Some conjectures on the gaps between consecutive primes

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Abstract

Five conjectures on the gaps between consecutive primes are formulated. One expresses the number of twins below a given bound directly by $\pi(N)$. These conjectures are compared with the computer results and a good agreement is found.

1. Introduction.

In 1922 Hardy and Littlewood [1] have proposed about 15 conjectures. The conjecture B of their paper states:

There are infinitely many primes pairs (p, p'), where $p' = p + d$, for every even d. If $\pi_d(N)$ denotes the number of pairs less than N, then

$$\pi_d(N) \sim 2c_2 \frac{N}{\log^2(N)} \prod_{p|d} \frac{p-1}{p-2}. \tag{1}$$

Here the constant c_2 is defined in the following way[1]:

$$c_2 \equiv \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.66016\ldots \tag{2}$$

The computer results of the search for pairs of primes separated by a distance d and smaller than N for $N = 2^{22}, 2^{24}, \ldots, 2^{40} \approx 1.1 \times 10^{12}$ are shown in the Fig.1. The characteristic oscillating pattern of points is caused by the product

$$J(d) = \prod_{p|d, p>2} \frac{p-1}{p-2} \tag{3}$$

appearing in (1). The above notation $\pi_d(N)$ denotes prime pairs not necessarily successive. In this paper we are interested in the investigation of a behaviour of the number of pairs of *consecutive* primes p_n, p_{n+1} with difference $p_{n+1} - p_n = d$. To make more clear distinction from the case treated by the Hardy–Littlewood conjecture, we exchange the appearance of index d and N in our notation:

Definition 1:

$$h_N(d) = \text{number of pairs } \ p_n, p_{n+1} < N \text{ with } d = p_{n+1} - p_n. \tag{4}$$

[1]The author believes there should exists heuristic explanation why the value of c_2 is so close to 2/3.

Here the letter h stays as an abbreviation for "histogram". We note that, skipping the only triple (3, 5, 7) of three consecutive primes separated by 2, the identities $\pi_d(N) \equiv h_N(d)$ hold for $d = 2$ (Twin primes) and $d = 4$, which is naturally to name "Cousin primes".

The problem of the appearance of gaps between consecutive primes has a long history. Two main questions related to that problem can be distinguished: One concerns the estimation of the difference:

$$d_n = p_{n+1} - p_n. \tag{5}$$

The growth rate of the form $d_n = \mathcal{O}(p_n^\theta)$ with different θ was proved in the past. The Riemann Hypothesis implies $\theta = \frac{1}{2} + \epsilon$ for any $\epsilon > 0$ and a few results with θ closest to 1/2 are: the result of Mozzochi [2] $\theta = \frac{1051}{1920}$, Lou and Yao obtained $\theta = 6/11$ [3] and recently Baker and Harman have improved it to $\theta = 0.535$ [4]. The second group of papers deals mainly with first appearance of a given gap of length d, see e.g. [5], [6], [7], [8] or the explicit formula giving the largest gap between consecutive primes [9], [10], [11], [12]. In 1974 there appeared the paper by Brent [13], where the statistical properties of the distribution of gaps between consecutive primes were studied both theoretically and numerically. Brent have applied the inclusion–exclusion principle and obtained from (1) the formula for $h_N(d)$ (in his notation $Q_N(r)$). However his result (formula (4) in [13]) had not a closed form and he had to produce on the computer the table of constants appearing in his formula (4).

In this paper I am going to formulate four conjectures on the gaps between consecutive primes and a new formula giving the number of twins below a given bound N expressed directly by the number of primes smaller than N.

2. Gaps between consecutive primes

I have counted on a computer the number of gaps between consecutive primes up to $N = 2^{44} \approx 1.76 \times 10^{13}$. During the computer search the data representing the function $h_N(d)$ were stored at values of N, forming the geometrical progression with the ratio 4, i.e. at $N = 2^{20}, 2^{22}, \ldots, 2^{42}, 2^{44}$. Such a choice of the intermediate thresholds as powers of 2 was determined by the employed computer program, because the primes were coded as bits. The resulting curves are plotted on the semi–logarithmic axes in the Fig.2. The straight lines are the least–square fits of the assumed exponential decrease of $h_N(d)$ with d to the actual values.

In the plot of $h_N(d)$ in the Fig.2 a lot of regularities can be observed. The pattern of points in Fig.2 *does not depend on* N: for each N the arrangements of circles is the same, only the intercept increases and the slope decreases. Comparison of the Fig.1 and Fig.2 and the fact that the points in Fig.2 lie around the straight lines on the semi-logarithmic scale suggest the following *Ansatz* for $h_N(d)$:

$$h_N(d) \sim B(N) \prod_{p|d, p>2} \frac{p-1}{p-2} e^{-A(N)d}. \tag{6}$$

The essence point of the present consideration consists in a possibility of determining the unknown functions $A(N)$ and $B(N)$ by *assuming only the above exponential* *decrease of $h_N(d)$ with d and employing two identities fulfilled by $h_N(d)$.* First of all, the number of all gaps is by 1 smaller than the number of all primes smaller than N:

$$\sum_d h_N(d) \;=\; \pi(N) - 1, \tag{7}$$

where $\pi(N)$ denotes the number of primes smaller than N. The second selfconsistency condition comes from the observation, that the sum of differences between consecutive primes $p_n \leq N$ is equal to the largest prime $\leq N$ and for large N we can write:

$$\sum_d h_N(d)\; d \;\approx\; N. \tag{8}$$

The problem is that for the *Ansatz* (6) I am not able to write down in the closed form the selfconsistency conditions (7) and (8), because the product (3) behaves very erratically. However, $J(d)$ takes small values of the order 1; e.g. for $N < 2^{44}$ it takes values between 1 and 3.2 (3.2 appears only when $d = 210$, $d = 420$ or $d = 630$ — the largest gap for $N < 2^{44}$ was 716). To calculate the sums in (7) and (8) I replace the product $J(d)$ in (6) by its mean value s defined as:

$$s = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \prod_{p|2k, p>2} \frac{p-1}{p-2}. \tag{9}$$

Summing up geometrical series from the above identities (7) and (8) I get:

$$A(N) = \frac{\pi(N)}{N}, \tag{10}$$

$$B(N) = 2\frac{\pi^2(N)}{sN}. \tag{11}$$

Comparison with the Hardy and Littlewood conjecture for Twins $d = 2$ for $\log(N) \gg d$ gives $1/s = c_2$. In this heuristic way we arrive at the identity:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \prod_{p|2k, p>2} \frac{p-1}{p-2} = \frac{1}{\prod_{p>2}(1 - \frac{1}{(p-1)^2})} = 1.514\ldots.$$

This equality indeed can be rigorously proved: it follows e.g. from the eq.(33) in [14] or eq. (17.12) in [15] in the limit $N \to \infty$. It can be also shown to be true by an application of the Ikehara-Delange theorem [16].

Finally we state the main

Conjecture 1

$$h_N(d) = 2c_2 \frac{\pi^2(N)}{N} \prod_{p|d, p>2} \frac{p-1}{p-2} e^{-d\pi(N)/N} + error\ term(N, d) \quad \text{for } d \geq 6 \tag{12}$$

while for Twins ($d=2$) and Cousins ($d=4$) the identities $h_N(d)=\pi_d(N)$ hold. The formula (12) consists of three terms. The first one depends only on N, the second only on d, but the third term depends both on d and N.

Putting in (12) $\pi(N)\sim N/\log(N)$ and comparing with the original Hardy–Littlewood conjecture we have that the number $h_d(N)$ of *successive* primes $(p_{n+1},\ p_n)$ smaller than N and of the difference d $(=p_{n+1}-p_n)$ is diminished by the factor $\exp(-d/\log(N))$ in comparison with *all* pairs of primes (p,p') apart in the distance d $(=p'-p)$:

$$h_N(d)=\pi_d(N)e^{-d/\log(N)}+error\ term(N,d),\quad \text{for } d\geq 6. \tag{13}$$

The above relation is illustrated by the Figures 1 and 2. Taking N very large at fixed d (such, that $\log(N)\gg d$) we get from (13) the intuitively obvious relation $h_N(d)\approx\pi_d(N)$ — the plots in the Fig.2 tend to the horizontal line for increasing N.

By the Prime Number Theorem with the smallest error term obtained under the assumption of the Riemann Hypothesis we have

$$\pi(N)=\int_0^N\frac{du}{\log(u)}+\mathcal{O}(N^{1/2}\log(N)). \tag{14}$$

On the other hand (1) is being generalized as

$$\pi_d(N)\sim 2c_2\prod_{p|d}\frac{p-1}{p-2}\int_0^N\frac{du}{\log^2(u)}\equiv 2c_2\prod_{p|d}\frac{p-1}{p-2}\mathrm{Li}_2(N) \tag{15}$$

where the error term is unknown. Taking into account (14) and (15) we see that the above equality (13) does not hold in higher orders of $1/\log(N)$ because:

$$\frac{1}{N}\left(\int_0^N\frac{du}{\log(u)}\right)^2\neq\int_0^N\frac{du}{\log^2(u)}. \tag{16}$$

However it is tempting to compare the number of Twins predicted by the term $\pi^2(N)/N$ appearing in front of exponent in (12) with the actual values of Twins obtained from the computer search. It turns out that the difference $\pi_2(N)-\pi^2(N)/N$ is of positive sign and on the log-log plot the points obtained from the computer lie almost perfectly on the straight line of the slope $0.83\approx 5/6$, see Fig.3. It has to be contrasted with the fact, that $\pi_2(x)-2c_2\mathrm{Li}_2(N)$ changes sign, already at low values of N of the order 10^6, see [17] and up to $N=2^{42}$ there are over 90000 sign changes of the $\pi_2(N)-2c_2\mathrm{Li}_2(N)$ [18]. Additionally we know from the Littlewood theorem that the difference $\pi(x)-\mathrm{Li}(x)$ changes infinitely often the sign. Comparison of the regular behaviour of the difference $\pi_2(N)-2c_2\frac{\pi^2(N)}{N}$ seen on the Fig.2 with these facts leads to the conjecture

Conjecture 2

$$\pi_2(N)=2c_2\frac{\pi^2(N)}{N}+\alpha N^{5/6}+error\ term(N), \tag{17}$$

where $\alpha = 0.000393$ was obtained from fitting the parameters in the Fig.3. Let me notice that the ratio of the above estimation (17) for the number of Twins with $\pi(N)$ given by (14) and the usual one given by

$$\pi_2(N) \sim 2c_2\mathrm{Li}_2(N) \tag{18}$$

tends to one when $N \to \infty$. The comparison of theseconjectures (17) and (18) with the actual computer datais provided in the Table I.

TABLE **I**

N	$\pi_2(N)$	$c_2\pi^2(N)/N + \alpha N^{5/6}$	ratio	$c_2\mathrm{Li}_2(N)$	ratio
2^{20}	8535	8506	1.0033857	8583	0.994423
2^{22}	27995	27680	1.0113654	27813	1.006531
2^{24}	92246	91780	1.0050760	92035	1.002292
2^{26}	309561	309293	1.0008672	309743	0.999414
2^{28}	1056281	1056401	0.9998861	1057167	0.999162
2^{30}	3650557	3650149	1.0001118	3651227	0.999817
2^{32}	12739574	12738612	1.0000755	12739548	1.000002
2^{34}	44849427	44846898	1.0000564	44844247	1.000116
2^{36}	159082253	159100516	0.9998852	159083374	0.999993
2^{38}	568237005	568289266	0.9999080	568223658	1.000023
2^{40}	2042054332	2042240611	0.9999088	2042048046	1.000003
2^{42}	7378928530	7379315290	0.9999476	7378892252	1.000005
2^{44}	26795709320	26795829623	0.9999955	26795507391	1.000008

The *error term* in (17) cannot be determined from the available computer data — the difference $\pi_2(N) - 2c_2\frac{\pi^2(N)}{N} - \alpha N^{\frac{5}{6}}$ changes the sign, see the Table I. Let us notice, that due to the special form of (17) the part $\sqrt{N}$ of the error term in the estimation (14) cancels with N in the denominator leaving only the uncertainties contained in the unknown *error term*(N) in (17) plus the term $\mathcal{O}(\log^2(N))$ surviving from (14).

For large N from (17) we get the remarkable equation

$$\pi_2(N) \sim 2c_2\frac{\pi^2(N)}{N} \tag{19}$$

relating $\pi_2(N)$ directly with $\pi(N)$. Because $\mathrm{Li}(N)$ in (14) as well as Li_2(N) in (18) monotonically increases, while there are local fluctuations in the "density" of primes and Twins, the above formula incorporates all irregularities in the distribution of primes into the estimation for the number of Twins.

3. Two applications

As the application of the Conjecture 1 I can obtain the length $G(N)$ of the largest gap between consecutive primes below a given bound N. Simply, the largest gap G

appears only once, so it is equal to the value at which $h_N(d)$ crosses the d-axis on the Fig.2:

$$h_N(G(N)) = 1. \tag{20}$$

Using the general form of the functions $A(N)$ and $B(N)$ (10) and (11) we obtain

Conjecture 3

$$G(N) \sim g(N) = \frac{N}{\pi(N)}(2\log(\pi(N)) - \log(N) + c_0), \tag{21}$$

where $c_0 = \log(2c_2) \approx 0.278$. The above prediction (3) expresses $G(N)$ directly by $\pi(N)$ and for the estimation $\pi(N) \sim N/\log(N)$ conjectured by Gauss it gives:

$$G(N) \sim g_1(N) = \log(N)(\log(N) - 2\log\log(N) + c_0). \tag{22}$$

To my knowledge the above dependence of $G(N)$ is new, the most similar I have seen in [11], where Cadwell gave the heuristic arguments that

$$G(N) \sim \log(N)(\log(N) - \log\log(N)). \tag{23}$$

The formula (22) for large N passes onto the Cramer [9] conjecture (see also [10]):

$$G(N) \sim \log^2(N). \tag{24}$$

The examination of the formula (21) with the "experimental" results and the formula (24) is given in the Fig.4. Let me notice, that the values of $N \approx 10^{14}$ searched by direct checking are small for the asymptotic of eq.(22), because $\log 10^{14} = 32.24\ldots$, $\log\log 10^{14} = 3.47\ldots$, however large for modern computers, but nevertheless the agreement between the prediction and the actual data is quite good. The Fig.4 should be compared with the figure on the page 12 in [19]. To produce this plot I have put in (21) the formula (14) for $\pi(N)$.

There are serious doubts about the validity of the Cramer conjecture [20]. A. Granville believes that the actual $G(N)$ can be larger than that given by (24), namely he claims [20] that there are infinitely many pairs of primes p_n, p_{n+1} for which:

$$p_{n+1} - p_n = G(p_n) > 2e^{-\gamma}\log^2(p_n) = 1.12292\ldots\log^2(p_n). \tag{25}$$

However, by assuming the Hypothesis H of Sierpinski and Schinzel [21] it can be proven that for *almost* all prime gaps $G(N) < \log^2(N)$. In connection with this let me remark that on the Fig.4 the curve $G(N)$ always lies below the Cramer conjecture (24) (what agrees with the corollary from the Sierpinski–Schinzel Hypothesis) and for (25) to be true there should be such regions were $G(N) > 1.123\log^2(N)$. In fact, the estimation (21) I have obtained using the formula for $h_N(d)$ outside the regime of its applicability — for large d the corresponding values of $h_N(d)$ are small and display large fluctuations (see Fig.2): $h_N(d)$ describes accurately only the "bulk" values of the number of gaps. There are large fluctuations between actual data and the formula (21) seen in the Fig.4, but there are also points, where the "staircase" like plot of $G(N)$ crosses the analytical curve $g(N)$, thus there are sign changes of the difference $G(N) - g(N)$. The open problem is whether discrepancies between $G(N)$ and $g(N)$ will grow with N? If they do, the skepticism about Cramer conjecture can be justified. But the opposite possibility is, that for much larger N the fluctuations will become invisible, i.e. if the Fig.4 can be plotted say for $N = 10^{1000}$ on the same sheet of paper, the discrepancies between actual data and the formula (21) (as well Cramer conjecture) will disappear — these three curves will collapse onto the one curve.

The glance at the Fig.4 suggests that the following "average" property may hold true:

Conjecture 4

$$\lim_{N\to\infty} \frac{\int_2^N G(x)dx}{\int_2^N g(x)dx} = 1. \qquad (26)$$

The integral of the difference $G(N) - g(N)$ oscillates and only the ratio of surfaces below curves $G(N)$ and $g(N)$ has the chance to tend to 1.

As the final application of the formula (12) and our computer data we consider the conjecture made by D.R. Heath-Brown in [22], see also problem A8 in [23]. Heath-Brown guessed that

$$\sum_{p_n \leq N} (p_n - p_{n-1})^2 \sim 2N \log(N). \qquad (27)$$

Treating the product $J(d)$ exactly the same way as in derivation of (10) and (11) from (12) I get:

Conjecture 5

$$\sum_{p_n < N} (p_n - p_{n-1})^2 = \sum_d d^2 h_N(d) = \frac{2N^2}{\pi(N)} + error\ term(N). \qquad (28)$$

Again, the above formula involves directly $\pi(N)$ and for $\pi(N) \sim N/\log(N)$ pass over onto the (27). The comparison with the computer data is given in Table II and it suggests that (28) better reproduces data obtained by computer than (27), however ratios of both predictions to the actual computer data tend to 1 with increasing N.

TABLE **II**

The sum of squares of gaps between consecutive primes. In the second column the numbers obtained by a computer are given, while in the third one values obtained from eq.(27) and in the fifth from eq.(28) are presented. The fourth and sixth column contains the appropriate ratios.

N	$\sum d_n^2$	eq.(27)	ratio	eq.(28)	ratio
2^{20}	22171764	29072700	0.7626	26772796	0.8281
2^{22}	100275380	127919880	0.7839	118820844	0.8439
2^{24}	444929864	558195838	0.7971	522110155	0.8522
2^{26}	1959715564	2418848633	0.8102	2275494249	0.8612
2^{28}	8565851940	10419655651	0.8221	9849466113	0.8697
2^{30}	37168128504	44655667077	0.8323	42385671111	0.8769
2^{32}	160316134724	190530846196	0.8414	181487345070	0.8833
2^{34}	687851546612	809756096335	0.8495	773707462336	0.8890
2^{36}	2938092559092	3429555231536	0.8567	3285796459875	0.8942
2^{38}	12499933597196	14480344310930	0.8632	13906838608376	0.8988
2^{40}	52993288896472	60969870782865	0.8692	58681260682528	0.9031
2^{42}	223959886541176	256073457288032	0.8746	246938313792889	0.9069
2^{44}	943825347126668	1073069725778421	0.8796	1036598392711419	0.9105

Acknowledgment: I thank Prof. E. Bombieri and Prof. A.Granville for e-mail feedback and Prof. W.Narkiewicz for discussions. I also thank Prof. B.J. Birch for telling me that $0.83 \approx 5/6$ and Prof. T. Nicely for sending me new maximal prime gaps.

Figure Caption:

Figure 1 The plot illustrating the Hardy–Littlewood conjecture. There is a logarithmical scale on the y-axis , while on the N-axis there is a linear scale. There are oscillations of the period 6 clearly visible. Additionally, there are structures of the length 30 overimposed. Let us mention "local" spikes at $d = 30, 60, \ldots$. Especially well profound are spikes at $d = 210 = 2 \cdot 3 \cdot 5 \cdot 7$ and 420.

Figure 2 The plot showing the dependence of the histogram $h_N(d)$ on d at $N = 2^{20}, 2^{22}, \ldots, 2^{44}$. There is a logarithmical scale on the y-axis , while on the N-axis there is a linear scale. The straight lines are the best fits obtained by means of the least-square method. Let us mention "local" spikes at $d = 30, 60, \ldots$. Especially well profound are spikes at $d = 210 = 2 \cdot 3 \cdot 5 \cdot 7$ and for $N = 2^{40}$, $N = 2^{42}$ and $N = 2^{44}$ at its multiplicity $d = 420$.

Figure 3 The plot of the difference $\pi_2(N) - 2c_2\pi^2(N)/N$ from the computer data. For the least – square fit a few first points were discarded (in fact only two points determine a straight line!).

Figure 4 Plot of $G(N)$ for N up to 1.35×10^{15}. The results from the computer search are drawn by thin solid line, eq.(21) is bold line and the conjecture of Shanks (24) is shown by dashed line. Most of the points plotted on this figure come from my own search up to $2^{44} = 1.76 \times 10^{13}$. Gaps larger than 2^{44} I have taken from [7] and [8].

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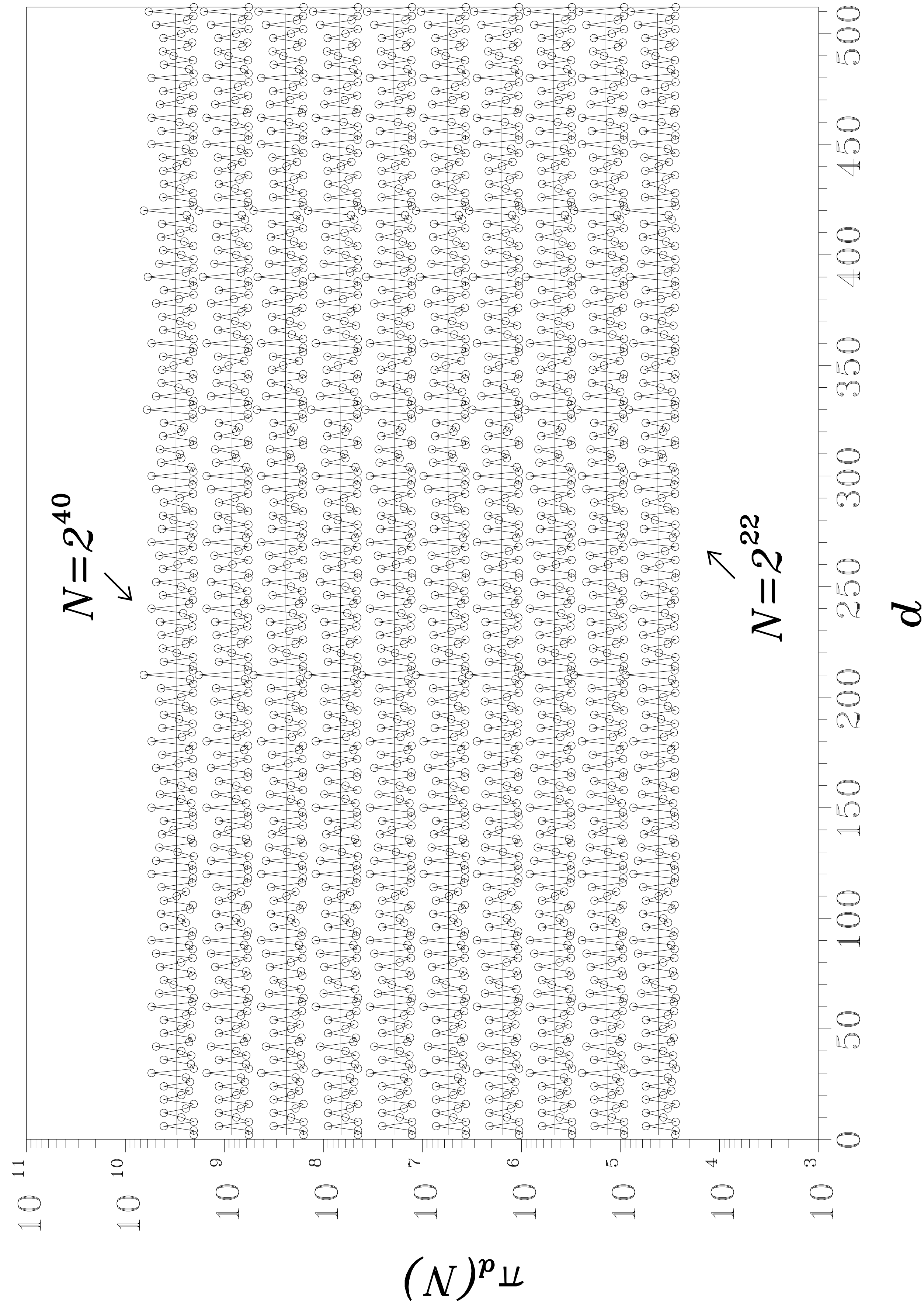
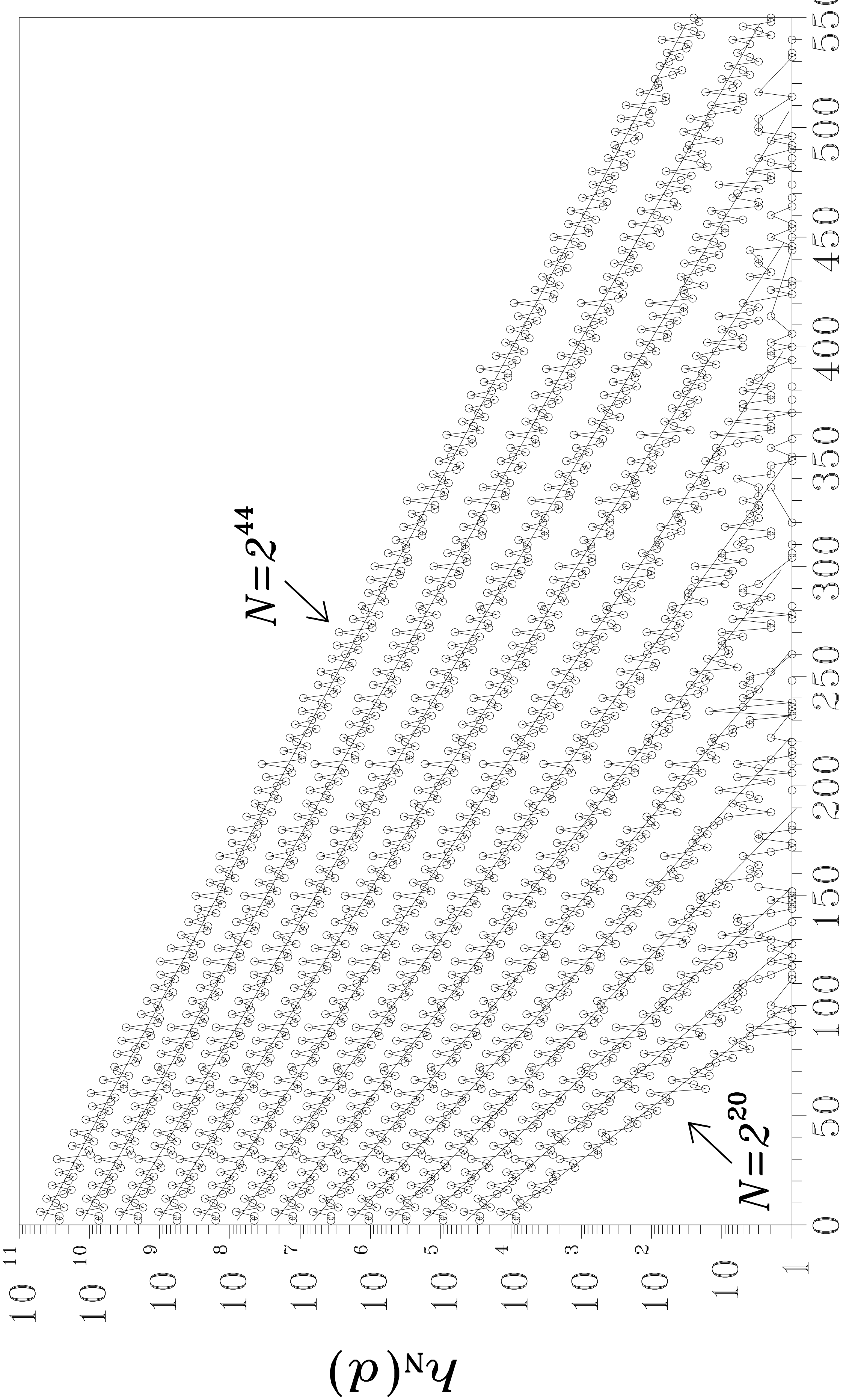


N=2^40
N=2^22
d
π_d(N)
0
50
100
150
200
250
300
350
400
450
500
10^3
10^4
10^5
10^6
10^7
10^8
10^9
10^10
10^11

Fig.1



N=2^44
N=2^20
d
h_N(d)
0
50
100
150
200
250
300
350
400
450
500
550
1
10
10^2
10^3
10^4
10^5
10^6
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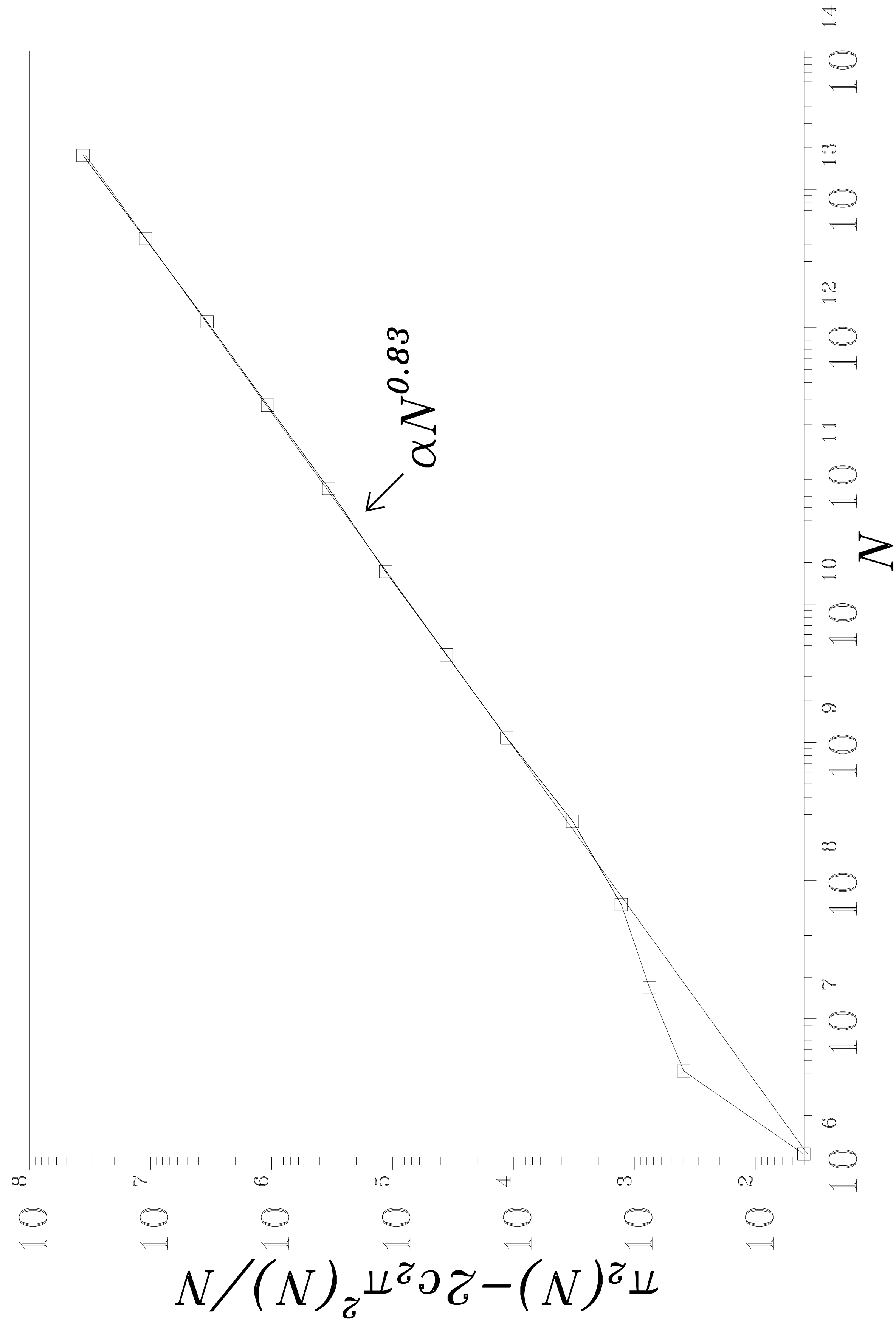
10^9
10^10
10^11

Fig.2


$\alpha N^{0.83}$
N
$\pi_2(N) - 2c_2\pi^2(N)/N$
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}
10^{8}
10^{6}
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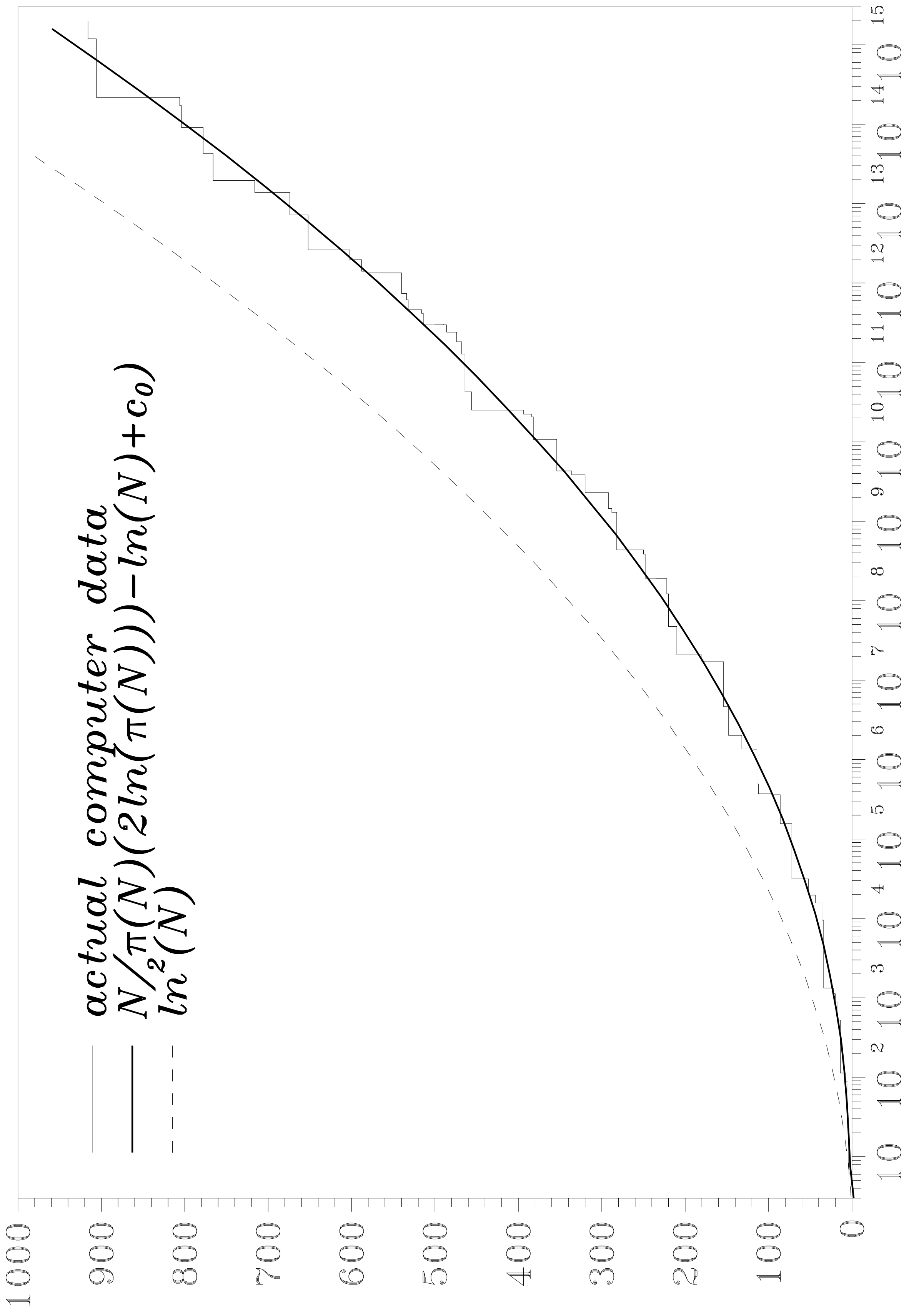
10^{14}

Fig.3



actual computer data
N/π(N)(2ln(π(N))−ln(N)+c0)
ln2(N)
N
G(N)
0
100
200
300
400
500
600
700
800
900
1000

Fig.4