m_{ij} = 0$ if $j > i+1$, with row and column sum vector $\left(1, 3, 6, \ldots, \binom{n+1}{2}\right)$. For instance, when $n = 2$ there are the two matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix},$$

while an example for $n = 5$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 1 & 0 & 4 & 10 \end{bmatrix}.$$

Show that $g(n) = C_1C_2\cdots C_n$.

(b) [2+] Let $f(n)$ be the number of ways to write the vector

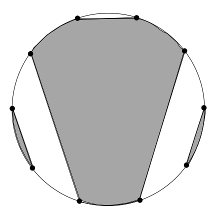
$$\left(1, 2, 3, \ldots, n, -\binom{n+1}{2}\right) \in \mathbb{Z}^{n+1}$$

as a sum of vectors $e_i - e_j$, $1 \leq i < j \leq n+1$, without regard to order, where e_k is the kth unit coordinate vector in $\mathbb{Z}^{n+1}$. For instance, when $n = 2$ there are the two ways $(1, 2, -3) = (1, 0, -1) + 2(0, 1, -1) = (1, -1, 0) + 3(0, 1, -1)$. Assuming (a), show that $f(n) = C_1C_2\cdots C_n$.

(c) [3–] Let CR_n be the convex polytope of all $n \times n$ doubly-stochastic matrices $A = (a_{ij})$ satisfying $a_{ij} = 0$ if $i > j+1$. It is easy to see that CR_n is an integral polytope of dimension $\binom{n}{2}$. Assuming (a) or (b), show that the relative volume of CR_n (as defined §4.6) is given by

$$\nu(\mathrm{CR}_n) = \frac{C_1 C_2 \cdots C_{n-1}}{\binom{n}{2}!}.$$

6.C9 [3–] Join $4m+2$ points on the circumference of a circle with $2m+1$ nonintersecting chords, as in Exercise 6.19(n). Call such a set of chords a *net*. The circle together with the chords forms a map with $2m+2$ (interior) regions. Color the regions red and blue so that adjacent regions receive different colors. Call the net *even* if an even number of regions are colored red and an even number blue, and *odd* otherwise. The figure below shows an odd net for $m = 2$.



Let $f_e(m)$ (respectively, $f_o(m)$) denote the number of even (respectively, odd) nets on $4m+2$ points. Show that

$$f_e(m) - f_o(m) = (-1)^{m-1} C_m.$$

6.C10 (a) [2+] Show that

$$C(x)^q = \sum_{n \geq 0} \frac{q}{n+q} \binom{2n-1+q}{n} x^n.$$

(b) [2+] Show that

$$\frac{C(x)^q}{\sqrt{1-4x}} = \sum_{n \geq 0} \binom{2n+q}{n} x^n.$$

6.C11 (a) [2–] Give a generating function proof of the identity

$$\sum_{k=0}^{n} C_{2k} C_{2(n-k)} = 4^n C_n. \tag{9}$$

(b) [5–] Give a bijective proof.

6.C12 (a) [2+] Find all power series $F(t) \in \mathbb{C}[\![t]\!]$ such that if

$$\frac{1 - x + xtF(t)}{1 - x + x^2 t} = \sum_{n \geq 0} f_n(x) t^n,$$

then $f_n(x) \in \mathbb{C}[x]$.

(b) [2+] Find the coefficients of the polynomials $f_n(x)$.

6.C13 (a) [3–] Find the unique continuous function $f(x)$ on $\mathbb{R}$ satisfying for all $n \in \mathbb{N}$:

$$\int_{-\infty}^{\infty} x^n f(x) dx = \begin{cases} C_k, & \text{if } n = 2k \\ 0, & \text{if } n = 2k+1. \end{cases}$$

(b) [3–] Find the unique continuous function $f(x)$ for $x > 0$ satisfying for all $n \in \mathbb{N}$:

$$\int_0^{\infty} x^n f(x) dx = C_n.$$

6.C14 [2+] The *Fibonacci tree* F is the rooted tree with root v, such that the root has degree one, the child of every vertex of degree one has degree two, and the two children of every vertex of degree two have degrees one and two. Figure 3 shows the first six levels of F. Let $f(n)$ be the number of closed walks in F of length $2n$ beginning at v. Show that

$$f(n) = \frac{1}{2n+1} \binom{3n}{n},$$

the number of ternary trees with n vertices. (See Exercises 5.45, 5.46, and 5.47(b) for further occurences of this number.)

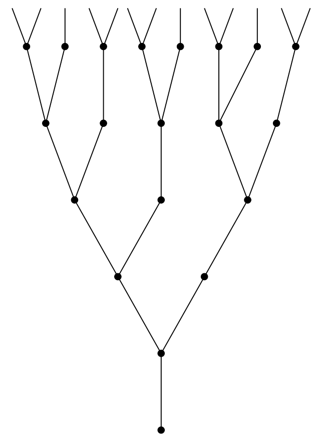


Figure 3: The Fibonacci tree

SOLUTIONS

6.19(ooo) Traverse the tree in preorder. When going down an edge (i.e., away from the root) record 1 if this edge goes to the left or straight down, and record -1 if this edge goes to the right. This gives a bijection with (p^4).

(ppp) The proof follows from the generating function identity

$$\begin{aligned} C(x)^2 &= \frac{C(x)-1}{x} \\ &= \sum_{k\geq 0} x^k \sum_{i=0}^{k} O(x)^i 2^{k-i} E(x)^{k-i} \\ &= \frac{1}{(1-2xE(x))(1-xO(x))}. \end{aligned}$$

This result is due to L. Shapiro, private communication dated 26 December 2001, who raises the question of giving a simple bijective proof. In a preprint entitled "Catalan trigonometry" he gives a simple bijective proof of the related identity

$$O(x) = x(O(x)^2 + E(x)^2)$$

and remarks that there is a similar proof of

$$E(x) = 1 + 2xE(x)O(x).$$

For a further identity of this nature, see Exercise 6.C11.

(qqq) This result is due to L. Shapiro, private communication dated 24 May 2002.

(rrr) Add one further up step and then down steps until reaching $(2n, 0)$. This gives a bijection with the Dyck paths of (i).

(sss) Deleting the first UD gives a bijection with (i) (Dyck paths of length $2n$). This result is due to David Callan, private communication dated 3 November 2004.

(ttt) In the two-colored Motzkin paths of (yyy) replace the step $(1, 1)$ with the sequence of steps $(1, 1) + (1, 1) + (1, -1)$, the step $(1, -1)$ with $(1, 1) + (1, -1) + (1, -1)$, the red step $(1, 0)$ with $(1, 1) + (1, -1)$, and the blue step $(1, 0)$ with $(1, 1) + (1, 1) + (1, -1) + (1, -1)$.

(uuu) Every Dyck path P with at least two steps has a unique factorization $P = XYZ$ such that Y is a Dyck path (possibly with 0 steps), $\text{length}(X) = \text{length}(Z)$, and XZ is a Dyck path (with at least two steps) of the type being counted. Hence if $f(n)$ is the number of Dyck paths being counted and $F(x) = \sum_{n\geq 1} f(n-1)x^n$, then

$$C(x) = 1 + F(x)C(x).$$

It follows that $F(x) = xC(x)$, so $f(n) = C_n$ as desired. This result is due to Sergi Elizalde (private communication, September, 2002).

(vvv) To obtain a bijection with the Dyck paths of (i) add a $(1, 1)$ step immediately following a path point $(m, 0)$ and a $(1, -1)$ step at the end of the path (R. Sulanke, private communication from E. Deutsch dated 4 February 2002).

(www) To obtain a bijection with (vvv) contract a region under a peak of height one to a point (E. Deutsch and R. Sulanke, private communication from E. Deutsch dated 4 February 2002).

(xxx) *First solution.* Fix a permutation $u \in \mathfrak{S}_3$, and let $T(u)$ be the set of all u-avoiding permutations (as defined in Exercise 6.39(l)) in all $\mathfrak{S}_n$ for $n \geq 1$. Partially order $T(u)$ by setting $v \leq w$ if v is a subsequence of w (when v and w are written as words). One checks that $T(u)$ is isomorphic to the tree T. Moreover, the vertices in $T(u)$ of height n consist of the u-avoiding permutations in $\mathfrak{S}_{n+1}$. The proof then follows from Exercise 6.19(ee,ff).

Second solution. Label each vertex by its degree. A saturated chain from the root to a vertex at level $n-1$ is thus labelled by a sequence $(b_1, b_2, \ldots, b_n)$. Set $a_i = i + 2 - b_i$. This sets up a bijection between level $n-1$ and the sequences $(a_1, \ldots, a_n)$ of (s).

Third solution. Let $f_n(x) = \sum_v x^{\deg(v)}$, summed over all vertices of T of height $n-1$. Thus $f_1(x) = x^2$, $f_2(x) = x^2 + x^3$, $f_3(x) = x^4 + 2x^3 + 2x^2$, etc. Set $f_0(x) = x$. The definition of T implies that we get $f_{n+1}(x)$ from $f_n(x)$ by substituting $x^2 + x^3 + \cdots + x^{k+1} = x^2(1-x^k)/(1-x)$ for x^k. Thus

$$f_{n+1}(x) = \frac{x^2(f_n(1) - f_n(x))}{1-x},\ n \geq 0.$$

Setting $F(x,t) = \sum_{n\geq 0} F_n(x) t^n$, there follows

$$\frac{x - F(x,t)}{t} = \frac{x^2}{1-x}(F(1,t) - F(x,t)).$$

Hence

$$F(x,t) = \frac{x - x^2 + x^2 t F(1,t)}{1 - x + x^2 t}.$$

Now use Exercise 6.C12.

The trees $T(u)$ were first defined by J. West, *Discrete Math.* **146** (1995), 247–262 (see also *Discrete Math.* **157** (1996), 363–374) as in the first solution above, and are called *generating trees.* West then presented the labeling argument of the second solution, thereby giving new proofs of (ee) and (ff). For further information on generating trees, see C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps, *Discrete Math.*, to appear; available at

`algo.inria.fr/flajolet/Publications/publist.html`.

(yyy) Replace a step $(1,1)$ with $1,1$, a step $(1,-1)$ with $-1,-1$, a red step $(1,0)$ with $1,-1$, a blue step $(1,0)$ with $-1,1$, and adjoin an extra 1 at the beginning and -1 at the end. This gives a bijection with (r) (suggested by R. Chapman). The paths being enumerated are called *two-colored Motzkin paths*. See for instance E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani, *Lecture Notes in Comput. Sci.* **959**, Springer, Berlin, 1995, pp. 254–263.

(zzz) Let π be a noncrossing partition of $[n]$. Denote the steps in a Motzkin path by U (up), D (down) and L (level). In the sequence $1, 2, \ldots, n$, replace the smallest element of a nonsingleton block of π with the two steps LU. Replace the largest element of a nonsingleton block of π with DL. Replace the element of a singleton block with LL. Replace an element that is neither the smallest nor largest element of its block with DU. Remove the first and last terms (which are always L). For instance, if $\pi = 145\text{–}26\text{–}3$, then we obtain $ULULLDUDLD$. This sets up a bijection with (pp). E. Deutsch (private communication dated 16 September 2004) has also given a simple bijection with the Dyck paths (i). The bijection given here is a special case of a bijection appearing in W. Chen, E. Deng, R. Du, R. Stanley, and C. Yan, Crossings and nestings of matchings and partitions, `math.CO/0501230`.

(a^4) Replace each step $(1,1)$ or $(0,1)$ with the step $(1,1)$, and replace each step $(1,0)$ with $(1,-1)$. We obtain a bijection with the paths of (rrr).

(b^4) See R. K. Guy, *J. Integer Sequences* **3** (2000), article 00.1.6, available at

`www.research.att.com/`
`~njas/sequences/JIS/VOL3/GUY/catwalks.html`,

and R. K. Guy, C. Krattenthaler, and B. Sagan, *Ars Combinatorica* **34** (1992), 3–15.

(c^4) Replace the left-hand endpoint of each arc with a 1 and the right-hand endpoint with a -1. We claim that this gives a bijection with the ballot sequences of (r). First note that if we do the same construction for the noncrossing matchings of (o), then it is very easy to see that we get a bijection with (r). Hence we will give a bijection from (o) to (c^4) with the additional property that the locations of the left endpoints and right endpoints of the arcs are preserved. (Of course any bijection between (o) and (c^4) would suffice to prove the present item; we are showing a stronger result.)

Let M be a noncrossing matching on $2n$ points. Suppose we are given the set S of left endpoints of the arcs of M. We can recover M by scanning the elements of S from right-to-left, and attaching each element i to the *leftmost* available point to its right. In other words, draw an arc from i to the first point to the right of i that does not belong to S and to which no arc has been already attached. If we change this algorithm by attaching each element of S to the *rightmost* available point to its right, then it can be checked that we obtain a nonnesting matching and that we have defined a bijection from (o) to (c^4).

I cannot recall to whom this argument is due. Can any reader provide this information? For further information on crossings and nestings of matchings, see W. Chen, E. Deng, R. Du, R. Stanley, and C. Yan, Crossings and nestings of matchings and partitions, `math.CO/0501230`, and the references given there. help!

(d^4) Let $f : \mathbb{P} \to \mathbb{P}$ be any function satisfying $f(i) \leq i$. Given a ballot sequence $\alpha = (a_1, \ldots, a_{2n})$ as in (r), define the corresponding *f-matching* M_α as follows. Scan the 1's in α from right-to-left. Initially all the 1's and -1's in α are unpaired. When we encounter $a_i = 1$ in α, let j be the number of unpaired -1's to its right, and draw an arc from a_i to the $f(j)$th -1 to its right (thus pairing a_i with this -1). Continue until we have paired a_1, after which all terms of α will be paired, thus yielding the matching M_α. By construction, the number of f-matchings of $[2n]$ is C_n. This gives infinitely many combinatorial interpretations of C_n, but of course most of these will be of no special interest. If $f(i) = 1$ for all i, then we obtain the noncrossing matchings of (o). If $f(i) = i$ for all i, then we obtain the nonnesting matchings of (c^4). If $f(i) = \lfloor i/2 \rfloor$ for all i, then we obtain the matchings of the present item. Thus these matchings are in a sense "halfway between" noncrossing and nonnesting matchings.

(e^4) Reading the the points from left-to-right, replace each isolated point and each point which is the left endpoint of an arc with 1, and replace each point which is the right endpoint of an arc with -1. We obtain a bijection with (p^4).

(f^4) Label the points $1, 2, \ldots, n$ from left-to-right. Given a noncrossing partition of $[n]$ as in (pp), draw an arc from the first element of

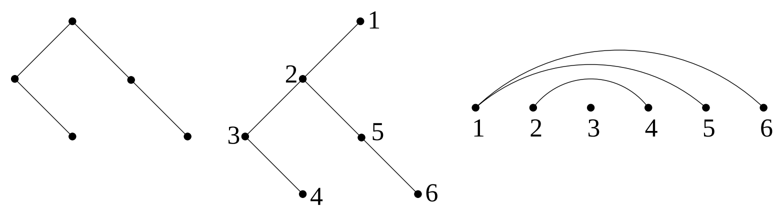


Figure 4: The bijecton for Exercise 6.19(g^4)

each block to the other elements of that block, yielding a bijection with the current item. This result is related to research on network testing done by Nate Kube (private communication from Frank Ruskey dated 9 November 2004).

(g^4) Given a binary tree with n vertices as in (c), add a new root with a left edge connected to the old root. Label the $n+1$ vertices by $1, 2, \ldots, n{+}1$ in preorder. For each right edge, draw an arc from its bottom vertex to the top vertex of the first left edge encountered on the path to the root. An example is shown in Figure 4. On the left is a binary tree with $n = 5$ vertices; in the middle is the augmented tree with $n{+}1$ vertices with the preorder labeling; and on the right is the corresponding set of arcs. This result is due to David Callan, private communication, 23 March 2004.

(h^4) See S. Elizalde, Statistics on Pattern-Avoiding Permutations, Ph.D. thesis, M.I.T., June 2004 (Proposition 3.5.3(1)).

(i^4) Three proofs are given by H. Niederhausen, Catalan traffic at the beach, preprint available at

`www.math.fau.edu/Niederhausen/html/Papers/`
`CatalanTraffic.ps`.

(j^4) Let P be a parallelogram polyomino of the type being counted. Linearly order the maximal vertical line segments on the boundary of P according to the level of their bottommost step. Replace such a line segment appearing on the right-hand (respectively, left-hand) path of the boundary of P by a 1 (respectively, -1), but omit the final line segment (which will always be on the left). For instance, for the first parallelogram polyomino shown in the statement of the problem, we get the sequence $(1, 1, -1, -1, 1, -1)$.

This sets up a bijection with (r). This result is due to E. Deutsch, S. Elizalde, and A. Reifegerste (private communication, April, 2003).

(k^4) If we consider the number of chord diagrams (n) containing a fixed horizontal chord, then we obtain the standard quadratic recurrence for Catalan numbers. An elegant "bijectivization" of this argument is the following. Fix a vertex v. Given a nonintersecting chord diagram with a distinguished horizontal chord K, rotate the chords so that the left-hand endpoint of K is v. This gives a bijection with (n). Another way to say this (suggested by R. Chapman) is that there are n different chord slopes, each occuring the same number of times, and hence C_n times.

(l^4) Kepler towers were created by X. Viennot, who gave a bijection with the Dyck paths (i). Viennot's bijection was written up by D. Knuth, Three Catalan bijections, Institut Mittag-Leffler preprint series, 2005 spring, #04; `www.ml.kva.se/preprints/0405s`. The portion of this paper devoted to Kepler walls is also available at `www-cs-faculty.stanford.edu/~knuth/programs/viennot.w`.

(m^4) Immediate from the generating function identity

$$C(x) = \frac{1}{1 - xC(x)} = 1 + xC(x) + x^2C(x)^2 + \cdots.$$

This result is due to E. Deutsch (private communication dated 8 April 2005).

(n^4) Add 1 to the terms of the sequences of (w). Alternatively, if $(b_1, \ldots, b_{n-1})$ is a sequence of (s), then let $(a_1, \ldots, a_n) = (n + 1 - b_{n-1}, b_{n-1} - b_{n-2}, \ldots, b_2 - b_1)$.

(o^4) Partial sums of the sequences in (r). The sequences of this exercise appear explicitly in E. P. Wigner, *Ann. Math.* **62** (1955), 548–564.

(p^4) In (rrr) replace an up step with 1 and a down step with -1.

(q^4) Suppose that the reverse sequence $b_1 \cdots b_{2n-2} = a_{2n-2} \cdots a_1$ begins with k -1's. Remove these -1's, and and for each $1 \le i \le k$ remove the rightmost b_j for which $b_{k+1} + b_{k+2} + \cdots + b_j = i$. This yields a sequence of $k + 1$ ballot sequences as given by (r). Place a 1 at the beginning and -1 at the end of each of these ballot sequences and concatenate, yielding a bijection with (r). This result (stated in terms of lattice paths) is due to David Callan, private communication dated 26 February 2004.

Example. Let $a_1 \cdots a_{14} = + - - + - - + + + + - + - -$ (writing $+$ for 1 and $-$ for -1), so $b_1 \cdots b_{14} = - - + - + + + + - - + - - +$. Remove b_1, b_2, b_5, b_{14}, yielding the ballot sequences $+-$, $+ + + - - + - -$, and $\emptyset$. We end up with the ballot sequence $+ + - - + + + + - - + - - + + -$.

(r[4]) Consider the pairs of lattice paths of (l) and the lines L_i defined by $x+y=i$, $1 \leq i \leq n$. Let S denote the set of all lattice squares contained between the two paths. The line L_i will pass through the interior of some b_i elements of S. Set $a_i = \pm b_i$ as follows: (i) $a_1 = b_1 = 1$, (ii) $a_i a_i > 0$ if $b_i \neq b_{i-1}$, and (iii) if $b_i = b_{i-1}$, then $a_i = a_{i-1}$ if the top lattice square in S that L_i passes through lies above the top lattice square in S that L_{i-1} passes through, and otherwise $a_i = -a_{i-1}$. This sets up a bijection with (l).

(s[4]) In the tree T of (xxx), label the root by 0 and the two children of the root by 0 and 1. Then label the remaining vertices recursively as follows. Suppose that the vertex v has height n and is labelled by j. Suppose also that the siblings of v with labels less than j are labelled $t_1, \ldots, t_i$. It follows that v has $i+2$ children, which we label $t_1, \ldots, t_i, j, n$. See Figure 5 for the labelling up to height 3. As in the second solution to Exercise 6.19(xxx), a saturated chain from the root to a vertex at level $n-1$ is thus labelled by a sequence $(a_1, a_2, \ldots, a_n)$. It can be seen that this sets up a bijection between level $n-1$ and the sequences we are trying to count. The proof then follows from (xxx). This exercise is due to Z. Sunik, *Electr. J. Comb.* **10** (2003), N5. Sunik also points out that the number of elements labelled j at level n is equal to $C_j C_{n-j}$.

(t[4]) Given a plane tree with n edges, traverse the edges in preorder and record for each edge except the last the degree (number of successors) of the vertex terminating the edge. It is easy to check that this procedure sets up a bijection with (e). This result is due to David Callan, private communication dated 3 November 2004.

(u[4]) Let T be a plane tree with $n+1$ vertices labelled $1, 2, \ldots, n+1$ in preorder. Do a depth first search through T and write down the

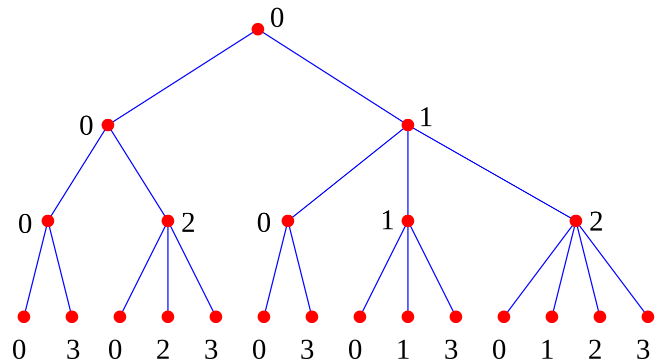


Figure 5: The tree for Exercise 6.19(s^4)

vertices in the order they are visited (including repetitions). This establishes a bijection with (e). The sequences of this exercise appear implicitly in E. P. Wigner, *Ann. Math.* **62** (1955), 548–564, viz., as a contribution $X_{a_1a_2}X_{a_2a_3}\cdots X_{a_{2n-1}a_{2n}}X_{a_{2n}a_1}$ to the $(1,1)$-entry of the matrix X^{2n}. Exercise 5.19(o^4) is related.

(v^4) The sequences $1, 1+a_n, 1+a_n+a_{n-1}, \dots, 1+a_n+a_{n-1}+\cdots+a_2$ coincide with those of (s). See R. Stanley, *J. Combinatorial Theory* **14** (1973), 209–214 (Theorem 1).

(w^4) Partially order the set $P_n = \{(i,j) : 1 \leq i < j \leq n\}$ componentwise. Then the sets $\{(i_1,j_1),\dots,(i_k,j_k)\}$ are just the antichains of P_n and hence are equinumerous with the order ideals of P_n (see the end of Section 3.1). But P_n is isomorphic to the poset $\mathrm{Int}(\boldsymbol{n-1})$ of Exercise 6.19(bbb), so the proof follows from this exercise.

This result is implicit in the paper A. Reifegerste, On the diagram of 132-avoiding permutations, `math.CO/0208006`. She observes that if $(i_1\cdots i_k, j_1\cdots j_k)$ is a pair being counted, then there is a unique 321-avoiding permutation $w \in \mathfrak{S}_n$ whose *excedance set* $E_w = \{i : w(i) > i\}$ is $\{i_1,\dots i_k\}$ and such that $w(i_k) = j_k$ for all k. Conversely, every 321-avoiding $w \in \mathfrak{S}_n$ gives rise to a pair being counted. Thus the proof follows from Exercise 6.19(ee).

(x^4) Let L be a lattice path as in (h). Let $(0,0) = v_0, v_1, \dots, v_k = (n,n)$ be the successive points at which L intersects the diagonal $y = x$. Let L' be the path obtained by reflecting about $y = x$ the portions of L between each v_{2i-1} and v_{2i}. The horizontal steps of L' then correspond to the moves of the first player, while the vertical steps correspond to the moves of the second player.

This result and solution are due to Lou Shapiro, private communication dated 13 May 2005. Shapiro stated the result in terms of the game of Parcheesi, but since many readers may be unfamiliar with this game we have given a more mundane formulation.

(y^4) Since $n+1$ and n are relatively prime, each equivalence class has exactly $2n+1$ elements. Hence the number of classes is $\frac{1}{2n+1}\binom{2n+1}{n} = C_n$. This fact is the basis for the direct combinatorial proof that there are C_n ballot sequences (as defined in Corollary 6.2.3(ii)) of length $2n$; see Example 5.3.12.

(z^4) Let $C_\lambda = \cdots c_{-2}c_{-1}c_0c_1c_2\cdots$ be the code of the partition λ, as defined in Exercise 7.59, where $c_0 = 1$ and $c_i = 0$ for $i < 0$. Let $S_\lambda = \{i : c_i = 1\}$. For instance, if $\lambda = (3,1,1)$ then $S_\lambda = \mathbb{N} - \{1,2,5\}$. It is easy to see (using Exercise 7.59(b)) that λ is an n-core and an $(n+1)$-core if and only if S_λ is a set counted by (r^5), and the proof follows. This cute result is due to J. Anderson, *Discrete Math.* **248** (2002), 237–243. Anderson obtains the more general result that if m and n are relatively prime, then the number of partitions λ that are both m-cores and n-cores is $\frac{1}{m+n}\binom{m+m}{m}$. J. Olsson and D. Stanton (in preparation) show that in addition the largest $|\lambda|$ for which λ is an m-core and n-core is given by $(m^2-1)(n^2-1)/24$.

(a^5) We claim that each equivalence class contains a unique element $(a_1,\dots,a_n)$ satisfying $a_1+a_2+\cdots+a_i \geq i$ for $1 \leq i \leq n$. The proof then follows from (n^4). To prove the claim, if $\alpha = (a_1,\dots,a_n) \in S_n$, then define $\alpha' = (a_1-1,\dots,a_n-1,-1)$. Note that the entries of α' are ≥ -1 and sum to -1. If $E = \{\alpha_1,\dots,\alpha_k\}$ is an equivalence class, then it is easy to see that the set $\{\alpha'_1,\dots,\alpha'_k\}$ consists of all conjugates (or cyclic shifts) that end in -1 of a single word α'_1, say. It follows from Lemmma 5.3.7 that there is a unique conjugate (or cyclic shift) β of α'_1 such that all partial sums of β, except for the sum of all the terms, are nonnegative. Since the last component of β is -1, it follows that $\beta = \alpha'_j$ for a unique j. Let $\alpha_j = (a_1,\dots,a_n)$. Then α_j will be the unique element of E satisfying $a_1+\cdots+a_i \geq i$, as desired.

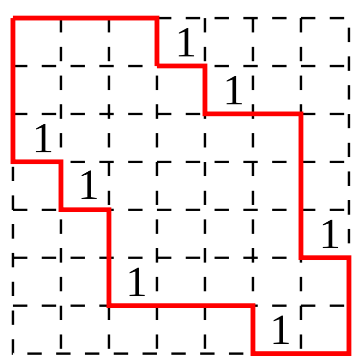
(b^5) Let $\alpha = (\alpha_1,\dots,\alpha_k)$, $\beta = (\beta_1,\dots,\beta_k)$. Define a Dyck path by going up α_1 steps, then down β_1 steps, then up α_2 steps, then down β_2 steps, etc. This gives a bijection with (i), due to A. Reifegerste, The excedances and descents of bi-increasing permutations, preprint (Cor. 3/8); `math.CO/0212247`.

(c^5) See J. Françon and G. Viennot, *Discrete Math.* **28** (1979), 21–35.

(d^5) See N. A. Loehr, *Europ. J. Combinatorics* **26** (2005), 83–93. This contrived-looking interpretation of C_n is actually closely related to Exercise 6.25(i) and the (q,t)-Catalan numbers of Garsia and Haiman.

(e^5) See N. A. Loehr, *ibid.*

(f^5) Replace an excedance of w with a 1 and a nonexcedance with a -1, except for the nonexcedance $2n+1$ at the end of w. This sets up a bijection with (r). There is also a close connection with (j^4). If P is a parallelogram polyomino of the type counted by (j^4), then place P in a $(2n+1)\times(2n+1)$ square M. Put a 1 in each square immediately to the right of the bottom step in each maximal vertical line on the boundary, except for the rightmost such vertical line. Put a 0 in the remaining squares of M. This sets up a bijection between (j^4) and the permutation matrices corresponding to the permutations counted by the present exercise. An example is given by the figure below, where the corresponding permutation is 4512736. This result is due to E. Deutsch, S. Elizalde, and A. Reifegerste (private communication, April, 2003).



(g^5) Let $a_1a_2\cdots a_{2n}$ be a permutation being counted, and associate with it the array

$$\begin{array}{ccccc} a_2 & a_4 & a_6 & \cdots & a_{2n} \\ a_1 & a_3 & a_5 & \cdots & a_{2n-1} \end{array}.$$

This sets up a bijection with (ww), standard Young tableaux of shape (n, n). This result it due to E. Deutsch and A. Reifegerste, private communication dated 4 June 2003. Deutsch and Reifegerste also point out that the permutations being counted have an alternative description as those 321-avoiding permutations in $\mathfrak{S}_{2n}$ with the maximum number of descents (or equivalently, excedances), namely n.

(h^5) By Corollary 7.13.6 (applied to permutation matrices), Theorem 7.23.17 (in the case $i = 1$), and Exercise 7.28(a) (in the case where A is a symmetric permutation matrix of trace 0), the RSK algorithm sets up a bijection between 321-avoiding fixed-point-free involutions in $\mathfrak{S}_{2n}$ and standard Young tableaux of shape (n, n). Now use (ww). There are also numerous ways to give a more direct bijection.

(i^5) As in (h^5), the RSK-algorithm sets up a bijection between 321-avoiding involutions in $\mathfrak{S}_{2n-1}$ with one fixed point and standard Young tableaux of shape $(n, n-1)$; and again use (ww).

(j^5) In the solution to (ii) it was mentioned that a permutation is stack-sortable if and only if it is 231-avoiding. Hence a permutation in $\mathfrak{S}_{2n}$ can be sorted into the order $2n, 2n-1, \ldots, 1$ on a stack if and only if it is 213-avoiding. Given a 213-avoiding fixed-point-free involution in $\mathfrak{S}_{2n}$, sort it in reverse order on a stack. When an element is put on the stack record a 1, and when it is taken off record a –1 (as in the solution to (ii)). Then we obtain exactly the sequences $a_1, a_2, \ldots, a_n, -a_n, \ldots, -a_2, -a_1$, where $a_1, a_2, \ldots, a_n$ is as in (r), and the proof follows. Moreover, E. Deutsch (private communication, May, 2001) has constructed a bijection with the Dyck paths of (i).

(k^5) Similar to (j^5).

(l^5) Obvious bijection with (ggg).

(m^5) Given a standard Young tableau T of the type being counted, construct a Dyck path of length $2n$ as follows. For each entry

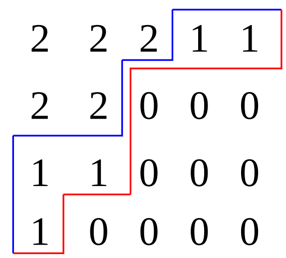


Figure 6: A plane partition and two lattice paths

$1, 2, \ldots, m$ of T, if i appears in row 1 then draw an up step, while if i appears in row 2 then draw a down step. Afterwards draw an up step followed by down steps to the x-axis. This sets up a bijection with (i).

(n^5) The bijection of (m^5) yields an *elevated* Dyck path, i.e., a Dyck path of length $2n + 2$ which never touches the x-axis except at the beginning and end. Remove the first and last step to get a bijection with (i).

(o^5) Remove all entries except $3, 5, 7, \ldots, 2n-1$ and shift the remaining entries in the first row one square to the left. Replace $2i + 1$ with i. This sets up a bijection with SYT of shape (n, n), so the proof follows from Exercise 6.19(ww). This result is due to T. Chow, H. Eriksson, and K. Fan, in preparation. This paper also shows the more difficult result that the number of SYT of shape (n, n, n) such that adjacent entries have opposite parity is the number $B(n-1)$ of Baxter permutations of length $n - 1$ (defined in Exercise 6.55).

(p^5) Given the plane partition π, let L be the lattice path from the lower left to upper right that has only 2's above it and no 2's below. Similarly let L' be the lattice path from the lower left to upper right that has only 0's below it and no 0's above. See Figure 6 for an example. This pair of lattice paths coincides with those of Exercise 6.19(m)

(q^5) In the two-colored Motzkin paths of (yyy), number the steps $1, 2, \ldots, n - 1$ from left to right. Place the upsteps $(1, 1)$ in A, the downsteps $(1, -1)$ in B, and the red flatsteps $(1, 0)$ in C. This result is due to David Callan, priviate communication dated 26 February 2004.

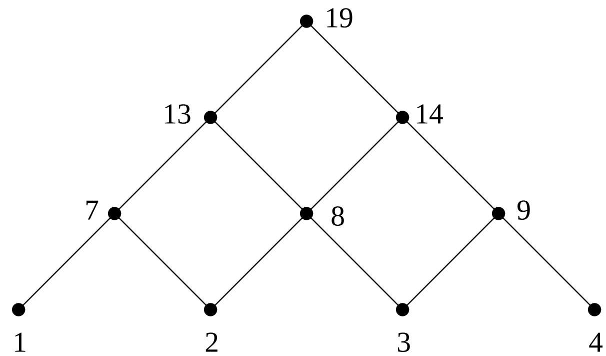


Figure 7: The poset T_5

(r[5]) Let S_n be the submonoid of $\mathbb{N}$ (under addition) generated by n and $n+1$. Partially order the set $T_n = \mathbb{N} - S_n$ by $i \le j$ if $j - i \in S_n$. Figure 7 illustrates the case $n = 5$. It can be checked that $T_n \cong \mathrm{Int}(\boldsymbol{n-1})$, as defined in Exercise 6.19(bbb). Moreover, the subsets S being counted are given by $\mathbb{N}-I$, where I is an order ideal of T_n. The proof follows from Exercise 6.19(bbb). This result is due to Mercedes H. Rosas, private communication dated 29 May 2002.

(s[5]) Analogous to (jjj), using $\frac{1}{n}\binom{2n}{n+1} = C_n$. This problem was suggested by S. Fomin.

(t[5]) Let a_i be the multiplicity of $e_i - e_{i+1}$ in the sum. The entire sum is uniquely determined by the sequence $a_1, a_2, \ldots, a_{n-1}$. Moreover, the sequences $0, a_1, a_2, \ldots, a_{n-1}$ that arise in this way coincide with those in (u). This exercise is an unpublished result of A. Postnikov and R. Stanley.

(u[5]) Let H be a polyomino of the type being counted, say with ℓ rows. Let $a_i + 1$ be the width (number of columns) of the first i rows of H, and let $\alpha_i = a_i - a_{i-1}$ for $1 \le i \le \ell$ (with $a_0 = 0$). Similarly let $b_i + 1$ be the width of the last i rows of H, and let $\beta_i = b_{\ell-i+1} - b_{\ell-i}$ for $1 \le i \le \ell$ (with $b_0 = 1$). This sets up a bijection with the pairs (α, β) of compositions counted by (b[5]). This argument is due to A. Reifegerste, *ibid.* By a refinement of this argument she also shows that the number of polyominoes of the type being counted with ℓ rows is the Narayana number $N(n, \ell) = \frac{1}{n}\binom{n}{\ell}\binom{n}{\ell-1}$ of Exercise 6.36.

Another bijection was provided by E. Deutsch (private communication dated 15 June 2001). Namely, given the polynomino H, let $a_1+1, \ldots, a_\ell+1$ be the row lengths and $b_1+1, \ldots, b_{\ell-1}+1$ be the lengths of the overlap between the successive rows. Let D be the Dyck path of length $2n$ with successive peaks at heights $a_1, \ldots, a_\ell$ and successive valleys at heights $b_1, \ldots, b_{\ell-1}$. This sets up a bijection with Dyck paths of length $2n$, as given in (i). (Compare with the solution to (l).)

(v^5) There is a simple bijection with the binary trees T of (c). The root of T corresponds to the rectangle containing the upper right-hand corner of the staircase. Remove this rectangle and we get two smaller staircase tilings, making the bijection obvious. This result is the case $d = 2$ of Theorem 1.1 of H. Thomas, New combinatorial descriptions of the triangulations of cyclic polytopes and the second higher Stasheff-Tamari posets, preprint available at `www.math.uwo.ca/~hthomas2`.

(w^5) Let A be an antichain of the poset of intervals of the chain $\boldsymbol{n-1}$. The number of such antichains is C_n by (bbb), since for any poset there is a simple bijection between its order ideals I and antichains A, viz., A is the set of maximal elements of I. (See equation (2) of Section 3.1.) Construct from A a poset P on the points $1, 2, \ldots, n+1, 1', 2', \ldots, (n+1)'$ as follows. First, $1 < 2 < \cdots < n+1$ and $1' < 2' < \cdots < (n+1)'$. If $[i, j] \in A$, then define $i < (j+2)'$ and $j < (i+2)'$. This gives a bijection between (bbb) and (w^5). This result is due to J. Stembridge, private communication dated 22 November 2004.

(x^5) Let

$$(1, 1, \ldots, 1, -n) = \sum_{i=1}^{n-1} (a_i(e_i - e_{i+1}) + b_i(e_i - e_{n+1})) + a_n(e_n - e_{n+1})$$

as in (t^5). Set $m_{ii} = a_i$ and $m_{in} = b_i$. This uniquely determines the matrix M and sets up a bijection with (t^5). See A. Postnikov and R. Stanley, *ibid.*

(y^5) The solution to (lll) sets up a bijection between order ideals of $\mathrm{Int}(\boldsymbol{n-1})$ and *all* regions into which the cone $x_1 \geq x_2 \geq \cdots \geq x_n$ is divided by the hyperplanes $x_i - x_j = 1$, for $1 \leq i < j \leq n$. In this bijection, the bounded regions correspond to the order ideals containing all singleton intervals $[i,i]$. It is easy to see that such order ideals are in bijection with *all* order ideals of $\mathrm{Int}(\boldsymbol{n-2})$. Now use (bbb). This result was suggested by S. Fomin.

(z^5) See L. J. Billera and G. Hetyei, *J. Combinatorial Theory (A)* **89** (2000), 77–104; `math.CO/9706220` (Corollary 4).

6.25 (j) The degree of $G(k,n+k)$ is the number $f^{(n^k)}$ of standard Young tableaux of the rectangular shape (n^k) (see e.g. R. Stanley, *Lecture Notes in Math.* **579**, Springer, Berlin, 1977, pp. 217-251 (Thm. 4.1) or L. Manivel, *Symmetric Functions, Schubert Polynomials and Degeneracy Loci*, American Mathematical Society and Société Mathématique de France, 1998), and the proof follows from Exercise 6.19(ww).

(k) This result was conjectured by J.-C. Aval, F. Bergeron, N. Bergeron, and A. Garsia. The "stable case" (i.e., $n \to \infty$) was proved by J.-C. Aval and N. Bergeron, Catalan paths, quasi-symmetric functions and super-harmonic spaces, preprint; `math.CO/0109147`. The full conjecture was proved by J.-C. Aval, F. Bergeron, and N. Bergeron, Ideals of quasi-symmetric functions and super-covariant polynomials for S_n, preprint; `math.CO/0202071`.

(l) This is a result of Alexander Woo, `math.CO/0407160`. Woo conjectures that Ω_w is the "most singular" Schubert variety, i.e., the point X_{w_0} (which always has the largest multiplicity for any Schubert variety Ω_v) of Ω_w has the largest multiplicity of any point on any Schubert variety of $\mathrm{GL}(n,\mathbb{C})/B$.

(m) A matrix $A \in \mathrm{SL}(n,\mathbb{C})$ satisfying $A^{n+1}=1$ is diagonalizable with eigenvalues ζ satisfying $\zeta^{n+1}=1$. The conjugacy class of A is then determined by its multiset of eigenvalues. It follows that the number of conjugacy classes is the number of multisets of $\mathbb{Z}/(n+1)\mathbb{Z}$ whose elements sum to 0. Now use (jjj). For the significance of this result and its generalization to other Lie groups, see D. Z. Djokovíc, *Proc. Amer. Math. Soc.* **80** (1980), 181–184. Further discussion appears in Lecture 5 of S. Fomin and N. Reading, Root systems and generalized associahedra, available at

`www.math.lsa.umich.edu/~fomin/Papers`.

6.38 (n) See O. Guibert, E. Pergola, and R. Pinzani, *Ann. Combinatorics* **5** (2001), 153–174.

6.C1 (a) These results appear in M. Bousquet-Mélou and G. Schaeffer, Walks on the slit plane, preprint; `math.CO/0012230` (Theorem 7), and M. Bousquet-Mélou, Walks on the slit plane: other approaches, *Advances in Applied Math.*, to appear; `math.CO/0104111` (Theorem 19). The proofs are obtained from the formula

$$\sum_{n\geq 0}\sum_{i,j} a_{i,j}(n)x^i y^j t^n = \frac{\left(1-2t(1+\bar{x})+\sqrt{1-4t}\right)^{1/2}\left(1+2t(1-\bar{x})+\sqrt{1+4t}\right)^{1/2}}{2(1-t(x+\bar{x}+y+\bar{y}))}, \qquad (10)$$

where $\bar{x}=1/x$ and $\bar{y}=1/y$. Equation (1) is also given a bijective proof in the second paper (Proposition 2).

(b) Equation (2) was conjectured by Bousquet-Mélou and Schaeffer, *ibid.*, p. 11. This conjecture, as well as equation (3), was proved G. Xin, Proof of a conjecture on the slit plane problem; `math.CO/0304178`. The proof is obtained from (10) as in (a).

(c) The situation is analogous to (a). The results appear in the two papers cited in (a) and are based on the formula

$$\sum_{n\geq 0}\sum_{i,j} b_{i,j}(n)x^i y^j t^n = \frac{\left(1-8t^2(1+\bar{x}^2)+\sqrt{1-16t^2}\right)^{1/2}}{\sqrt{2}(1-t(x+\bar{x})(y+\bar{y}))}.$$

The case $i=1$ of (4) is given a bijective proof in Bousquet-Mélou and Schaeffer, *ibid.*, Proposition 7.

(d) Let X_n be the set of all closed paths of length $2n$ from $(0,0)$ to $(0,0)$ that intersect the half-line L defined by $y=x,\ x\geq 1$. Given $P\in X_n$, let k be the smallest integer such that P intersects L after k steps, and let Q be the path consisting of the first k steps of P. Let P' be the path obtained from P by reflecting Q about the line $y=x$. Then $P'\in X_n$, $w(P')=w(P)\pm 1$, and the map $P\mapsto P'$ is an involution. Any path P of length $2n$ from $(0,0)$ to $(0,0)$ not contained in X_n satisfies $w(P)=0$. It follows that $f(n)$

is the number of closed paths of length $2n$ from $(0,0)$ to $(0,0)$ not intersecting L.

Now consider the linear change of coordinates $(x,y) \mapsto \frac{1}{2}(-x-y, x-y)$. This transforms a closed path of length $2n$ from $(0,0)$ to $(0,0)$ not intersecting L to a closed path from $(0,0)$ to $(0,0)$ with $2n$ diagonal steps $(\pm 1, \pm 1)$, not intersecting the negative real axis. Now use equation (5).

This result was stated by R. Stanley, Problem 10905, Problems and Solutions, *Amer. Math. Monthly* **108** (2001), 871. The published solution by R. Chapman, **110** (2003), 640–642, includes a self-contained proof of equation (5).

6.C2 (a) The enumeration of rooted planar maps subject to various conditions is a vast subject initiated by W. T. Tutte. The result of this problem appears in *Canad. J. Math.* **15** (1963), 249–271. A good introduction to the subject, with many additional references, can be found in [**3.16**, §2.9]. There has been a revival of interest in the enumeration of maps, motivated in part by connections with physics. At present there is no comprehensive survey of this work, but two references that should help combinatorialists get into the subject are D. M. Jackson, *Trans. Amer. Math. Soc.* **344** (1994), 755–772, and D. M. Jackson, in *DIMACS Series in Discrete Mathematics and Computer Science* **24** (1996), 217–234.

(b) Let G be a rooted planar map with n edges and p vertices. We can define a dual n-edge and p'-vertex map G', such that by Euler's formula $p' = n + 2 - p$. From this it follows easily that the answer is given by $M(n)(n+2)/2$, where $M(n)$ is the answer to (a). For a more general result of this nature, see V. A. Liskovets, *J. Combinatorial Theory (B)* **75** (1999), 116–133 (Prop. 2.6). It is also mentioned on page 150 of L. M. Koganov, V. A. Liskovets and T. R. S. Walsh, *Ars Combinatoria* **54** (2000), 149–160.

6.C3 (a) See T. Nakamigawa, *Theoretical Computer Science* **235** (2000), 271-282 (Corollary 6) and A. Dress, J. Koolen, and V. Moulton, *European J. Combin.* **23** (2002), 549–557. Another proof was given by J. Jonsson in the reference cited below.

(b) See J. Jonsson, Generalized triangulations and diagonal-free subsets of half-moon shapes, preprint,

`http://www.math.kth.se/~jakobj/combin.html#deltank`.

The proof interprets the result in terms of lattice path enumeration and applies the Gessel-Viennot theory of nonintersecting lattice paths (§2.7).

6.C4 This result was first proved by I. Dumitriu using random matrix arguments. An elegant bijective proof was then given by E. Rassart, as follows. We want a bijection φ from (1) quadruples (D, D', e, e'), where D and D' are Dyck paths with $2n$ steps and e and e' are up edges of D and D' ending at the same height i, and (2) Dyck paths E with $4n$ steps from $(0, 0)$ to $(4n, 0)$ which do not touch the point $(2n, 0)$. We first transform (D, e) into a partial Dyck path L ending at height i. Let f be the down edge paired with e (i.e., the first down edge after e beginning at height i), and flip the direction of each edge at or after f. Let e_2 be the first up edge to the left of e ending at height $i - 1$, and let f_2 be the down edge paired with e_2 in the original path D. Flip all edges of the current path at or after f_2. Continue this procedure, letting e_j be the first edge to the left of e_{j-1} such that e_j has height one less than e_{j-1}, etc., until no edges remain. We obtain the desired partial Dyck path L ending at height i. Do the same for (D', e'), obtaining another partial Dyck path L' ending at height i. Reverse the direction of L' and glue it to the end of L. This gives the Dyck path E. Figure 8 shows an example of the correspondence $(D, e) \mapsto L$. We leave the construction of φ^{-1} to the reader. See I. Dumitriu and E. Rassart, *Electronic J. Combinatorics* **10**(1) (2003), R-43; `math.CO/0307252`.

6.C5 (a) The most straightforward method is to observe that all three power series $F(t_0, t_1, \ldots; x)$ satisfy

$$F(t_0, t_1, \ldots; x) = \frac{1}{1 - t_0 x F(t_1, t_2, \ldots; x)}$$

with the initial condition

$$F(t_0, t_1, \ldots; 0) = 1.$$

A general combinatorial theory of continued fractions is due to P. Flajolet, *Discrete Math.* **32** (1980), 125–161. The equivalence of

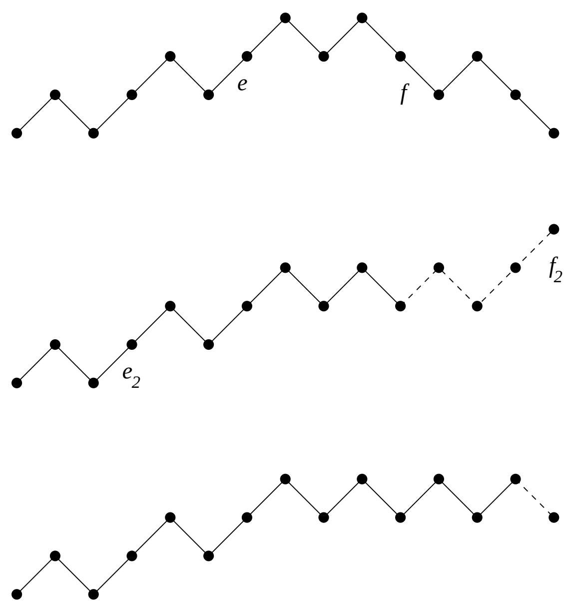


Figure 8: The bijection $(D, e) \mapsto L$ in the solution to Exercise 6.C4

(i) and (iii) is equivalent to a special case of Corollary 2 of this paper.

(b) Put $t_0 = t_1 = \cdots = t_n = 1$ and $t_i = 0$ for $i > n$. By the case $j = 2$ of Exercise 1.3, the generating function of part (ii) above becomes the left-hand side of equation (7). On the other hand, by Exercise 6.19(e) the coefficient of x^i for $i \leq n+1$ in the generating function of part (iii) becomes the coefficient of x^i in the right-hand side of (7) (since a tree with $i \leq n+1$ vertices has height at most n). Now let $n \to \infty$.

It is not difficult to give a direct proof of (7). The generating function $F_n(x) = \sum_{i \geq 0} (-1)^i \binom{n-i}{i} x^i$ satisfies

$$F_{n+2}(x) = F_{n+1}(x) - xF_n(x). \tag{11}$$

From Theorem 4.1.1 (or directly from Exercise 6.C12(b)) it follows that

$$F_n(x) = \frac{C(x)^{-n-1} - (xC(x))^{n+1}}{\sqrt{1-4x}}.$$

Hence the fraction in the left-hand side of (7) is given by

$$\frac{\sum_{i\geq 0}(-1)^i\binom{n-i}{i}x^i}{\sum_{i\geq 0}(-1)^i\binom{n+1-i}{i}x^i} = C(x)\frac{1-x^{n+1}C(x)^{2(n+1)}}{1-x^{n+2}C(x)^{2(n+2)}},$$

and (7) follows.

Equation (7) (with the numerator and denominator on the left-hand side defined by (11)) was given by V. E. Hoggatt, Jr., Problem H-297, *Fibonacci Quart.* **17** (1979), 94; solution by P. S. Bruckman, **18** (1980), 378.

(c) This conjecture is due to A. Postnikov. It has been checked for $n \leq 500$. It is also true whenever $\nu_2(C_n) \leq 2$, a simple consequence of (a) when $t_0 = t_1 = \cdots = 1$ and the fact that $(2i-1)^2 \equiv 1 \pmod 8$, and has been proved whenever $\nu_2(C_n) = 3$ by R. Stanley (unpublished).

6.C6 See the reference given in the solution to Exercise 6.19(t^5).

6.C7 This result was conjectured by F. Brenti in 1995 and first proved by D. Zeilberger,

`http://www.math.rutgers.edu/~zeilberg/mamarim/`

`mamarimhtml/catalan.html`.

Zeilberger's proof consists essentially of the statement

$$\begin{aligned} B_{2n}(q) &= (-1)^n\frac{1}{2n+1}\binom{2n+1}{n}q^n \\ &\qquad + \sum_{i=0}^{n-1}(-1)^i\frac{2n-2i+1}{2n+1}\binom{2n+1}{i}(q^i+q^{2n-i}) \\ B_{2n+1}(q) &= \sum_{i=0}^{n}(-1)^{i+1}\frac{n-i+1}{n+1}\binom{2n+2}{i}q^i. \end{aligned}$$

Later (as mentioned by Zeilberger, *ibid.*) Brenti found a combinatorial interpretation of the polynomials $B_m(q)$ which implies his conjecture.

6.C8 (a,c) Let M' denote the part of M below the main diagonal. It is easy to see that M' uniquely determines M. It was shown by C. S. Chan, D. P. Robbins, and D. S. Yuen, *Experiment. Math.* **9** (2000), 91–99, that the polytope CR_{n+1} (called the *Chan-Robbins polytope* or *Chan-Robbins-Yuen polytope*) can be subdivided into simplices, each of relative volume $1/\binom{n+1}{2}!$, which are naturally indexed by the matrices M'. It follows that $\binom{n+1}{2}!\,\nu(\mathrm{CR}_{n+1}) = g(n)$. Chan and Robbins had earlier conjectured that $\binom{n+1}{2}!\,\nu(\mathrm{CR}_{n+1}) = C_1C_2\cdots C_n$. (Actually, Chan and Robbins use a different normalization of relative volume so that $\nu(\mathrm{CR}_{n+1}) = C_1C_2\cdots C_n$.) This conjecture was proved by D. Zeilberger, *Electron. Trans. Numer. Anal.* **9** (1999), 147–148, and later W. Baldoni-Silva and M. Vergne, Residue formulae for volumes and Ehrhart polynomials of convex polytopes, preprint; `math.CO/0103097` (Thm. 33).

(b) Given the matrix $M = (m_{ij})$, there exist unique nonnegative integers $a_1, \ldots, a_n$ satisfying

$$\sum_{1\le i<j\le n} m_{ji}(e_i - e_j) + \sum_{i=1}^{n} a_i(e_i - e_{n+1}) = (1, 2, \ldots, n, -\binom{n+1}{2}).$$

This sets up a bijection between (a) and (b). This result is due to A. Postnikov and R. Stanley (unpublished), and is also discussed by W. Baldoni-Silva and M. Vergne, *ibid.* (§8).

6.C9 Let the vertices be $1, 2, \ldots, 4m+2$ in clockwise order. Suppose that there is a chord between vertex 1 and vertex $2i$.

Case 1: i is odd. The polygon is divided into two polygons, one (say P_1) with vertices $2, \ldots, 2i-1$ and the other (say P_2) with vertices $2i+1, \ldots, 4m+2$. Choose $i-1$ noncrossing chords on the vertices $2, 3, \ldots, 2i-1$. Thus we have a net N_1 on P_1. Choose a coloring of the faces of N_1. By "symmetry" half the nets on P_2 will have an even number of blue faces and half an odd number (with respect to the coloring of the faces of N_1), so the number of nets with a chord $(1, 2i)$ and an even number of blue faces minus the number of nets with a chord $(1, 2i)$ and an odd number of blue faces is 0. (It's easy to make this argument completely precise.)

Case 2: i is even, say $i = 2j$. The number of nets N_1 on P_1 with an even number of blue faces is $f_e(j-1)$. The number with an odd number is $f_o(j-1)$. Similarly the number of nets N_2 on P_2 with an even number of blue faces is $f_e(m-j)$ and with an odd number of blue faces is $f_o(m-j)$. Hence the number of nets on the original $4m+2$ points with a chord $(1,4j)$ and an even number of blue faces is $f_e(j-1)f_e(m-j)+f_o(j-1)f_o(m-j)$. Thus

$$f_e(m)=\sum_{j=1}^{m}(f_e(j-1)f_e(m-j)+f_o(j-1)f_o(m-j)).$$

Similarly

$$f_o(m)=\sum_{j=1}^{m}(f_e(j-1)f_o(m-j)+f_o(j-1)f_e(m-j)).$$

Hence

$$f_e(m)-f_o(m)=\sum_{j=1}^{m}(f_e(j-1)-f_o(j-1))(f_e(m-j)-f_o(m-j)),$$

which is just the recurrence satisfied by Catalan numbers C_m (initial condition $C_0=1$) and thus also by $D_m=(-1)^{m+1}C_m$ (initial condition $D_0=-1$). Since $f_e(0)-f_o(0)=0-1=-1$, the proof follows.

This result was first proved using techniques from algebraic geometry by A. Eremenko and A. Gabrielov, The Wronski map and real Schubert calculus, preprint available at

`www.math.purdue.edu/~agabriel/preprint.html`

and `www.math.purdue.edu/~eremenko/newprep.html`. A bijective proof was given by S.-P. Eu, Coloring the net, preprint, based on the preprint S.-P. Eu, S.-C. Liu, and Y.-N. Yeh, Odd or even on plane trees.

6.C10 By polynomial interpolation it suffices to prove the two identities when q is a positive integer.

(a) Let $G(x)=xC(x)$. Then $G(x)=(x-x^2)^{\langle -1\rangle}$, and the proof follows easily from the Lagrange inversion formula (Theorem 5.4.2). Alternatively, $C(x)^q$ (when $q\in\mathbb{P}$) is the generating function for plane binary forests with q components. Now use Theorem 5.3.10 in the case $n=2k+q$, $r_0=k+q$, $r_2=k$ (and all other $r_i=0$).

(b) This can be obtained from (a) by differentiating $(xC(x))^q$ with respect to x, or alternatively from the identity

$$xC(x)^q = \frac{C(x)^{q-1}}{2} - \frac{(1-4x)C(x)^{q-1}}{2\sqrt{1-4x}}.$$

6.C11 (a) It is straightforward to verify that $E(x)^2 = C(4x^2)$, which is equivalent to (9). This result is due to L. Shapiro, private communication dated 24 May 2002.

6.C12 (a) It is easy to see that if $F(t)$ exists, then it is unique. Now it follows from $xC(t)^2 - C(t) + 1 = 0$ that

$$\frac{1-x+xC(t)}{1-x+x^2t} = \frac{1}{1-xtC(t)}.$$

Hence $F(t) = C(t)$.

(b) We have

$$\frac{1}{1-xtC(t)} = \sum_{k\geq 0} x^k t^k C(t)^k.$$

Hence by (a) and Exercise 6.C10(a) there follows

$$\begin{aligned} [x^k]f_n(x) &= [t^n]t^kC(t)^k \\ &= [t^{n-k}]C(t)^k \\ &= \frac{k}{n}\binom{2n-k-1}{n-k} = \frac{k}{n}\binom{2n-k-1}{n-1}. \end{aligned}$$

6.C13 (a) *Answer:* $f(x) = \sqrt{4-x^2}/2\pi$ for $-2 \leq x \leq 2$, and $f(x) = 0$ for $|x| \geq 2$. This result is the basis of Wigner's famous "semicircle law" for the distribution of eigenvalues of certain classes of random real symmetric matrices (*Ann. Math.* **62** (1955), 548–569, and **67** (1958), 325–327). Wigner did not rigorously prove the uniqueness of $f(x)$, but this uniqueness is actually a consequence of earlier work of F. Hausdorff, *Math. Z.* **16** (1923), 220–248.

(b) *Answer:* $f(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise, an easy consequence of (a).

6.C14 Let w be the vertex of F adjacent to v. Let G denote F with v removed, and let $g(n)$ be the number of closed walks in G of length $2n$ beginning at w. Write $A(x) = \sum_{n\geq 0} f(n)x^n$ and $B(x) = \sum_{n\geq 0} g(n)x^n$. The definitions of F and G yield

$$\begin{aligned} A(x) &= 1 + xA(x)B(x) \\ B(x) &= 1 + xB(x)^2 + xA(x)B(x). \end{aligned}$$

Eliminating $B(x)$ gives $A(x) = 1+xA(x)^3$, the equation satisfied by the generating function for ternary trees by number of vertices. This result first appeared in D. Bisch and V. Jones, *Invent. math.* **128** (1997), 89–157, and D. Bisch and V. Jones, in *Geometry and Physics (Aarhus, 1995)*, Lecture Notes in Pure and Appl. Math., vol. **184**, Dekker, New York, 1997.

CHRONOLOGY OF NEW PROBLEMS (beginning 12/17/01)

A number in brackets is the number of items (combinatorial interpretations of C_n) in Exercise 6.19 up to that point.

6.19(xxx)	December 17, 2001
6.19(s^4)	December 17, 2001
6.38(n)	December 17, 2001
6.C10	December 17, 2001
6.C12	December 17, 2001
6.19(k^4)	January 29, 2002
6.C5	January 29, 2002
6.19(i^4)	March 20, 2002 [90]
6.19(ppp)	April 29, 2002
6.19(vvv)	April 30, 2002
6.19(www)	April 30, 2002
6.19(r^5)	May 31, 2002
6.19(qqq)	June 1, 2002
6.C11	June 1, 2002
6.19(l^5)	June 2, 2002
6.19(y^4)	June 4, 2002
6.C2	June 8, 2002
6.19(p^5)	July 1, 2002
6.19(w^4)	August 9, 2002

6.C4 August 9, 2002

6.19(uuu) October 23, 2002 [100]

6.19(v^5) October 23, 2002

6.19(n^4) October 27, 2002

6.25(l) December 20, 2002

6.19(v^4) March 12, 2003

6.19(c^5) March 12, 2003

6.19(j^4) April 6, 2003

6.19(f^5) April 6, 2003

6.C1(b) (updated) April 17, 2003

6.19(g^5) June 4, 2003

6.C3 October 30, 2003

6.C14 November 11, 2003

6.19(h^4) April 20, 2004

6.19(a^5) May 12, 2004 (solution modified 5/16/04)

6.19(m^5) June 1, 2004 [110]

6.19(n^5) June 2, 2004

6.C1(d) (updated) June 3, 2004

6.C13(b) June 3, 2004

6.19(o^5) June 28, 2004

6.19(q^4) August 22, 2004

6.19(q^5) August 22, 2004

6.19(g^4) August 22, 2004

6.19(s^5) September 2, 2004

6.19(y^5) September 2, 2004

6.19(zzz) September 20, 2004

6.19(c^4) November 13, 2004

6.C7 November 24, 2004

6.19(w^5) November 25, 2004 [120]

6.25(m) December 17, 2004

6.19(t^4) December 18, 2004

6.19(sss) December 18, 2004

6.19(d^4) December 20, 2004

6.19(f^4) December 22, 2004

6.19(d^5) January 16, 2005

6.19(e^5) January 16, 2005

6.19(z^4) February 11, 2005

6.19(l^4) February 28, 2005 [2^7]

6.19(m^4) April 8, 2005

6.19(x^4) May 27, 2005 [130]