RAMSEY'S THEORY

You probably have heard of this interesting fact: among any six people in the world, there exist three who know each other or three who don't know each other. Actually there are many other similar results. For example, among any nine people there exist three who know each other or four who don't know each other, and among any fourteen people there exist three who know each other or five who don't know each other. Here we are going to generalize the above results and to study the patterns behind them.

Let $p, q \geq 2$ be two given integers. We define $R(p,q)$ as the smallest integer *n* for which among any edge colouring of the complete graph on order *n*, K_n, by red or blue, there exist a red complete graph on order *p*, K_p, or a blue complete graph on order *q*, K_q. Besides, we denote E_n as a null graph of degree *n*.

These numbers are known as **Ramsey numbers**. For example, $R(3,3)=6$, $R(3,4)=9$, $R(3,5)=14$ etc. are all Ramsey numbers. From the definition, we see that the Ramsey numbers have the following two properties:

1. For any positive integers $p, q \geq 2$, we have $R(p,q) = R(q,p)$.
2. For any positive integer $q \geq 2$, we have $R(2,q) = q$.

To date few Ramsey numbers are known. In addition to $R(2,q) = q$ mentioned above, it is not too difficult to find the values of the smaller Ramsey numbers like $R(3,3)$, $R(3,4)$, $R(3,5)$, … But beyond that we are in a long journey of searching for Ramsey numbers. Here we will first find $R(3,3)$.

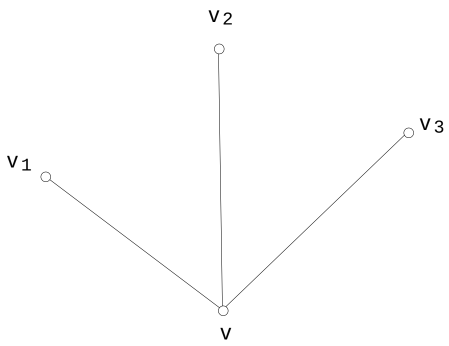
Example 1.

Prove that among any six people in the world, there exist three who know each other or three who don't know each other.

Solution.

The question is equivalent to proving that if we colour each edge of the complete graph on order 6, K_6, by red or blue, there exists a mono-coloured triangle. (In other words, we are to prove that $R(3,3)=6$.) The proof goes as follows.

Pick a vertex v in K_6. There are five edges incident to v, each of which is red or blue. By the pigeon-hole principle, three of these edges are of the same colour. Without loss of generality, we may assume that vv_1, vv_2 and vv_3 are red. (Red edges are represented by solid lines in the diagram below.)

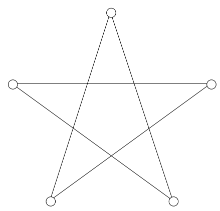


Consider the edges between the vertices v_1, v_2 and v_3. If any of these edges is red, then there will be a red triangle. Otherwise, v_1v_2, v_2v_3 and v_3v_1 are all blue, so $v_1v_2v_3$ is a blue triangle.

This shows that the graph necessarily contains a mono-coloured triangle. (In fact we can even prove that there exist at least two mono-coloured triangles in the above diagram.)

Theorem 1.
$R(3,3) = 6$.

Proof. From Example 1, we know that $R(3,3) \le 6$. It remains to construct an example of K_5 which does not contain K_3 and E_3. An example is given below.



This example establishes the fact that $R(3,3) > 5$, thereby proving the theorem.

Q.E.D.

Establishing a lower bound for $R(3,3)$ by construction as shown above is straightforward. However, when the number of vertices becomes larger and larger, such constructions become more and more difficult. For instance, to construct a 3-colouring of the edges of K_{16} without a mono-coloured triangle is by no means easy.

Nowadays, mathematicians generally agree that unless new algorithms are found, computing the exact values of $R(p,q)$ is difficult despite advances in computer technology. Consequently, computing upper and lower bounds for $R(p,q)$ become increasingly important. A simple result is given below.

Theorem 2.

For any $p, q \geq 3$, we have $R(p,q) \leq R(p-1,q) + R(p,q-1)$.

Proof. Denote $n = R(p-1,q) + R(p,q-1)$. We shall establish below that any 2-coloring of K_n by red and blue necessarily contains a red K_p or a blue K_q.

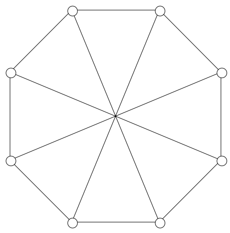
Take an arbitrary vertex *v* and consider the $n-1$ edges incident to it. Note that among these there must be at least $R(p-1,q)$ red edges or $R(p,q-1)$ blue edges. Otherwise the total number of edges incident to *v* is at most $[R(p-1,q)-1]+[R(p,q-1)-1] = n-2$, which is impossible.

Suppose there are $R(p-1,q)$ red edges incident to *v*. Then, among these 'red neighbours' of *v* there is a red K_{p-1} or a blue K_q. If there is a blue K_q then we are done. If there is a red K_{p-1}, then this K_{p-1} together with *v* form a red K_p.

In the same way, we can prove that if there are $R(p,q-1)$ incident to *v*, there must be a red K_p or a blue K_q.

Q.E.D.

With the help of this theorem we can easily find the values of $R(3,4)$, $R(4,4)$ and $R(3,5)$. For example, we have $R(3,4) \leq R(2,4) + R(3,3) = 4 + 6 = 10$. After some trials, it is not difficult to construct a graph of order 8 which does not contain K_3 or E_4. For example:



Consequently, $R(3,4) > 8$. On the other hand, we are unable to construct a graph of order 9 which does not contain K_3 or E_4 after many trials, so we guess that $R(3,4) = 9$.

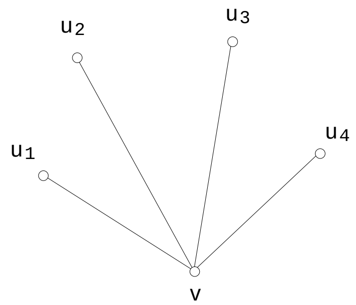
Theorem 3.

$R(3,4)=9$.

Proof. From the above discussions, we know that $R(3,4)>8$. So it suffices to prove that any 2-colouring of K_9 necessarily contains a red K_3 or a blue K_4.

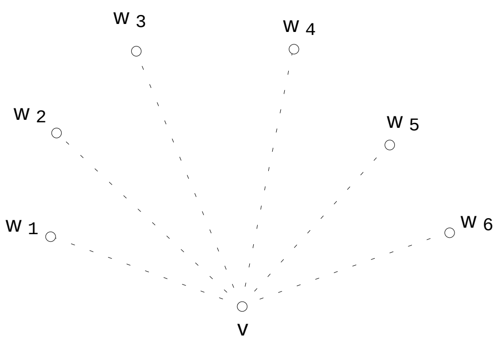
We shall consider two cases:

Case 1: Some vertex *v* in K_9 is incident to four red edges (as shown below).



In this case, if any two of u_1, u_2, u_3, u_4 are connected by a red edge u_iu_j, then u_i, u_j and *v* form a red K_3. Otherwise all edges connecting u_1, u_2, u_3, u_4 are blue, so they form a blue K_4.

Case 2: Some vertex *v* in K_9 is incident to six blue edges (as shown below).



Among the edges connecting w_1, w_2, …, w_6, there must be a red K_3 or a blue K_3. So together with *v*, there must be a red K_3 or a blue K_4.

Finally, we shall prove that at least one of the two cases will occur. If not, each vertex is incident to exactly three red edges and five blue edges. Removing all blue edges from the graph, each vertex is of degree 3, so the sum of the degrees of all vertices is equal to $3\times 9=27$, contradicting the fact that the sum of degrees must be even.

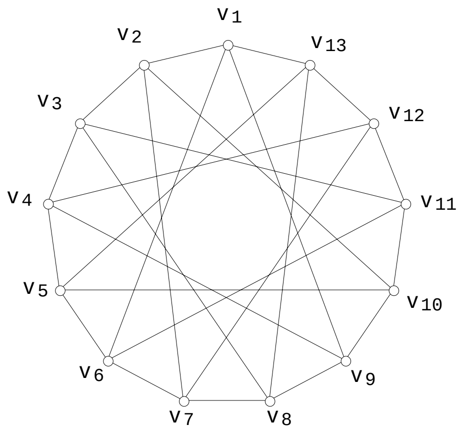
Q.E.D.

Next we shall consider $R(3,5)$. First, note that $R(3,5) \le R(2,5) + R(3,4) = 5 + 9 = 14$. Now we shall prove that $R(3,5) = 14$, by constructing a graph of order 13 without K_3 or E_5.

Theorem 4.

$R(3,5) = 14$.

Proof. Clearly, $R(3,5) \le R(2,5) + R(3,4) = 5 + 9 = 14$. The following graph of order 13 does not contain K_3 or E_5:



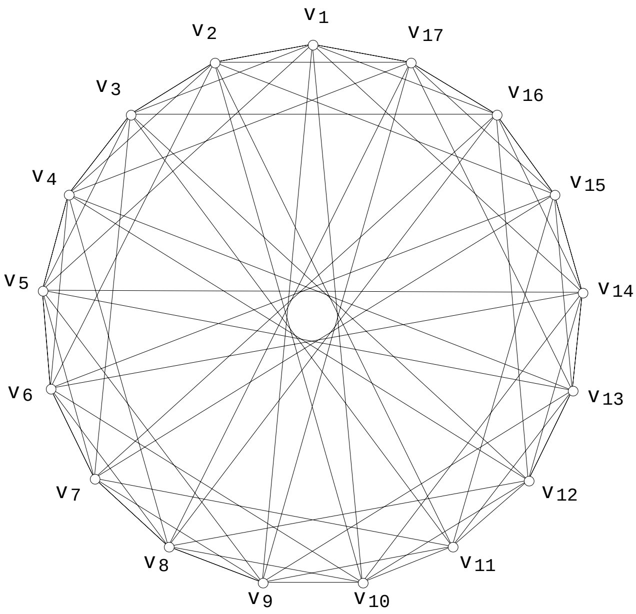
The method of construction is as follows: v_i and v_j are adjacent if and only if $|i - j| \in \{1, 5, 8, 12\}$. It is easy to see that the graph contains no K_3 or E_5. So $R(3,5) = 14$.

Q.E.D.

Theorem 5.

$R(4,4) = 18$.

Proof. As in the proof of the previous theorem, we have $R(4,4) \le R(4,3) + R(3,4) = 9 + 9 = 18$, while the diagram below guarantees that $R(4,4) > 17$.



In the figure, v_i and v_j are adjacent if and only if $|i-j| \in \{1, 2, 4, 8, 9, 13, 15, 16\}$.

Q.E.D.

Here we shall not discuss the other exact values of $R(p,q)$. The table below lists some known values or upper and lower bounds of some $R(p,q)$:

p \ q	3	4	5	6	7	8	9	10	11	12
3	6	9	14	18	23	28	36	40 43	46 51	51 60
4		18	25	35 41	49 61	53 84	69 115	80 149	96 191	106 238
5			43 49	58 87	80 143	95 216	114 316	442		
6				102 165	298	495	780	1171		

- In the table, double-underlined numbers are exact values, and the upper and lower bounds are inclusive.

'Suppose an evil alien would tell mankind "Either you tell me [the value of R(5,5)] or I will exterminate the human race."… It would be best in this case to try to compute it, both by mathematicians and with a computer.

If he would ask [for the value of R(6,6)], the best thing would be to destroy him before he destroys us, because we couldn't [determine R(6,6)].'

—Erdös

Exercises

1. Prove that for any integers $p, q \geq 2$ we have $R(p,q) \leq C_{p-1}^{p+q-2}$. (This is known as the Erdös-Szekeres upper bound.)

   Use this result to prove that $R(p,p) \leq \dfrac{4^{p-1}}{\sqrt{p-1}}$. This upper bound was kept for 50 years until the Czech mathematician V. Rödl and later the Danish mathematician A. Thomason improved it to

$$R(p,p) \leq \frac{A \times C_{p-1}^{2p-2}}{\sqrt[3]{p-1}}$$

   where A is a positive constant, in 1986 and 1988 respectively.

2. Prove that for $p \geq 3$ we have $R(p,p) > 2^{\frac{p}{2}}$. In 1975 Spencer even proved that

$$R(p,p) > \left(\frac{\sqrt{2}}{e} + o(1)\right) p 2^{\frac{p}{2}}.$$

   Here $o(1)$ is a function of p which converges to zero as p tends to infinity.