The Mertens constant

$$B_3 \equiv \lim_{x\to+\infty}\left(\log(x) - \sum_{\mathrm{prime}\,p\le x}\frac{\log(p)}{p}\right) \tag{1}$$

may be evaluated using

$$B_3 = \gamma + \sum_{\mathrm{prime}\,p\le A}\frac{\log(p)}{p(p-1)} + \sum_{k>1}\mu(k)\left(\frac{\zeta'(k)}{\zeta(k)} + \sum_{\mathrm{prime}\,p\le A}\frac{\log(p)}{p^k-1}\right). \tag{2}$$

This result is independent of A, since

$$\sum_{k>1}\frac{\mu(k)}{x^k-1} = -\frac{1}{x(x-1)} \tag{3}$$

for all $|x|>1$. As $A\to+\infty$, we obtain the slowly convergent sum

$$B_3 = \gamma + \sum_{\mathrm{prime}\,p}\frac{\log(p)}{p(p-1)}. \tag{4}$$

For $A<2$, we obtain

$$B_3 = \gamma + \sum_{k>1}\mu(k)\frac{\zeta'(k)}{\zeta(k)} \tag{5}$$

with a summand that is $\mathrm{O}(2^{-k})$ as $k\to\infty$. In general, the Möbius summand in (2) is $\mathrm{O}(P^{1-k})$, where P is the smallest prime that is greater than A.

To derive (4) from (1), we may use the prime number theorem to approximate

$$\frac{\zeta'(s)}{\zeta(s)} + \sum_{\mathrm{prime}\,p\le x}\frac{\log(p)}{p^s-1} = -\sum_{\mathrm{prime}\,p>x}\frac{\log(p)}{p^s-1} \approx -\int_x^\infty\frac{\mathrm{d}y}{y^s} = \frac{x^{1-s}}{1-s} \tag{6}$$

for $s>1$ and large x. Using $\zeta(s) = 1/(s-1)+\gamma+\mathrm{O}(s-1)$, we obtain

$$\gamma = \lim_{x\to+\infty}\left(\log(x) - \sum_{\mathrm{prime}\,p<x}\frac{\log(p)}{p-1}\right) \tag{7}$$

as $s\to 1$. This shows that (1) agrees with (4) and hence with (2) and (5).

With $A = 6779$, Pari-GP evaluated B_3 to 5000 decimal places in 67 minutes on a 2.2 GHz Opteron, using the `sumalt` routine for the alternating sums in

$$\frac{\zeta'(k)}{\zeta(k)} = \frac{1}{2^{k-1}-1}\left(-\log(2) + \frac{2^{k-1}}{\zeta(k)}\sum_{n>1}(-1)^n\frac{\log(n)}{n^k}\right). \tag{8}$$

The result was checked by varying A and also by increasing the working precision.

With $A = 13523$, Pari-GP took 700 minutes to compute the 10001 decimal places in `http://physics.open.ac.uk/~dbroadhu/cert/cohenb3.txt`

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