Three Connections to Continued Fractions

Ezra Brown

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Introduction

It is often the case that seemingly unrelated parts of mathematics turn out to have unexpected connections. In this paper, we explore three puzzles and see how they are related to continued fractions, an area of mathematics with a distinguished history within the world of number theory.

Puzzle 1: A Mistake. A typesetting error produced $g-1$ instead of g^{-1}, and a student was quick to point out that, for the problem at hand, it didn't matter. What is the value of g?

Puzzle 2: A Whole Lot of Cows. About 22 centuries ago, Archimedes wrote a letter in which he challenged his fellow mathematicians to determine the size of a certain herd of cattle. To do this involves solving the equation $x^2 - 410286423278424y^2 = 1$ in nonzero integers x and y. What is the connection, and why does this equation even have such a solution?

Puzzle 3: A Mystery. I once read a book on number theory that contained a tantalizing problem whose solution eluded me for years. In this book, it stated that the expression

$$2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \cdots}}}}$$

was equal to a certain real number θ. What is θ and why is this expression equal to θ?

The quantity g in Puzzle 1 produces a continued fraction, the solution to Puzzle 2 uses a continued fraction, and the expression in Puzzle 3 *is* a continued fraction.

Don't know what a continued fraction is? Don't worry—you'll find out.

A TEXnical Error and Simple Continued Fractions

In typing out a recent problem set in TEX, I inadvertently typed `$g-1$` (which typesets as $g-1$) instead of `$g∧{-1}$` (which typesets as g^{-1}. One student was quick to point out that, for the problem at hand, it didn't matter. Knowing this and knowing that the answer is positive, what is g?

Now $g-1 = g^{-1} = \frac{1}{g}$, so $g = 1 + \frac{1}{g}$. If we multiply both sides by g and transpose, we are led to the equation $g^2 - g - 1 = 0$. This has two solutions, namely $g = \frac{1 \pm \sqrt{5}}{2}$; but $g > 0$, so the answer is $g = \frac{1+\sqrt{5}}{2}$. End of story — or is it? Let's look a bit deeper.

Substituting this value for g in the expression on the right, we see that $g = 1 + \cfrac{1}{1+\cfrac{1}{g}}$; do it again and we get $g = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{g}}}$; if we continue this process, we find that

$$g = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} \tag{1}$$

— that is, if the expression on the right makes sense.

Turns out, it *does* make sense: it is what we call a *simple continued fraction*, or *scf* for short. The three dots indicate that the pattern repeats forever, so that we have an *infinite* scf. In order to understand what this is, we need to talk about the finite ones first.

A *finite simple continued fraction* is an expression of the form

$$x = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \ldots + \cfrac{1}{b_{k-1} + \cfrac{1}{b_k}}}}},$$

where the b_i's are integers and $b_i \geq 1$ for $i \geq 1$. This notation is not easy to use, so we customarily write $x = \langle b_0, b_1, b_2, \ldots, b_k \rangle$ to represent the above scf. We'll call the b_i's *partial quotients* of x. Now since b_i is a positive integer for $i \geq 1$, we see that $0 \leq x - b_0 < 1$ and so $b_0 = [x]$, the greatest integer in x. Let us write $x_0 = x$ and $x_{n+1} = \frac{1}{x_n - b_n}$ for $n \geq 1$; the numbers x_i are called the *complete quotients* of x. It turns out that $b_n = [x_n]$ for all n, and it is but a short step to the following theorem.

Theorem 1. *(a) Every positive rational number has exactly two representations as a finite scf, differing only in the last place—if $x = \langle b_0, b_1, b_2, \ldots, b_k \rangle$ with $b_k > 1$, then the other finite scf representing x is $\langle b_0, b_1, b_2, \ldots, b_k - 1, 1 \rangle$.*

(b) Every finite scf represents a rational number.

Part (a) is a consequence of Euclid's Algorithm for finding greatest common divisors; here is an example:

$$\begin{aligned} &\frac{355}{113} = 3 + \frac{16}{113}; \quad \frac{113}{16} = 7 + \frac{1}{16}, \text{ and so} \\ &\frac{355}{113} = \langle 3, 7, 16 \rangle. \end{aligned} \tag{2}$$

Part (b) follows from the fact that simplifying a finite scf involves only finitely many arithmetic operations involving integers and rationals, so that the end result is a rational number. Here is an example:

$$\langle 2, 1, 2, 1, 1, 4, 1, 1, 6 \rangle = \frac{1264}{465} \tag{3}$$

Checking the arithmetic in the above examples is a good idea; writing out their decimal equivalents might be revealing.

If $x = \langle b_0, b_1, b_2, \dots, b_k \rangle$, then for $n \le k$, the theorem tells us that the scf $C_n = \langle b_0, b_1, b_2, \dots, b_n \rangle$ is a rational number called the n^{th} *convergent* to x. This brings up a problem with continued fractions: how do we calculate the convergents? Dealing with an 8-deep fraction is tedious at best, so is there a shortcut? In fact, there is.

Theorem 2. *Let $b_0, b_1, \dots$ be real numbers with $b_i \ge 1$ for $i \ge 1$. Define the numbers P_n and Q_n as follows:*

$$\begin{aligned} P_{-1} &= 1, P_{-2} = 0; Q_{-1} = 0, Q_{-2} = 1; \\ P_n &= b_n P_{n-1} + P_{n-2}, n \ge 0; \\ Q_n &= b_n Q_{n-1} + Q_{n-2}, n \ge 0. \end{aligned}$$

Then the successive convergents to the scf $\langle b_0, b_1, b_2, \dots, b_k \rangle$ are $C_n = \dfrac{P_n}{Q_n}$, for $n \ge 1$.

For example, for that number g defined in Equation (1), the numbers P_n and Q_n, for $n \ge 0$, are $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$ and $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$, respectively — the Fibonacci numbers. Their ratios approximate the number $g = (1 + \sqrt{5})/2$ commonly known as the golden mean ("g" as in "golden").

Here is a sketch of the proof of Theorem 2: For the base case, notice that $C_0 = b_0 = b_0/1$ and indeed $P_0 = b_0 = b_0 \cdot 1 + 0 = b_0 P_{-1} + P_{-2}$ and $Q_0 = 1 = b_0 \cdot 0 + 1 = b_0 Q_{=1} + Q_{-2}$. Then, with the induction hypothesis in hand — namely, that the above formulas are true for all $n \le k$ and for all scf's — we notice that

$$\langle b_0, b_1, b_2, \dots, b_{k+1} \rangle = \langle b_0, b_1, b_2, \dots, b_k + \frac{1}{b_{k+1}} \rangle$$

and apply the formulas to the right-hand side.

The convergents exhibit some curious behavior. For example, the successive convergents in (3) begin as follows:

$$\frac{2}{1}, \frac{3}{1}, \frac{8}{3}, \frac{11}{4}, \frac{19}{6}, \frac{87}{32}, \dots$$

Notice that $19 \cdot 32 - 87 \cdot 7 = -1, 11 \cdot 7 - 19 \cdot 4 = 1, 8 \cdot 4 - 11 \cdot 3 = -1$, and in general, it appears that the following is true:

Theorem 3. *If P_n and Q_n are as in Theorem 2, then $P_n Q_{n+1} - P_{n+1} Q_n = (-1)^{n+1}$ for $n \geq 0$.*

(Exercise: use the formulas to prove it.)

Infinite scf's first turn up in Europe in the sixteenth and seventeenth centuries in the work of Bombelli (1526-1573) and Cataldi (1548-1626), with Wallis (1616-1703) and Huyghens (1629-1695) first working out the theory ([2], Chaps. 2 and 3). They have broad applications in number theory, both to the approximation of irrational numbers by rationals (the Mystery) and to the solution of certain quadratic equations in integers (the Cows). We may define the *infinite simple continued fraction* $\langle b_0, b_1, b_2, \dots \rangle$ by

$$x = \langle b_0, b_1, b_2, \dots \rangle = \lim_{k \to \infty} \langle b_0, b_1, b_2, \dots, b_k \rangle,$$

provided this limit exists. If $b_k \geq 1$ for all $k \geq 1$, the limit does exist:

Theorem 4. *If $b_0, b_1, b_2, \dots$ is a sequence of numbers such that $b_i \geq 1$ for $i \geq 1$, and if the numbers P_n and Q_n are defined as above, then:*

(a) $\dfrac{P_0}{Q_0} < \dfrac{P_2}{Q_2} < \dfrac{P_4}{Q_4} < \dots$, *and* $\dfrac{P_1}{Q_1} > \dfrac{P_3}{Q_3} > \dfrac{P_5}{Q_5} > \dots$;

(b) $\dfrac{P_{2n}}{Q_{2n}} < \dfrac{P_{2n+1}}{Q_{2n+1}}$ *for all $n \geq 0$; and*

(c) $\lim_{k \to \infty} \langle b_0, b_1, b_2, \dots, b_k \rangle$ *exists.*

Proof. Let us sketch the proof. To prove (a), use the results from Theorems 2 and 3: for example, begin one of the base cases by observing that $0 < \dfrac{P_2}{Q_2} = \dfrac{b_2 P_1 + P_0}{b_2 Q_1 + Q_0}$ and $b_2 \geq 1$; then, clear of fractions and apply some algebra. The proof of (b) is simply a restatement of Theorem 3, paying special attention to the parity of the subscripts. Now, by (a), the convergents $\dfrac{P_{2n}}{Q_{2n}}$ form an increasing sequence; by (b) this sequence is bounded above by any odd convergent. By the completeness of the real numbers, the sequence $\left(\dfrac{P_{2n}}{Q_{2n}}\right)$ converges to their least upper bound x. It turns out that x is also the limit of the sequence of odd convergents, and to show this is not hard. $\square$

Now, let's try computing the scf for an irrational number — say, $\sqrt{19}$. We know it is infinite, but let's just see what happens. To begin, we notice that $[\sqrt{19}] = 4$, so that

$\sqrt{19} = 4 + \sqrt{19} - 4$, and the algorithm proceeds as follows.

$$\begin{aligned}
\sqrt{19} &= 4 + (\sqrt{19} - 4), \\
\frac{1}{\sqrt{19}-4} &= \frac{\sqrt{19}+4}{3} = 2 + \frac{\sqrt{19}-2}{3}, \\
\frac{3}{\sqrt{19}-2} &= \frac{\sqrt{19}+2}{5} = 1 + \frac{\sqrt{19}-3}{5}, \\
\frac{5}{\sqrt{19}-3} &= \frac{\sqrt{19}+3}{2} = 3 + \frac{\sqrt{19}-3}{2}, \\
\frac{2}{\sqrt{19}-3} &= \frac{\sqrt{19}+3}{5} = 1 + \frac{\sqrt{19}-2}{5}, \\
\frac{5}{\sqrt{19}-2} &= \frac{\sqrt{19}+2}{3} = 2 + \frac{\sqrt{19}-4}{3}, \\
\frac{3}{\sqrt{19}-4} &= \sqrt{19}+4 = 8 + (\sqrt{19}-4), \\
\frac{1}{\sqrt{19}-4} &= \frac{\sqrt{19}+4}{3} = 2 + \frac{\sqrt{19}-2}{3},
\end{aligned}$$

and hey, look, it repeats: $\sqrt{19} = \langle 4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8 \ldots \rangle$, which we abbreviate as $\sqrt{19} = \langle 4, \overline{2, 1, 3, 1, 2, 8} \rangle$. It is also true that $\sqrt{2} = \langle 1, \overline{2} \rangle$, and rational approximations to $\sqrt{2}$ were known almost 4 millenia ago. Calculate the scf for several values of $\sqrt{D}$, where D is any positive nonsquare integer, and you will be convinced that the following theorem is true:

Theorem 5. *There exists a least positive integer k, called the period length, such that the scf expansion of $\sqrt{D}$ is given by $\sqrt{D} = \langle N, \overline{b_1, \ldots, b_{k-1}, 2N} \rangle$.*

This result dates from the seventeenth century and grew out of correspondence between Pierre Fermat (1601-1665) and the English mathematician William Brouncker (1620-1684). In fact, these so-called *periodic continued fractions* are precisely those that represent quadratic irrationalities — and this is a theorem due to, of all people, that startling prodigy Évariste Galois (1811-1832):

Theorem 6. (Galois) *The simple continued fraction $x = \langle b_0, b_1, b_2, \ldots \rangle$ represents a quadratic surd — i.e. an irrational root of a quadratic equation with rational coefficients — if and only if x is periodic.*

Computer algebra systems often have built-in functions to find the scf expansion of quadratic surds. But it is fun to write your own; try it and see! And now, on to the cattle!

Archimedes and the Cattle

Archimedes was the greatest mathematician in antiquity, and one measure of his greatness was that he thought BIG. He moved the earth with a big stick, and it turns out that such a stick needed to be about 10^{35} millimeters long. He filled the universe with grains of sand and counted the grains (roughly 10^{51} of them), then imagined an even bigger universe full of sand and counted those grains (about 10^{63} grains). Finally, in a letter to his friend Eratosthenes he posed the problem of finding the size of a certain herd of cattle, which size pitifully dwarfs all of the previous numbers. (Maybe he wrote the letter — and maybe he didn't; more about that later.) Here are the details:

Archimedes asks us to find the numbers W, X, Y and Z of white, black, brown, and spotted bulls, and the numbers w, x, y and z of white, black, brown, and spotted cows, subject to the following nine conditions:

$$\begin{array}{ll} 1.\quad W = (1/2+1/3)X + Z, & 2.\quad X = (1/4+1/5)Y + Z, \\ 3.\quad Y = (1/6+1/7)W + Z, & 4.\quad w = (1/3+1/4)(X+x), \\ 5.\quad x = (1/4+1/5)(Y+y), & 6.\quad y = (1/5+1/6)(Z+z), \\ 7.\quad z = (1/6+1/7)(W+w), & 8.\quad W+X \text{ is a square, and} \\ 9.\quad Y+Z \text{ is a triangular number.} & \end{array} \tag{4}$$

These amount to seven linear equations and two nonlinear equations in eight unknowns. so that it is not obvious that a solution even exists. Solving the linear equations 1 through 7 is fairly straightforward, and with a computer algebra system such as *Mathematica* we can do it in the blink of an eye. It turns out that the first seven variables are rational multiples of z with common denominator 5439213; if we put $z = 5439213v$, we obtain integer values for all eight variables, namely

$$\begin{array}{llll} W = 10366482v, & X = 7460514v, & Y = 7358060v, & Z = 4149387v, \\ w = 7206360v, & x = 4893246v, & y = 3515820v, & z = 5439213v, \end{array} \tag{5}$$

where v is an integer-valued parameter. At a minimum, Archimedes now has about 50 million head of cattle.

To satisfy condition 8, we want $W + X = 17826996v$ to be a square. Since $17826996 = 4 \cdot 4456749$, with the latter factor squarefree, this will occur if we put $v = 4456749s^2$. This yields $W = 46200808287018s^2$ with similarly magnified values for the other seven variables. At this point, with about 225 trillion head of cattle in the herd, Archimedes has clearly overrun the planet—and he still must satisfy condition 9, namely that $Y + Z$ must be triangular.

Now, the triangular numbers are $1, 3, 6, 10, 15, \ldots$ and have the general form $n(n+1)/2$, so this means that $Y + Z = 51285802909803s^2 = n(n+1)/2$ for some integer n.

Multiplying this equation by 8 and adding 1 yields the equation $410286423278424s^2 + 1 = (2n+1)^2$. So, if we set $t = 2n+1$, we conclude that satisfying conditions 1-9 amounts to solving the equation

$$t^2 - 410286423278424s^2 = 1, \tag{6}$$

for nonzero integers s and t.

It is apparent madness to prove that s exists, let alone ever find it. But it does, and we can, by means of continued fractions. It turns out that the convergents to the scf expansion of an irrational number are excellent approximations to that number. In particular, we have the following corollary to Theorem 3:

Theorem 7. *If $\alpha = \langle b_0, b_1, b_2, \dots \rangle$ is the scf for α, and if P_n and Q_n are as in Theorem 2, then $|P_n/Q_n - \alpha| < 1/Q_n^2$.*

For example, $|355/113 - \pi| = 0.00000026676\ldots < 0.0000783\ldots = 1/113^2$. Using this result, Lagrange (1736-1813) was able to prove that the scf for $\sqrt{D}$ encodes all of the solutions to the equation $x^2 - Dy^2 = 1$:

Theorem 8. (Lagrange) *Let D be a positive nonsquare integer, let $N = [\sqrt{D}]$, and let $\sqrt{D} = \langle N, \overline{b_1, \dots, b_{k-1}, 2N} \rangle$, where k is the period length. If P_n and Q_n are as in Theorem 2, and m is a positive integer, then*

$$P_{mk}^2 - DQ_{mk}^2 = \begin{cases} -1, & \text{if } mk \text{ is odd;} \\ 1, & \text{if } mk \text{ is even.} \end{cases}$$

Furthermore, if the integers x and y satisfy $x^2 - Dy^2 = \pm 1$, then there exists an integer m such that $x = P_{mk}$ and $y = Q_{mk}$, where k is as above.

For a proof, see [12], Section 13.4 or [7], Section 14.5. As an example, we found that $\sqrt{19} = \langle 4, \overline{2,1,3,1,2,8} \rangle$, with period length $k = 6$; a short calculation reveals that $P_6 = 170, Q_6 = 39$, and $170^2 - 19 \cdot 39^2 = 28900 - 19 \cdot 1521 = 28900 - 28899 = 1$.

We now see that Lagrange's Theorem will enable us to find a solution to Equation (6), so we need the scf expansion of $\sqrt{410286423278424}$. Now the period length of this scf is 203254, but *Mathematica* happily computes both this scf and — using the formulas from Theorem 2 — the values of s and t. The final value for $W = 46200808287018s^2$ is, as stated in the literature [4], a 206545-digit number. Written out in full, at 80 characters a line and 72 lines a page, it runs to 37 pages; it took about 35 seconds to calculate the scf, and about 5 minutes to find the value of W. The total number of cattle turns out to be

$$7760271406486818269530232 \cdots 8973723406626719455081800,$$

with the $\cdots$ representing 206495 missing digits. Lots of milking to be done here!

And now, let's solve that mystery.

Generalized Continued Fractions, and the Mystery Solved

More generally, we can look at so-called *generalized continued fractions*, i.e. expressions of the form

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ldots + \cfrac{a_k}{b_k + \ldots}}}},$$

where the a_i's and b_i's are numbers. Again, this notation is unwieldy, so we write this as

$$x = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+\cdots}\frac{a_k}{b_k+\cdots}. \tag{7}$$

If $a_k > 0$ and $b_k \geq 1$ for all k, then the above expression converges to a real number. Using induction, we can prove that the successive convergents to (7) are $\frac{P_n}{Q_n}$, for $n \geq 1$, where

$$\begin{aligned} &P_0 = b_0, P_{-1} = 1; Q_0 = 1, Q_{-1} = 0;\\ &P_n = b_n P_{n-1} + a_n P_{n-2}, n \geq 1;\\ &Q_n = b_n Q_{n-1} + a_n Q_{n-2}, n \geq 1. \end{aligned} \tag{8}$$

These generalized formulas clearly resemble the formulas from Theorem 2 for the convergents of simple continued fractions. Again, if the cf in equation (7) converges, then it is periodic if and only if it represents a quadratic irrational.

Notice that the a_n and the b_n are not necessarily integers, or even positive, or even rational (or, for that matter, even real). The great magical genius Ramanujan (1887-1920) had a particular liking for continued fractions with $b_n = 1$ and the a_n arbitrary real numbers, or even variables.

We can now solve the last puzzle, the one about the continued fraction

$$2 + \frac{2}{2+}\frac{3}{3+}\frac{4}{4+}\frac{5}{5+\cdots}. \tag{9}$$

It is said to represent a real number θ. But what is θ?

Since $\theta = 2 + \frac{2}{2+p}$, where p is a positive number, we see that $2 < \theta < 3$. From the formulas in (8), we see that $b_0 = 2$, and for $n \geq 1, b_n = a_n = n + 1$. Thus,

$$\begin{aligned}
P_0 &= 2,\\
P_1 &= 2 \cdot 2 + 2 \cdot 1 = 6,\\
P_2 &= 3 \cdot 6 + 3 \cdot 2 = 24,\\
P_3 &= 4 \cdot 24 + 4 \cdot 6 = 120,\\
P_4 &= 5 \cdot 120 + 5 \cdot 24 = 720,
\end{aligned}$$

and we boldly guess that $P_n = (n+2)!$ for $n \geq 0$. Furthermore,

$$\begin{aligned}
Q_1 &= 2 \cdot 1 + 2 \cdot 0 = 2,\\
Q_2 &= 3 \cdot 2 + 3 \cdot 1 = 9,\\
Q_3 &= 4 \cdot 9 + 4 \cdot 2 = 44,\\
Q_4 &= 5 \cdot 44 + 5 \cdot 9 = 265;
\end{aligned}$$

the fourth convergent to θ is $P_4/Q_4 = 720/265 = 2.71698\ldots$, the fifth is $P_5/Q_5 = 5040/1854 = 2.71844\ldots$, and we guess that $\theta = e$.

In order to prove it, we need to make sense of the sequence $2, 9, 44, 265, \ldots$. Notice that

$$\begin{aligned}
2 &= 3 - 1 = \frac{3!}{2!} - \frac{3!}{3!},\\
9 &= 4 \cdot 2 + 1 = 4\left(\frac{3!}{2!} - \frac{3!}{3!}\right) + 1 = \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!},\\
44 &= 5 \cdot 9 - 1 = 5\left(\frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!}\right) - 1 = \frac{5!}{2!} - \frac{5!}{3!} + \frac{5!}{4!} - \frac{5!}{5!},
\end{aligned}$$

and in general, we guess that

$$Q_n = (n+2)!\left(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n+2}\frac{1}{(n+2)!}\right) = (n+2)!\sum_{k=2}^{n+2}\frac{(-1)^k}{k!} \tag{10}$$

We'll leave it as an exercise to prove that the equation in (10) holds for all $n \geq 1$.

Having done so, we conclude that the nth convergent C_n to the continued fraction in (9) is given by

$$C_n = \frac{P_n}{Q_n} = \frac{(n+2)!}{(n+2)!\sum_{k=2}^{n+2}\frac{(-1)^k}{k!}} = \frac{1}{\sum_{k=2}^{n+2}\frac{(-1)^k}{k!}}.$$

Since $0! = 1! = 1$, we know that

$$\sum_{k=2}^{n+2}\frac{(-1)^k}{k!} = \frac{1}{1} - \frac{1}{1} + \sum_{k=0}^{n+2}\frac{(-1)^k}{k!} = \sum_{k=0}^{n+2}\frac{(-1)^k}{k!};$$

we also know that $\lim_{n\to\infty} \sum_{k=0}^{n+2} \frac{(-1)^k}{k!}$ exists by the Alternating Series Test. In fact, $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}$ is what we get by substituting $x = -1$ in the usual Maclaurin series for e^x.

Hence, $\theta = \lim_{n\to\infty} C_n = 1/e^{-1} = e$, and our guess that

$$2 + \frac{2}{2+}\frac{3}{3+}\frac{4}{4+}\frac{5}{5+\cdots} = e$$

was correct!

Questions

Where can I find out more about continued fractions? Most elementary number theory books have chapters devoted to continued fractions. See, for example, [6] (a classic), [7] (which also treats generalized continued fractions), [8] and [12]. Olds' book [10] is a very nice elementary introduction. Perron's book [11] will take you a long way into the subject—if you can read German. As for the history of continued fractions, both Volume 2 of Dickson's encyclopedic *History of the Theory of Numbers* [4] and Brezinski's work [2] contain tons of historical information; their styles are distinctly different. The above list is far from complete.

What else can you do with continued fractions? Among other things, you can use them to factor large integers. In the late 1960's, Brillhart and Morrison [3] developed an integer factoring algorithm called CFRAC, with which they factored the seventh Fermat number $F_7 = 2^{128} + 1$. Based on continued fractions, it was the world's principle large-number cracker until being superseded by the Quadratic Sieve in the early 1980's.

What is the simple continued fraction expansion for e? The finite scf in Equation (3) for the rational number $\frac{1264}{465}$ is its beginning. (That's why I asked you to check the arithmetic.) In fact, $e = \langle 2, 1, 2, 1, 1, 4, 1, 1, \dots, 2n, 1, 1, \dots \rangle$; for a proof, see [7], Section 11.6.

What about continued fractions for π? The finite scf in Equation (2) for $\frac{355}{113}$ is how it begins; that six-decimal-place approximation to π was known to the Chinese mathematician Tsu Ch'ung Chi (430-501). It proceeds, with no apparent pattern, as follows: $\pi = \langle 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \dots \rangle$. As of 1999, it was known to 20,000,000 terms.

If Archimedes didn't make up the Cattle Problem, who did — and when? This problem surfaces in 1773 when Gotthold Ephraim Lessing published the Greek text of a 24-verse epigram translated from an Arabic manuscript. This epigram was the

Cattle Problem as stated by Archimedes in a letter for the students at Alexandria which Archimedes sent to his friend Eratosthenes. A fair amount of ink has been spilled over the question, "What did Archimedes know, and when did he know it?" Most historians of mathematics agree that the problem is very likely due to Archimedes, although the epigram came later.

What's TEX? TEXis a mathematical typesetting program, but that is like saying that Hank Aaron was a homerun hitter. For it has become a language that mathematicians use to communicate with each other in emails, handwritten notes, and other informal settings. It was developed by Donald Knuth, the legendary computer scientist, who did not cash in on his wonderful product, but *gave it away* to the American Mathematical Society. Such feats of invention and altruism are as admirable as they are rare.

What is so interesting about Ramanujan's continued fractions? Among other features they look spectacular. One of my favorites is the following, which he included in a famous letter to the English mathematician G. H. Hardy:

$$\frac{1}{1+}\,\frac{e^{-2\pi\sqrt{5}}}{1+}\,\frac{e^{-4\pi\sqrt{5}}}{1+}\,\frac{e^{-6\pi\sqrt{5}}}{1+\ldots} = \left[\frac{\sqrt{5}}{1+\left(5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2}-1\right)^{1/5}} - \frac{\sqrt{5}+1}{2}\right] e^{2\pi/\sqrt{5}}$$

No, I don't know how to prove that these two expressions are equal. But there is some comfort: Hardy stated that when he first saw this equality, it defeated him completely.

Why is it called the Pell Equation if Pell had nothing to do with it? It was one of the few mistakes that the great Leonhard Euler (17078-1783) ever made. He wrongly attributed the equation to John Pell because it appeared in a book Pell wrote, but Pell had no connection with the equation. It would be appropriate to name it after either Brouncker, Fermat, or the Indian mathematician Bhaskara (1114-1185), all of whom studied the equation extensively. This is yet another example of Boyer's Law, which — what's Boyer's Law? That's another story.

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