DYNAMIC POLE ASSIGNMENT AND SCHUBERT CALCULUS

M. S. RAVI *, JOACHIM ROSENTHAL †, AND XIAOCHANG WANG ‡

Abstract. The output feedback pole assignment problem is a classical problem in linear systems theory. In this paper we calculate the number of complex dynamic compensators of order q assigning a given set of poles for a q-nondegenerate m-input, p-output system of McMillan degree $n = q(m+p-1)+mp$. As a corollary it follows that when this number is odd, the generic system can be arbitrarily pole assigned by output feedback with a real dynamic compensator of order at most q if and only if $q(m+p-1)+mp \geq n$.

Key words. output feedback pole assignment, dynamic compensator, holomorphic curves in Grassmannian, degree of variety

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1. Introduction. The output feedback pole assignment of linear systems with static or dynamic compensators is a classical problem in control theory and many theoretical and numerical research papers have been devoted to this problem.

Although the systems involved are linear, the problem is in fact not linear. It was Brockett and Byrnes [4] who first explained the pole assignment problem with static compensators as an intersection problem in a compactified set of static compensators, the Grassmann manifold $\mathrm{Grass}(m, m+p)$. In making the connection to the classical Schubert calculus they were able to show that there are

$$d(m,p) = \deg \mathrm{Grass}(m, m+p) = \frac{1!2!\cdots(p-1)!(mp)!}{m!(m+1)!\cdots(m+p-1)!} \tag{1.1}$$

complex static output feedback laws which assign a set of poles for a nondegenerate m-input, p-output linear system of McMillan degree $n = mp$. In particular if the number $d(m,p)$ is odd, pole assignment by real static feedback is possible, because the set of complex solutions has to be invariant under complex conjugation. Moreover even if $d(m,p)$ is even Wang showed in [25] using algebro geometric techniques that a real solution exists for the generic system as soon as $mp > n$.

People have been looking for similar results for the dynamic pole assignment problem for a long time. A first attempt was made by Byrnes in [5]. Recently Rosenthal interpreted in [16, 17] the pole assignment problem with dynamic compensators, again as an intersection problem in a compactified space of dynamic compensators which we denote by $K^q_{m,p}$. It was also proved in [17] that if a plant has McMillan degree $n = q(m+p-1)+mp$ and is q-nondegenerate then there exist

$$d(m,p,q) = \deg K^q_{m,p} \tag{1.2}$$

* Department of Mathematics, East Carolina University, Greenville, North Carolina 27858, (maravi@ecuvax.cis.ecu.edu).

† Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556-5683 (Joachim.Rosenthal@nd.edu). Supported in part by NSF grant DMS-9400965.

‡ Department of Mathematics, Texas Tech University, Lubbock, Texas 79409-2013 (mdxia@ttacs1.ttu.edu). Supported in part by NSF grant DMS-9224541.

complex dynamic feedback compensators of order q which assign a set of $n+q$ closed loop poles. At this point we want to mention that all major results derived in [4, 17, 25] are based on a careful study of the associated pole assignment map. (Compare with Section 2 for more details). Indeed the number $d(m,p,q)$ can also be viewed as the mapping degree of the associated pole assignment map and this map has geometrically the type of a central projection.

One goal of our paper is to derive a formula for $d(m,p,q)$. Historically the formula (1.1) for $d(m,p) = d(m,p,0)$ was first discovered in 1886 by Hannibal Schubert [18], a German high school teacher, using a symbolic formalism known as Schubert calculus. Using modern language the number $d(m,p)$ is equal to σ_1^{mp} where σ_1 denotes the first Chern class of the classifying bundle over the Grassmann manifold $\mathrm{Grass}(m, m+p)$. By applying Pieri's formula (see Section 3 for more details)

$$(1.3) \qquad (i_1, i_2, \ldots, i_m) \cdot \sigma_1 = \sum_{i_l - 1 > i_{l-1}} (i_1, \ldots, i_l - 1, \ldots, i_m)$$

repeatedly to $(p, p+2, p+3, \ldots, p+m) = \sigma_1$, one can compute the number $d(m,p) = \deg \mathrm{Grass}(m, m+p)$.

In [15] we defined a set of subvarieties of $K^q_{m,p}$ similar to the Schubert varieties of $\mathrm{Grass}(m, m+p)$, and proved a geometric formula similar to Pieri's formula (1.3). This enables us to express $d(m,p,q) = \deg K^q_{m,p}$ as the solution of a partial difference equation with boundary condition. In this paper (Section 3) we will solve this difference equation and we will derive a closed formula for $d(m,p,q)$ which is valid for all positive integers m, p and q. From this formula we finally will derive several new results which predict real and complex solutions assigning a specific set of closed loop poles. One of the main results of this paper is:

Theorem 1.1. *The poles of an m-input p-output q-nondegenerate linear system of McMillan degree*

$$(1.4) \qquad n = q(m+p-1) + mp$$

can be assigned arbitrarily by using output feedback with complex dynamic compensator of order at most q, and there are

$$(1.5) \qquad \begin{aligned} &d(m,p,q) \\ &= (-1)^{q(m+1)} (mp + q(m+p))! \sum_{n_1 + \cdots + n_m = q} \frac{\prod_{k<j} (j-k+(n_j-n_k)(m+p))}{\prod_{j=1}^{m} (p+j+n_j(m+p)-1)!} \end{aligned}$$

complex solutions for each set of poles. In particular, if $d(m,p,q)$ is odd a real solution always exists. Moreover when $d(m,p,q)$ is odd, the generic system can be arbitrarily pole assigned by output feedback with real dynamic compensators of order at most q if and only if

$$(1.6) \qquad n \leq q(m+p-1) + mp.$$

The variety $K^q_{m,p}$ which parameterizes the set of m-input, p-output compensators of McMillan degree q can also be viewed as a parameterization of the space of rational curves of degree q on the Grassmann variety $\mathrm{Grass}(m, m+p)$. This geometric link originates from the well known Hermann-Martin identification [12]. (Compare also with [6, 17]).

Surprisingly to us there has been recently a tremendous interest in the intersection theory of parameterized curves (of arbitrary genus) on Grassmann varieties and other homogeneous spaces [2, 8, 24, 31]. Researchers working in conformal quantum field theory conjectured several new intersection numbers and an interesting formula for all numbers $d(m,p,q)$, different from (1.5) was part of this conjecture. Readers interested in the Physics behind this conjecture are referred to Vafa [24]. The conjecture itself is formulated by Intriligator in [8] as well as in [2]. In [15] we were able to verify this conjecture for all numbers $d(m,p,q)$. More recently Siebert and Tian [22] presented a proof covering the conjecture for all spaces of parameterized curves on a Grassmann variety. For readers interested in these connections we will give some more details at the end of Section 3.

The paper is structured as follows: In the next section we review the notion of an autoregressive system. This class of systems generalizes the class of transfer functions and it allows us to define the pole placement map using the behavioral approach to systems modeling as proposed by Willems [28, 29]. In this framework the points of the variety $K^q_{m,p}$ naturally parameterize all autoregressive compensators of a fixed number of inputs, outputs and a bounded McMillan degree. We also restate the main results derived in [17], which were in part the motivation of this paper. We conclude this section with two new theorems (Theorem 2.14 and Theorem 2.15) which sharpens the main results derived in [17].

The main theorem (Theorem 1.1) will be proven in Section 3. The proof involves the review of the generalized Pieri formula which was derived in [15]. In order to derive the new formula (1.5) describing the degree of the pole placement map in the critical dimension we will solve the partial recurrence relation mentioned earlier. This leads not only to a closed formula for the degree of the pole placement map in the critical dimension but also to a formula of the degree of some generalized Schubert varieties (Theorem 3.5). The section will be concluded with several simplified formulas covering particular situations.

In Section 4 we concentrate on the question for which triples m,p,q the degree $d(m,p,q)$ is odd respectively even. In Theorem 4.2 and Corollary 4.4 we present a relatively simple combinatorial procedure which computes the mod 2 degree of the variety $K^q_{m,p}$ for arbitrary m,p,q. Using this procedure we prove the existence of odd degrees even if $\min(m,p) \geq 3$, covering in this way many multi-input, multi-output feedback situations. (If $\min(m,p) \geq 3$ the degree of all Grassmann varieties is even. In part because of this there do not exist any positive pole placement results over $\mathbb{R}$ in the critical dimension, i.e. when $n = mp$). We conclude the section with a complete description of all odd numbers $d(m,p,q)$ for $q = 0,1,2$.

Finally in the last section we merge the derived results and provide a collection of corollaries and consequences. In this section we also cover situations when the plant is represented by a "traditional" strictly proper transfer function or when the compensator is supposed to be a proper transfer function only.

2. The set of autoregressive systems $A^q_{m,p}$, the projective variety $K^q_{m,p}$ and the pole placement map. In this section we collect some mathematical preliminaries and simultaneously establish our notation. We will develop the theory using the behavioral approach of Willems [28] because we believe that the problem formulation in this setting is very natural. For the relation of this formulation to the traditional transfer function formulation we refer to [17, 28, 29].

First we review the notion of signal space, behavior and autoregressive system. For this let $\mathbb{K}$ denote either the set of real numbers or the set of complex numbers, i.e. $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{K}^{\mathbb{R}}$ denote the set of all functions $\psi : \mathbb{R} \to \mathbb{K}$. With respect to the usual addition and scalar multiplication of functions, $\mathbb{K}^{\mathbb{R}}$ is a real vector space. A linear subspace $\mathcal{H} \subset \mathbb{K}^{\mathbb{R}}$ which consists of functions which are arbitrarily many times differentiable will be called a signal space (see [3, 27]). In other words $\mathcal{H}$ is a linear subspace which is invariant under the linear transformation $\frac{d}{dt}$. Usually we will assume that $\mathcal{H} = C^\infty(\mathbb{R}, \mathbb{K})$, though other function spaces are well possible. (Compare with [3, page 76] and [28].)

Let $p(s)$ be a polynomial with coefficients in $\mathbb{K}$, i.e. $p(s) \in \mathbb{K}[s]$. Such a polynomial induces a linear transformation $\hat{p} : \mathcal{H} \to \mathcal{H}$, $w(t) \mapsto p(\frac{d}{dt})w(t)$. More generally consider a $p \times k$ polynomial matrix $P(s)$ with entries in $\mathbb{K}[s]$. $P(s)$ induces a linear transformation

$$\begin{aligned} \hat{P} : \quad \mathcal{H}^k &\longrightarrow \mathcal{H}^p \\ w(t) &\longmapsto P(\frac{d}{dt})w(t). \end{aligned} \tag{2.1}$$

Using the language of Willems [28] we call the kernel of the linear transformation $\hat{P}$ the behavior and will denote this subset of the signal set $\mathcal{H}^k$ by $\mathcal{B}$.

In general the behavior $\mathcal{B} = \ker(P(\frac{d}{dt}))$ is an infinite dimensional $\mathbb{R}$-vector space of the signal space $\mathcal{H}^k$. In the case that $P(s)$ is square and invertible it is however well known that the behavior $\mathcal{B}$ has real dimension $n = \deg\det P(s)$. Moreover the dynamics of this autonomous system is described by the roots of the characteristic polynomial $\det P(s) = 0$.

Recall that two $p \times k$ polynomial matrices $P(s)$ and $\tilde{P}(s)$ are called (row) equivalent if there is a unimodular matrix $U(s)$ with $\tilde{P}(s) = U(s)P(s)$. Clearly row equivalent matrices define the same behavior. On the other hand if the signal space is sufficiently rich, e.g. if $C^\infty(\mathbb{R}, \mathbb{R}) \subset \mathcal{H}$, one has the following result: (Compare with [3, Section 6.2] and [10, Theorem 3.9].)

LEMMA 2.1. *(cf. [19, Corollary 2.5]) If $C^\infty(\mathbb{R}, \mathbb{R}) \subset \mathcal{H}$ then $P(s)$ and $\tilde{P}(s)$ define the same behavior if, and only if they are row equivalent.*

Based on this result we define:

DEFINITION 2.2. An equivalence class of full rank $p \times k$ polynomial matrices is called an autoregressive system.

The class of autoregressive systems generalizes the class of transfer functions in the following way: Consider a proper or improper $p \times m$ transfer function $G(s)$. Assume $G(s)$ has a left (polynomial) coprime factorization $D^{-1}(s)N(s) = G(s)$. If $\tilde{D}^{-1}(s)\tilde{N}(s) =$

$G(s)$ is a second left coprime factorization then it is well known that the $p \times (m+p)$ polynomial matrices $(N(s)\ D(s))$ and $(\tilde{N}(s)\ \tilde{D}(s))$ are row equivalent. In other words $(N(s)\ D(s))$ defines an autoregressive system.

The following definition extends the notion of McMillan degree to the class of autoregressive systems:

DEFINITION 2.3. [17, 26, 28] The degree of an autoregressive system $P(s)$ is given by the maximal degree of the full size minors of $P(s)$.

Next we would like to introduce feedback. For this consider a $p \times (m+p)$ autoregressive system $P(s)$ (the plant) and an $m \times (m+p)$ autoregressive system $C(s)$ (the compensator). The closed loop system is then the dynamical system described through the system of autoregressive equations

$$\begin{pmatrix} P(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{pmatrix} \cdot w(t) \;=\; 0. \tag{2.2}$$

Note that the square polynomial matrix $\begin{pmatrix} P(s) \\ C(s) \end{pmatrix}$ is in general not of full rank, i.e. (2.2) describes not an autoregressive system as defined in Definition 2.2. In order to single out the compensators which give rise to a closed loop autoregressive system we define (compare with [20]):

DEFINITION 2.4. A compensator $C(s)$ is called admissible if the closed loop characteristic polynomial

$$\phi(s) := \det \begin{pmatrix} P(s) \\ C(s) \end{pmatrix} \;\neq\; 0. \tag{2.3}$$

We are now in a position to define the pole placement map. Let $P(s)$ be a $p \times (m+p)$ autoregressive system of McMillan degree n and denote with $A^q_{m,p}$ the set of all $m \times (m+p)$ autoregressive systems of McMillan degree at most q. Let $B_P \subset A^q_{m,p}$ be the set of autoregressive systems which are not admissible compensators. Finally identify the set of nonzero polynomials of degree at most d with the projective space $\mathbb{P}^d$. Then define:

DEFINITION 2.5. The pole placement map for a plant $P(s)$ is defined as the rational map given by:

$$\begin{array}{rrcl} \rho_P \;: & A^q_{m,p} - B_P & \longrightarrow & \mathbb{P}^{n+q} \\ & C(s) & \longmapsto & \phi(s) = \det \begin{pmatrix} P(s) \\ C(s) \end{pmatrix} \end{array} \tag{2.4}$$

We want to note at this point that the roots of $\phi(s)$ do not depend on the particular representation of the plant $P(s)$ or the compensator $C(s)$. Indeed if $\tilde{P}(s) = U_1(s)P(s)$ and $\tilde{C}(s) = U_2(s)C(s)$ then $\tilde{\phi}(s) = \det U_1(s) \cdot \det U_2(s) \cdot \phi(s)$. Finally the roots of $\phi(s)$ correspond to the poles of the closed loop system in the transfer function formulation. (See [17] for details).

For a given plant $P(s)$ we usually say that $P(s)$ is pole assignable (almost pole assignable) in the class of feedback compensators of degree at most q if the map ρ_P is onto (almost onto). Though many results are known when a system is pole assignable in the class of feedback compensators of order at most q over the complex numbers $\mathbb{C}$ [17], the question is still far from being solved over the reals and in the ungeneric situation. (Compare with [28]). Clearly a necessary condition for pole assignability is the following property:

DEFINITION 2.6. ([26, 28]) An autoregressive system $P(s)$ is called controllable or irreducible if the matrix $P(s)$ is of full row rank for all $s \in \mathbb{C}$.

Indeed if the system $P(s)$ is not controllable, the full size minors of $P(s)$ have a common factor which is necessarily a factor of the closed loop characteristic polynomial $\phi(s)$. Clearly, even if $P(s)$ is controllable one cannot expect that $P(s)$ is pole assignable in the set of feedback compensators of degree at most q. The following definition singles out an interesting class of systems which has the pole assignability property in the critical dimension (i.e. when $\dim A^q_{m,p} = \dim \mathbb{P}^{n+q}$) over the complex numbers.

DEFINITION 2.7. ([17]) A plant $P(s)$ is called q-nondegenerate if all compensators $C(s)$ of order at most q are admissible. To put it in other words $P(s)$ is q-nondegenerate if the set B_p introduced in (2.4) is empty.

In the last part of this section we establish the connection to our earlier work in [17, 26]. First we would like to point out the following observation. The pole placement map ρ_P as introduced in (2.4) depends actually only on the full size minors of $P(s)$ and $C(s)$. In other words if $C(s)$ and $\tilde{C}(s)$ have the same full size minors, then the resulting closed loop characteristic polynomial $\rho_P(C(s))$ and $\rho_P(\tilde{C}(s))$ have the same roots. Based on this fact we assign to each autoregressive system $C(s) \in A^q_{m,p}$ its full size minors, i.e. we consider the following Plücker map:

$$\begin{aligned} \pi \ : \qquad A^q_{m,p} &\longrightarrow \mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p}) \\ C(s) &\longmapsto c_1(s) \wedge \ldots \wedge c_m(s). \end{aligned} \tag{2.5}$$

Here $c_l(s)$ denotes the l-th row vector of the $m \times (m+p)$ matrix $C(s)$. Of course when describing the map π with respect to the standard basis

$$\{e_{i_1} \wedge \ldots \wedge e_{i_m} \mid 1 \le i_1 < \cdots < i_m \le m+p\} \tag{2.6}$$

it is well known that the coordinates are exactly the full size minors of the matrix $C(s)$. In particular the map π is well defined.

In the following, whenever we will work with coordinates, we will assume the standard basis (2.6). More specifically if

$$\pi(C(s)) = \sum_{i \in I(m)} f_i(s) \cdot e_{i_1} \wedge \ldots \wedge e_{i_m} \tag{2.7}$$

we will use the coordinates

$$f_i(s) = z_{(i;q)} s^q + z_{(i;q-1)} s^{q-1} + \cdots + z_{(i;0)}. \tag{2.8}$$

The map π is in general not an embedding as it is the case for the classical Plücker embedding (the case $q=0$). Indeed as shown in [17] $\pi(C(s))=\pi(\tilde{C}(s))$ if, and only if the matrices $C(s)$ and $\tilde{C}(s)$ are H-equivalent. (See [17] for details). On the other hand if $C(s)$ and $\tilde{C}(s)$ are both controllable (see Definition 2.6) then $C(s)$ and $\tilde{C}(s)$ are H-equivalent if, and only if they are row equivalent. The following Lemma summarizes those statements.

LEMMA 2.8. *π restricted to the set of controllable autoregressive systems is an embedding, in particular π is generically one-one.*

From the earlier remarks it is clear that the pole placement map ρ_P factors over the image of π. We introduce therefore the following notation:

DEFINITION 2.9. $K^q_{m,p}$ denotes the image of $A^q_{m,p}$ under the map π.

By definition the set $K^q_{m,p}$ is a subset of the projective space

$$\mathbb{P}^N := \mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p}).$$

Note that the Plücker coordinates $\{f_i(s)\}$ introduced in (2.8) satisfy a set of quadratic relations coming from the description of Grass$(m,m+p)$ in $\mathbb{P}^{\binom{m+p}{m}-1}$ [7, p. 65]. Those relations must hold for all $s \in \mathbb{K}$. Equating coefficients one gets a necessary set of quadratic relations for the coordinates $z_{(i;d)}$ as well. The following Theorem states that those relations define $K^q_{m,p}$.

THEOREM 2.10 ([17]). *$K^q_{m,p}$ is a projective (sub)-variety of $\mathbb{P}^N$. The defining relations are given by a set of homogeneous quadratic polynomials obtained from equating the coefficients in the Plücker relations. The variety $K^q_{m,p}$ is in general singular and has dimension $q(m+p)+mp$.*

The following example explains the situation.

EXAMPLE 2.11. ([16]) The only Plücker relation of Grass$(2,4)$ in $\mathbb{P}^5$ is given by

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0. \tag{2.9}$$

Let $f_{ij}(s) = z_{(ij;1)}s + z_{(ij;0)}$ and

$$f_{12}(s)f_{34}(s) - f_{13}(s)f_{24}(s) + f_{14}(s)f_{23}(s) = 0, \tag{2.10}$$

we then have three quadratic equations

$$\begin{aligned}
z_{(12;1)}z_{(34;1)} - z_{(13;1)}z_{(24;1)} + z_{(14;1)}z_{(23;1)} &= 0\\
z_{(12;1)}z_{(34;0)} - z_{(13;1)}z_{(24;0)} + z_{(14;1)}z_{(23;0)} + z_{(12;0)}z_{(34;1)} - z_{(13;0)}z_{(24;1)} + z_{(14;0)}z_{(23;1)} &= 0\\
z_{(12;0)}z_{(34;0)} - z_{(13;0)}z_{(24;0)} + z_{(14;0)}z_{(23;0)} &= 0
\end{aligned}$$

which defines the projective variety $K^1_{2,2}$ in $\mathbb{P}^{11}$. Because $\dim K^1_{2,2} = 8$ it follows that $K^1_{2,2}$ is a complete intersection and by Bézout's theorem [7] the degree is equal to $2^3 = 8$.

As we can describe the compensator $C(s)$ through the vector

$$c(s) = c_1(s) \wedge \ldots \wedge c_m(s) \tag{2.11}$$

we can describe the plant $P(s)$ through the vector

$$p(s) = p_1(s) \wedge \ldots \wedge p_p(s). \tag{2.12}$$

Finally the closed loop characteristic polynomial is given through the linear pairing

$$(2.13) \qquad < p(s), c(s) >:= c_1(s) \wedge \ldots \wedge c_m(s) \wedge p_1(s) \wedge \ldots \wedge p_p(s) = \phi(s).$$

Note that the linear pairing $<,>$ originally defined on $K^n_{p,m} \times K^q_{m,p}$ extends linearly to the product space $\mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p}) \times \mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p})$.

Next we show that the pole placement map ρ_P induces a central projection in the projective space $\mathbb{P}^N = \mathbb{P}(\mathbb{K}^{q+1} \otimes \wedge^m \mathbb{K}^{m+p})$. For this consider a fixed plant $P(s)$ represented through the vector $p(s) = p_1(s) \wedge \ldots \wedge p_p(s)$. Consider the subspace

$$(2.14) \qquad E_P := \{c(s) | < p(s), c(s) > \equiv 0\} \subset \mathbb{P}^N.$$

Then one has a central projection (compare with [17, 25])

$$(2.15) \qquad \begin{array}{rcl} L_P \ : \ \mathbb{P}^N - E_P & \longrightarrow & \mathbb{P}^{n+q} \\ f(s) & \longmapsto & < g(s), f(s) > \end{array}$$

Let χ_P be the restriction map $L_P \mid_{(K^q_{m,p} - E_P)}$, i.e.

$$(2.16) \qquad \chi_P \ : \ K^q_{m,p} - E_P \longrightarrow \mathbb{P}^{n+q}.$$

The next Lemma explains the relation between the maps χ_P, L_P and the pole placement map ρ_P.

LEMMA 2.12. *The pole placement map ρ_P introduced in* (2.4) *factors over the variety $K^q_{m,p}$ through*

$$(2.17) \qquad \rho_P = L_P \circ \pi.$$

The map ρ_P is onto (almost onto) if, and only if χ_P is. Finally a plant $P(s)$ is q-nondegenerate if, and only if $K^q_{m,p} \cap E_P = \emptyset$.

Proof. From the definition of the linear pairing $<,>$ it is clear that $\rho_P = L_P \circ \pi$. Moreover because

$$(2.18) \qquad \pi : A^q_{m,p} \to K^q_{m,p}$$

is onto the second statement follows. Finally if $P(s)$ is q-degenerate there is a compensator $C(s) \in A^q_{m,p}$ which is not admissible. But this is equivalent to the statement

$$(2.19) \qquad c_1(s) \wedge \ldots \wedge c_m(s) \in K^q_{m,p} \cap E_P.$$

□

This Lemma will allow us to study the pole assignment problem completely in the projective space $\mathbb{P}^N$. In the geometric picture the set $K^q_{m,p} \cap E_P$ will be of crucial importance. Note that

$$\pi(B_P) = K^q_{m,p} \cap E_P.$$

By abuse of notation we will denote $K^q_{m,p} \cap E_P$ with B_P as well, the last Lemma justifies this choice. The set B_P is sometimes called the base locus of the central projection χ_P

and by the last Lemma this set is empty if, and only if the plant $P(s)$ is q-nondegenerate. The following theorem gives the result which mainly motivated this paper.

THEOREM 2.13 ([17]). *For a q-nondegenerate system of McMillan degree $n = q(m+p-1)+mp$ the pole assignment map χ_P is onto over $\mathbb{C}$ and there are $\deg K^q_{m,p}$ (counted with multiplicity) complex dynamic compensators assigning each set of poles. In particular, a real solution always exits if $\deg K^q_{m,p}$ is odd.*

Proof. Since $B_P = \emptyset$ the pole placement map $\chi_P : K^q_{m,p} \to \mathbb{P}^{n+q}$ is finite morphism [21, Chapter I, §5, Theorem 7]. Therefore χ_P is onto over $\mathbb{C}$ [21, Chapter I, §5, Theorem 4] and $\deg \chi_P = \deg K^q_{m,p}$ [13, Corollary (5.6)]. □

Actually we can strengthen this result:

THEOREM 2.14. *Let P be a system of degree $n < q(m+p-1)+mp$. If*

$$\dim B_P = \dim E_P \cap K^q_{m,p} = q(m+p)+mp-n-q-1 \tag{2.20}$$

then χ_P is onto over $\mathbb{C}$, and over $\mathbb{R}$ if $\deg K^q_{m,p}$ is also odd.

Proof. Let H be the $q(m+p)+mp-n-q$ codimensional projective subspace in $\mathbb{P}^N$ such that

$$B_P \cap H = \emptyset \tag{2.21}$$

(such H exists by [13, Corollary (2.29)]), $\pi_1 : K^q_{m,p} \to \mathbb{P}^{q(m+p)+mp}$ be the central projection with center $E_P \cap H$ and $\pi_2 : \mathbb{P}^{q(m+p)+mp} - \pi_1(B_P) \to \mathbb{P}^{n+q}$ be the central projection with center $\pi_1(E_P)$. Then π_1 is onto over $\mathbb{C}$, and over $\mathbb{R}$ if $\deg K^q_{m,p}$ is also odd, and so is

$$\chi_P = \pi_2 \circ \pi_1.$$

□

THEOREM 2.15. *The pole assignment map χ_P is onto over $\mathbb{C}$ for the generic system if and only if*

$$n \leq q(m+p-1)+mp. \tag{2.22}$$

This condition is also sufficient over $\mathbb{R}$ if $\deg K^q_{m,p}$ is odd.

Proof. The necessity was proven by Willems and Hesselink in [30]. On the other hand if $n = q(m+p-1)+mp$ the generic system is q-nondegenerate by [17, Corollary 5.6] and the sufficiency follows from Theorem 2.13. If $n < q(m+p-1)+mp$ then it follows for the generic system from [17, Theorem 5.5] that

$$\dim B_P = q(m+p)+mp-n-q-1. \tag{2.23}$$

By Theorem 2.14 the sufficiency follows. □

3. The subvarieties Z_α of the variety $K^q_{m,p}$ and a closed formula of their degrees. In this section we derive a closed formula for the degree of a set of generalized Schubert subvarieties of the variety $K^q_{m,p}$. As a corollary we will obtain a formula for the mapping degree of the pole placement map in the critical dimension. For the convenience of the reader we quickly review some geometric aspects of the classical Pieri formula (1.3). For this consider the index set

$$I = \{i = (i_1, \ldots, i_m) | 1 \leq i_1 < \cdots < i_m\}$$

equipped with the partial order

$$(i_1, \ldots, i_m) \leq (j_1, \ldots, j_m) \quad \Leftrightarrow \quad i_l \leq j_l \forall l. \tag{3.1}$$

If a m-dimensional plane $P \in \mathrm{Grass}(m, m+p) \subset \mathbb{P}(\wedge^m \mathbb{K}^{m+p})$ is expanded in terms of the standard basis (2.6), i.e. if P is represented through the vector

$$x := \sum_{i \in I} x_i \cdot e_{i_1} \wedge \ldots \wedge e_{i_m} \tag{3.2}$$

we will call the coordinates x_i the Plücker coordinates (see [7, p. 64]) of the plane P. The set

$$S_i := \{x \in \mathrm{Grass}(m, m+p) | x_j = 0 \text{ for all } j \not\leq i\} \tag{3.3}$$

is called a Schubert variety. Let H_i be the hyperplane defined by setting $x_i = 0$ and let $|i| := \sum_{l=1}^m (i_l - l)$. Then the geometric version of Pieri's formula states that

$$S_i \cap H_i = \bigcup_{\substack{j \in I \\ j < i,\ |j| = |i| - 1}} S_j \tag{3.4}$$

and that the intersection multiplicity along each S_j is 1. In terms of the intersection ring, S_i represents a Schubert cycle $(i_1, i_2, \ldots, i_m)$ and H_i represents the Schubert cycle $\sigma_1 := (p, p+2, p+3, \ldots, p+m)$ and the geometric intersection is expressed through a formal multiplication as given in (1.3). Readers who want to learn more about Schubert calculus are referred to the excellent survey article of Kleiman and Laksov [9].

In [15] we proved a similar formula as given in (3.4) for subvarieties of $K^q_{m,p}$. In order to explain this generalized Pieri formula we first re-index the coordinates $z_{(i;d)}$ of $K^q_{m,p}$.

Definition 3.1. For each $(i; d)$, $i = (i_1, \ldots, i_m)$, $1 \leq i_1 < \cdots < i_m \leq m+p$, let $\alpha := (\alpha_1, \ldots, \alpha_m)$ be defined through:

$$\alpha_l = \begin{cases} [d/m](m+p) + i_{l+d-m[d/m]} & \text{for } l = 1, 2, \ldots, m[d/m] + m - d \\ ([d/m]+1)(m+p) + i_{l+d-m[d/m]-m} & \text{for } l = m[d/m] + m - d + 1, \ldots, m. \end{cases}$$

Using this reindexing we can associate to every coordinate $z_{(i;d)}$ of $K^q_{m,p}$ a new coordinate z_α. The following example shows the relation between the indices $(i; d)$ and α.

$$\begin{aligned} z_{(i;0)} &= z_i, \\ z_{(i;1)} &= z_{(i_2, \ldots, i_m, i_1 + m + p)}, \\ z_{(i;2)} &= z_{(i_3, \ldots, i_m, i_1 + m + p, i_2 + m + p)}, \\ &\vdots \\ z_{(i;m)} &= z_{(i_1 + m + p, \ldots, i_m + m + p)}, \\ z_{(i;m+1)} &= z_{(i_2 + m + p, \ldots, i_m + m + p, i_1 + 2(m+p))}, \\ &\vdots \end{aligned}$$

Note that the indices α belong to the index set

$$\tilde{I} := \{\alpha \in I | \alpha_m - \alpha_1 < m + p\} \tag{3.5}$$

which is by definition a subset of the index set I. In particular $\tilde{I}$ is also equipped with a partial order. Using this partial order we can now define an interesting set of subvarieties of $K^q_{m,p}$:

DEFINITION 3.2.

$$Z_\alpha := \{z \in K^q_{m,p} | z_\beta = 0 \text{ for all } \beta \not\leq \alpha\} \tag{3.6}$$

The main results of [15] are summarized as follows:

PROPOSITION 3.3 ([15]). *For each index α Z_α is a subvariety of dimension $|\alpha|$. If H_α is the hyperplane of $\mathbb{P}^N$ defined by $z_\alpha = 0$ then*

$$Z_\alpha \cap H_\alpha = \bigcup_{\substack{\beta \in \tilde{I} \\ \beta < \alpha,\ |\beta| = |\alpha| - 1}} Z_\beta \tag{3.7}$$

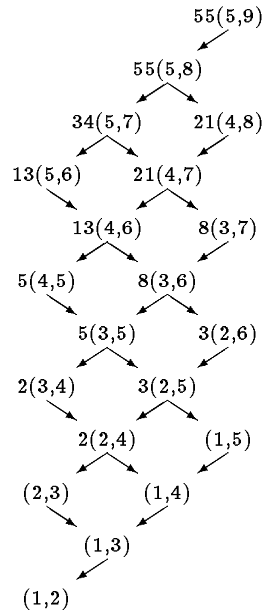
and the intersection multiplicity along each Z_β is 1.

Using Bézout's theorem ([7, Theorem 18.3]) the expression (3.7) translates into a partial recurrence relation, which the degrees of the varieties Z_α have to satisfy:

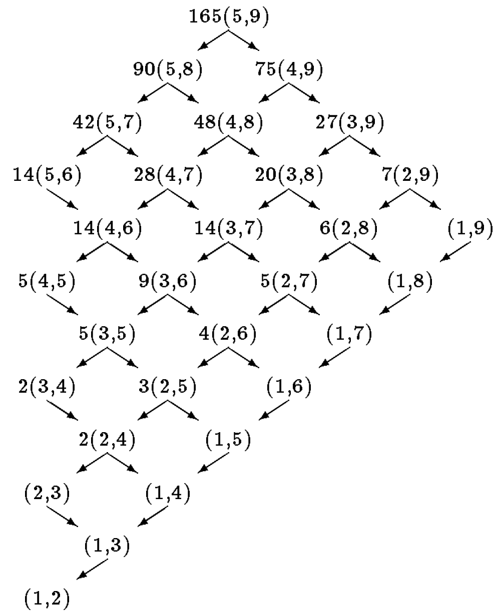
COROLLARY 3.4. *[15]*

$$\deg Z_\alpha = \sum_{\substack{\beta \in \tilde{I} \\ \beta < \alpha,\ |\beta| = |\alpha| - 1}} \deg Z_\beta. \tag{3.8}$$

The partial recurrence relation (3.8) has to be satisfied for the whole index set $\tilde{I}$. It is possible to depict this relation with the help of a Hasse diagram. A Hasse diagram corresponding to the variety Z_α is a directed graph, whose vertices are all $\beta \in \tilde{I}, \beta \leq \alpha$. The directed edges $\beta \rightarrow \gamma$ are precisely those ordered pairs such that β covers γ (i.e. $\beta > \gamma$ and $|\beta| = |\gamma| + 1$). Then according to Corollary 3.4, the degree of Z_α can be computed graphically in the following way: If we label the vertices in such a way that the number on $(1, 2, \ldots, m)$ is 1 and the number on β is the sum of the numbers on the vertices covered by β, then the number on α is $\deg Z_\alpha$. Following is an example of $Z_{(5,9)} = K^1_{2,3}$. Note that Rosenthal obtained $\deg K^1_{m,p} = 55$ by computing the coefficients of the Hilbert polynomial using the computer program CoCoA in [16]. For comparison we also include the Hasse diagram of the Schubert variety $S_{(5,9)}$, whose underlying diagram corresponds to all indices $i \in I, i \leq (5, 9)$.



Hasse Diagram of $Z_{(5,9)}$



Hasse Diagram of $S_{(5,9)}$

From above example one can see that the Hasse diagram of Z_α can be obtained by "cutting off" all the vertices of I which are not in $\tilde{I}$ in the Hasse diagram of S_α. If we use $d(\alpha_1, \ldots, \alpha_m)$ for the degree, then both $\deg Z_\alpha$ and $\deg S_\alpha$ satisfy the partial difference equation

$$d(\alpha_1, \ldots, \alpha_m) = \sum_{l=1}^{m} d(\alpha_1, \ldots, \alpha_l - 1, \ldots, \alpha_m) \tag{3.9}$$

subject to the initial condition

$$d(1, 2, \ldots, m) = 1 \tag{3.10}$$

and subject to the boundary conditions

$$d(0, \ldots, \alpha_m) = 0 \tag{3.11}$$

$$d(\ldots, k, k, \ldots) = 0. \tag{3.12}$$

$\deg Z_\alpha$ is subject to one more boundary condition, namely

$$d(k, \ldots, k + m + p) = 0. \tag{3.13}$$

The computation of the degrees of the varieties Z_α is therefore reduced to the solution of a partial difference equation with boundary conditions. The next theorem provides a closed formula for this problem:

Theorem 3.5.

$$\deg Z_\alpha = |\alpha|! \sum_{n_1 + \cdots + n_m = 0} \frac{\prod_{k<j} (\alpha_j - \alpha_k + (n_j - n_k)(m+p))}{\prod_j (\alpha_j + n_j(m+p) - 1)!} \tag{3.14}$$

with the convention that $1/k! = 0$ *if* $k < 0$.

Proof. Let $g(\alpha) = \deg S_\alpha$. Then (see [11, p. 103] and [23])

$$(3.15)\qquad g(\alpha) = |\alpha|! \frac{\prod_{k<j}(\alpha_j - \alpha_k)}{\prod_{j=1}^{m}(\alpha_j - 1)!} = |\alpha|! \det \begin{bmatrix} \frac{1}{(\alpha_1-1)!} & \frac{1}{(\alpha_1-2)!} & \cdots & \frac{1}{(\alpha_1-m)!} \\ \frac{1}{(\alpha_2-1)!} & \frac{1}{(\alpha_2-2)!} & \cdots & \frac{1}{(\alpha_2-m)!} \\ \vdots & \vdots & & \vdots \\ \frac{1}{(\alpha_m-1)!} & \frac{1}{(\alpha_m-2)!} & \cdots & \frac{1}{(\alpha_m-m)!} \end{bmatrix}$$

and (3.14) becomes

$$(3.16)\qquad \deg Z_\alpha = \sum_{n_1+\cdots+n_m=0} g(\alpha_1 + n_1(m+p), \ldots, \alpha_m + n_m(m+p)).$$

Let

$$d(\alpha) = \sum_{n_1+\cdots+n_m=0} g(\alpha_1 + n_1(m+p), \ldots, \alpha_m + n_m(m+p))$$

Then $d(\alpha)$ satisfies the equation (3.9) because $g(\alpha)$ does. Moreover since

$$d(\alpha_1, \ldots, \alpha_m) = g(\alpha_1, \ldots, \alpha_m)$$

for $\alpha_1 < \cdots < \alpha_m < m+p$, $d(\alpha)$ satisfies the condition (3.10) and (3.11) for $\alpha_m < m+p$. We only need to verify (3.12) and (3.13).

Notice that by (3.15)

$$(3.17)\qquad g(\ldots, \alpha_j, \ldots, \alpha_k, \ldots) = -g(\ldots, \alpha_k, \ldots, \alpha_j \ldots)$$

If $\alpha_j = \alpha_{j+1}$ then

$$\begin{aligned} &g(\ldots, \alpha_j + k_j(m+p), \alpha_{j+1} + k_{j+1}(m+p), \ldots) \\ &= -g(\ldots, \alpha_j + k_{j+1}(m+p), \alpha_{j+1} + k_j(m+p), \ldots). \end{aligned}$$

On the other hand if $\alpha_1 + m + p = \alpha_m$ then

$$\begin{aligned} &g(\alpha_1 + k_1(m+p), \ldots, \alpha_m + k_m(m+p)) \\ &= g(\alpha_m + (k_1 - 1)(m+p), \ldots, \alpha_1 + (k_m + 1)(m+p)) \\ &= -g(\alpha_1 + (k_m + 1)(m+p), \ldots, \alpha_m + (k_1 - 1)(m+p)). \end{aligned}$$

In either case $d(\alpha) = -d(\alpha)$; i.e. $d(\alpha) = 0$. □

Applying the formula (3.14) to $K^q_{m,p}$ we then have a formula for $\deg K^q_{m,p}$:

THEOREM 3.6.

$$(3.18)\qquad \begin{aligned} &\deg K^q_{m,p} \\ &= (-1)^{q(m+1)}(mp + q(m+p))! \sum_{n_1+\cdots+n_m=q} \frac{\prod_{k<j}(j-k+(n_j-n_k)(m+p))}{\prod_{j=1}^{m}(p+j+n_j(m+p)-1)!} \end{aligned}$$

Proof. Let $k = [q/m]$ and $r = q - km$. Then

$$K^q_{m,p} = Z_{(p+r+1+k(m+p),\dots,p+m+k(m+p),p+1+(k+1)(m+p),\dots,p+r+(k+1)(m+p))}.$$

So

$$\deg K^q_{m,p} = (\operatorname{sgn}\ \sigma) \sum_{n_1+\cdots+n_m=q} g(p+1+n_1(m+p),\dots,p+m+n_m(m+p))$$

where σ is the permutation

$$(r+1, r+2, \dots, m, 1, 2, \dots, r) \to (1, 2, \dots, m),$$

and the sign of this permutation is given through

$$\begin{aligned}\operatorname{sgn}\ \sigma &= (-1)^{r(m-r)} = (-1)^{(q-km)((k+1)m-q)} = (-1)^{2qkm+mq-q^2-k(k+1)m^2} \\ &= (-1)^{mq}(-1)^{q^2} = (-1)^{mq}(-1)^q.\end{aligned}$$

Therefore

$$\deg K^q_{m,p} = (-1)^{q(m+1)} \sum_{n_1+\cdots+n_m=q} g(p+1+n_1(m+p),\dots,p+m+n_m(m+p))$$

which is the formula (3.18). □

Combining Theorem 2.13, 2.15 and 3.6 we then have Theorem 1.1. □

We conclude this section with several simplified formulas. First recall the definition of the *Fibonacci* numbers given through the recurrence relation $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n > 1$. From Corollary3.4 it follows immediately that

$$\deg K^q_{2,3} = f_{5q+5}. \tag{3.19}$$

Using a well known expression for the Fibonacci sequence we therefore get

$$\deg K^q_{2,3} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{5(q+1)} - \left(\frac{1-\sqrt{5}}{2}\right)^{5(q+1)}\right). \tag{3.20}$$

Note that formula (3.20) has also been given by Intriligator [8, p. 3554] as an illustration of the conjectured intersection numbers arising from some computation in conformal quantum field theory. For $q = 1$ we get again $\deg K^1_{2,3} = 55$ (compare with the Hasse diagram of $Z_{(5,9)}$) and for $q = 2$ we get $\deg K^2_{2,3} = 610$.

In general, for $m = 2$, formula (3.18) can be simplified to

$$\deg K^q_{2,p} = (-1)^q (q(p+2)+2p)! \sum_{j=0}^{q} \frac{(q-2j)(p+2)+1}{(p+j(p+2))!(p+1+(q-j)(p+2))!}.$$

In order to illustrate "the nonlinear character" of the pole placement map we derived a table showing all degrees of the variety $K^q_{2,p}$ for $p = 1,\dots,9$ and $q = 0,\dots,5$:

$p\backslash q$	0	1	2	3	4	5
1	1	1	1	1	1	1
2	2	8	32	128	512	2048
3	5	55	610	6765	75025	832040
4	14	364	9842	265720	7174454	193710244
5	42	2380	147798	9112264	562110290	34673583028
6	132	15504	2145600	290926848	39541748736	5372862566400
7	429	100947	30664890	8916942687	2610763825782	763562937059280
8	1430	657800	435668420	266668876540	165745451110910	102703589621825280
9	4862	4292145	6186432967	7853149169635	10262482704258873	13319075453502743045

4. Odd or even degrees. In this section we introduce some methods which can be used to determine whether the deg $K^q_{m,p}$ is odd or even without computing the degree itself.

For $\mathrm{Grass}(m, m+p) = K^0_{m,p}$, a well known fact is that $\deg \mathrm{Grass}(m, p+m)$ is even whenever $\min(m,p) \geq 3$ [1]. This is not the case for general $K^q_{m,p}$, i.e. in a certain sense there are many more odd numbers for a fixed $q > 0$ than there are for $q = 0$.

The main result of this section is Theorem 4.2 which provides a short combinatorial description of all triples m, p, q which result in an odd degree. Using this theorem we derive several corollaries classifying the odd and even degree varieties.

In order to prepare for the main theorem we first rewrite formula 3.18:

$$\deg K^q_{m,p} = (-1)^{q(m+1)}(mp+q(m+p))! \sum_{n_1+\cdots+n_m=q} (-1)^{\frac{m(m-1)}{2}}$$

$$\det \begin{bmatrix} \frac{1}{(p-m+1+n_1(m+p))!} & \frac{1}{(p-m+2+n_1(m+p))!} & \cdots & \frac{1}{(p+n_1(m+p))!} \\ \frac{1}{(p-m+2+n_2(m+p))!} & \frac{1}{(p-m+3+n_2(m+p))!} & \cdots & \frac{1}{(p+1+n_2(m+p))!} \\ \vdots & \vdots & & \vdots \\ \frac{1}{(p+n_m(m+p))!} & \frac{1}{(p+1+n_m(m+p))!} & \cdots & \frac{1}{(p+m-1+n_m(m+p))!} \end{bmatrix} \tag{4.1}$$

$$= (-1)^{q(m+1)}(-1)^{\frac{m(m-1)}{2}} \sum_{n_1+\cdots+n_m=q} \sum_{\sigma} \mathrm{sgn}\ \sigma$$

$$\frac{(mp+q(m+p))!}{(p-m+\sigma(1)+n_1(m+p))!\cdots(p-1+\sigma(m)+n_m(m+p))!}. \tag{4.2}$$

Note that $\sum_{i=1}^m((p-m-1)+i+\sigma(i)+n_i(m+p)) = mp+q(m+p)$. It therefore follows that every summand in the expression (4.2) is a multinomial coefficient

$$\binom{k}{k_1,\ldots,k_m} := \frac{(k_1+k_2+\cdots+k_m)!}{k_1!k_2!\cdots k_m!}. \tag{4.3}$$

For multinomial coefficients there is a well known criterion frequently used by topologists which guarantees that such a coefficient is odd. We formulate this criterion as a lemma.

Lemma 4.1. *The multinomial coefficient* $\binom{k}{k_1,\ldots,k_m}$ *is odd if and only if there are no "carry overs" in the summation* $k_1+\cdots+k_m$ *when calculated using binary representation.*

Proof. Let

$$k = 2^{n_1} + \cdots + 2^{n_l},\ 0 \leq n_1 < \cdots < n_l.$$

Then

$$(x_1+\cdots+x_m)^k = \prod_{i=1}^{l}(x_1^{2^{n_i}} + \cdots + x_m^{2^{n_i}}) \mod 2.$$

□

In particular it follows from this Lemma that $\binom{k}{k_1,\ldots,k_m}$ is even as soon as two numbers among $\{k_1,\ldots,k_m\}$ are equal or two numbers are odd.

We will call a set $\{k_1, k_2, \ldots, k_m\}$ of positive integers *a disjoint binary partition of k* if the multinomial coefficient $\binom{k}{k_1,\ldots,k_m}$ is odd. To put it in other words $\{k_1, k_2, \ldots, k_m\}$ is a disjoint binary partition of k if $k_1+k_2+\cdots+k_m = k$, and if their binary representations

$$k_i = 2^{n_{i1}} + 2^{n_{i2}} + \cdots + 2^{n_{ir_i}},\ 0 \le n_{i1} < n_{i2} < \cdots < n_{ir_i}, \quad i = 1, 2, \ldots, m$$

have disjoint exponents; i.e. $n_{ij} \ne n_{rs}$ for all i, j, r, s.

THEOREM 4.2. *Let $a = \min(m, p)$. Then $\deg K^q_{m,p}$ is odd if and only if the number of disjoint binary partitions $\{k_1, \ldots, k_a\}$ of $q(m+p)+mp$ having the property that*

$$\{k_1, \ldots, k_a\} = \{m+p-1, m+p-3, \ldots, m+p-2a+1\} \mod m+p$$

is odd.

Before we give the proof we will illustrate Theorem 4.2 on several examples.

EXAMPLE 4.3.

a. $m = 2,\ p = 9,\ q = 4,\ q(m+p)+mp = 62 = 2+2^2+2^3+2^4+2^5$. The disjoint binary partitions equal to $\{10, 8\}$ mod 11 are:

$$\{2^5, 2+2^2+2^3+2^4\} = \{32, 30\},$$
$$\{2+2^2+2^4+2^5, 2^3\} = \{54, 8\},$$
$$\{2+2^3, 2^2+2^4+2^5\} = \{10, 52\}.$$

So $\deg K^4_{2,9} = \deg K^4_{9,2}$ is odd.

b. $m = 3,\ p = 4,\ q = 5,\ q(m+p)+mp = 47 = 1+2+2^2+2^3+2^5$. The disjoint binary partitions equal to $\{6, 4, 2\}$ mod 7 are:

$$\{2+2^2, 2^5, 1+2^3\} = \{6, 32, 9\},$$
$$\{2+2^5, 2^2, 1+2^3\} = \{34, 4, 9\},$$
$$\{1+2^2+2^3, 2^5, 2\} = \{13, 32, 2\},$$
$$\{1+2^3+2^5, 2^2, 2\} = \{41, 4, 2\}.$$

So $\deg K^5_{3,4} = \deg K^5_{4,3}$ is even.

c. $m = 3,\ p = 6,\ q = 3,\ q(m+p)+mp = 45 = 1+2^2+2^3+2^5$. There is only one disjoint binary partition equal to $\{8, 6, 4\}$ mod 9:

$$\{2^3, 1+2^5, 2^2\} = \{8, 33, 4\}$$

So $\deg K^3_{3,6} = \deg K^3_{6,3}$ is odd.

d. $m = 5,\ p = 6,\ q = 3,\ q(m+p)+mp = 63 = 1+2+2^2+2^3+2^4+2^5$. There is only one disjoint binary partition equal to $\{10, 8, 6, 4, 2\}$ mod 11:

$$\{2^5, 2^3, 1+2^4, 2^2, 2\} = \{32, 8, 17, 4, 2\}$$

So $\deg K^3_{5,6} = \deg K^3_{6,5}$ is odd.

Proof. Without loss of generality, assume $m \leq p$. Consider again the description of the degree of the variety $K^q_{m,p}$ as it was provided in formula (4.2). It is our goal to show that in the summation mod 2 the only relevant permutation is $\sigma = id$. In other words we will show by a clever "book keeping" that all other multinomial coefficients are either 0 or cancel each other.

First assume σ is not an idempotent, i.e. $\sigma^2 \neq id$ or $\sigma \neq \sigma^{-1}$. In this case one immediately verifies that the sets

$$\{((p-m-1)+i+\sigma(i)+n_i(m+p)) \mid i=1,\ldots,m\}$$

and

$$\{((p-m-1)+i+\sigma^{-1}(i)+n_{\sigma^{-1}(i)}(m+p)) \mid i=1,\ldots,m\}$$

are equal as unordered sets. But this just means that the corresponding multinomial coefficients in the summation (4.2) cancel each other mod 2. So it follows that we only have to sum over idempotent permutations.

Assume therefore that $\sigma^2 = id$. If $\sigma \neq id$ then σ contains a pure transposition, i.e. there are two distinct integers a, b having the property that $\sigma(a) = b$ and $\sigma(b) = a$. Consider the ordered set

$$(((p-m-1)+i+\sigma(i)+n_i(m+p)) \mid i=1,\ldots,m).$$

If $n_a = n_b$ then the corresponding multinomial coefficient is zero since two numbers in the binary partition (namely the numbers at positions a and b) are equal. On the other hand if $n_a \neq n_b$ then the corresponding multinomial coefficient cancels with the multinomial coefficient obtained by interchanging n_a and n_b.

It therefore follows that the only relevant summand in (4.2) is $\sigma = id$. The mod 2 degree of $K^q_{m,p}$ reduces therefore to the evaluation of the mod 2 sum of

$$\sum_{n_1+\cdots+n_m=q} \binom{(mp+q(m+p))}{p-m+1+n_1(m+p), p-m+3+n_2(m+p), \ldots, p+m-1+n_m(m+p)}. \tag{4.4}$$

Since a summand

$$\binom{(mp+q(m+p))}{p-m+1+n_1(m+p), p-m+3+n_2(m+p), \ldots, p+m-1+n_m(m+p)}$$

is odd if and only if $\{p-m+1+n_1(m+p), \ldots, p+m-1+n_m(m+p)\}$ is a disjoint binary partition of $mp+q(m+p)$, the $\deg K^q_{m,p}$ is odd if and only if the number of disjoint binary partitions equal to $\{p-m+1, p-m+3, \ldots, p+m-1\}$ mod $m+p$ of $q(m+p)+mp$ is odd. □

For the Grassmann variety it is possible to identify the first Chern class c_1 (respectively the first Stiefel Whitney class w_1) of the classifying bundle with the the first elementary symmetric function

$$x_1+\cdots+x_m \in \mathcal{Z}[x_1,\ldots,x_m].$$

The degree (respectively the mod 2 degree) of the Grassmann variety is then represented through the coefficient of a certain monom (see [23] for details) in the expansion of

$$(x_1 + \cdots + x_m)^{\dim \mathrm{Grass}(m,m+p)}.$$

For the mod 2 degree of the variety $K^q_{m,p}$ Theorem 4.2 gives a way to do a similar computation. For this consider the polynomial ring $\mathcal{Z}_2[x_1, \ldots, x_m]$, the ideal

$$I := \left\langle x_1^{m+p} - 1, \ldots, x_m^{m+p} - 1 \right\rangle$$

and the factor ring $R := \mathcal{Z}_2[x_1, \ldots, x_m]/I$. Then we have:

COROLLARY 4.4. *If $m \le p$ the mod 2 degree of the variety $K^q_{m,p}$ is equal to the coefficient of the monom $x_1^{m+p-1} x_2^{m+p-3} \ldots x_m^{p-m+1}$ in the expansion of*

$$(x_1 + \cdots + x_m)^{\dim K^q_{m,p}} \in R.$$

Proof. From the proof of Theorem 4.2 it follows that the mod 2 degree of $K^q_{m,p}$ is equal to the sum of certain multinomial coefficients of the form

$$\binom{\dim K^q_{m,p}}{k_1, \ldots, k_m} = \binom{q(m+p) + mp}{k_1, \ldots, k_m}.$$

Since the mod $(m+p)$ identification of disjoint binary partitions in Theorem 4.2 corresponds to the ideal theoretic identification of monoms in the factor ring R the total mod 2 number of identified monoms is exactly the mod 2 degree of $K^q_{m,p}$. □

In practice one often can use the "freshman's dream"

$$(x_1 + \cdots + x_m)^{2^k} = x_1^{2^k} + \cdots + x_m^{2^k} \mod 2.$$

The following examples illustrate the corollary.

EXAMPLE 4.5.

a. $m = 2$, $p = 3$, $q = 3$. In this case $\dim K^3_{2,3} = 21$ and we know from the table at the end of section 3 that $\deg K^3_{2,3} = 6765$. Using the corollary we compute

$$(x + y + z)^{21} = (x^{16} + y^{16} + z^{16})(x + y + z)^5 = (x^2 + y^2 + z^2)^3.$$

Since the coefficient in front of the monom $x^4 y^2$ is indeed 1 we conclude once more that $K^3_{2,3}$ is of odd degree.

b. $m = 3$, $p = 4$, $q = 69$, $q(m+p) + mp = 495 = 1 + 2 + 2^2 + 2^3 + 2^5 + 2^6 + 2^7 + 2^8$. Since the dimension is quite large we reduce mod 7 already in the first step:

$$(1, 2, 2^2, 2^3, 2^5, 2^6, 2^7, 2^8) = (1, 2, 4, 1, 4, 1, 2, 4) \mod 7.$$

Using this reduction we have:

$$\begin{aligned}(x + y + z)^{495} &= (x + y + z)^3 (x^2 + y^2 + z^2)^2 (x^4 + y^4 + z^4)^3 \\ &= (x + y + z)(x^4 + y^4 + z^4).\end{aligned}$$

Since there is no monom $x^6 y^4 z^2$ in this expansion we conclude that $K^{69}_{3,4}$ and $K^{69}_{4,3}$ both have an even degree.

COROLLARY 4.6. *Let* $\min(m,p)>1$. *Then any of the following conditions implies that* $\deg K^q_{m,p}$ *is even.*

a) $m+p$ *is even.*

b) $mp+3>(m+p)(q+2)$.

c) $\min(m,p)\geq q+2+\sqrt{q^2+4q+1}$.

d) The binary number of $q(m+p)+mp$ *has less than* m *1's besides the digit on the* 2^0 *position.*

e) $2^{\min(m,p)+1}>q(m+p)+mp+2$.

f) $\sum_{i=1}^{l} r_i<mp$ *where* $r_i\in[0,m+p)$ *is the number equals the* 2^{n_i} *mod* $m+p$ *in the binary representation* $q(m+p)+mp=2^{n_1}+\cdots+2^{n_l}$.

g) $m+p=2^k-1$.

Proof. Without loss of generality assume that $m\leq p$.

a) When $m+p$ is even, all the integers $p-m+1+n_1(m+p),\ldots,p+m-1+n_m(m+p)$ are odd. By the remark after Lemma 4.1 all multinomial coefficients appearing in (4.4) are even.

b) Let $2^r\leq q(m+p)+mp<2^{r+1}$. Then a necessary condition for $\{p-m+1+n_1(m+p),\ldots,p+m-1+n_m(m+p)\}$ to be a disjoint binary partition is

$$p-m+2i-1+n_i(m+p)\geq 2^r$$

for some i. In particular

$$p+m-1+q(m+p)\geq 2^r\geq(1/2)(q(m+p)+mp+1)$$

which implies

$$(m+p)(q+2)\geq mp+3.$$

c) Consider $-(m^2-2(q+2)m+3)$. It has two roots: $q+2\pm\sqrt{q^2+4q+1}$. So when $m\geq q+2+\sqrt{q^2+4q+1}$,

$$-(m^2-2(q+2)m+3)\leq 0.$$

The degree is even if $m=p$ by a). If $m<p$ (note that $q+2-m<0$),

$$\begin{aligned}(m+p)(q+2)-mp-3&=-(m^2-2(q+2)m+3)+(q+2-m)(p-m)\\&<-(m^2-2(q+2)m+3)\\&\leq 0.\end{aligned}$$

So c) implies b).

d) Under the condition, $q(m+p)+mp$ can not have a disjoint binary partition $\{k_1,\ldots,k_m\}$ such that none of the k_i is 1.

e) The smallest number such that d) is not satisfied is $2^{m+1}-2$. So e) implies d).

f) A necessary condition for

$$(k_1,\ldots,k_m)=(p-m+1,\ldots,p+m-1)\mod m+p,\quad k_i>0$$

is $\sum_{i=1}^m k_i\geq mp$.

g) Notice that $m+p$ is odd. So $2 \le m < p$. For any $n \ge k$, let $n = ak + r$, $0 \le r < k$. Then

$$2^n = 2^r(1 + 2^k + \cdots + 2^{k(a-1)})(2^k - 1) + 2^r.$$

So $2^n = 2^r \bmod m+p$. Let the binary representation of $q(m+p)+mp$ be $2^{n_1} + 2^{n_2} + \cdots + 2^{n_l}$ and consider

$$\prod_{i=1}^{l}(x_1^{2^{n_i}} + \cdots + x_m^{2^{n_i}}).$$

By replacing 2^{n_i} with 2^{r_i} for $r_i = n_i \bmod k$, $r_i \in [0,k)$, and using the property

$$(x_1^{2^r} + \cdots + x_m^{2^r})^2 = (x_1^{2^{r+1}} + \cdots + x_m^{2^{r+1}}) \mod 2$$

repeatedly, one ends up in

$$\prod_{i=1}^{j}(x_1^{2^{r_i}} + \cdots + x_m^{2^{r_i}}) \tag{4.5}$$

with $\{r_1, \ldots, r_j\} \subset [0,k)$ distinct. The polynomial (4.5) has degree at most $1 + 2 + \cdots + 2^{k-1} = m+p$ which is always less than mp under the condition $2 \le m < p$. By the same argument as in the proof of f), the degree is even.

□

An immediate corollary of Theorem 4.2 is the result of [1]: $\deg \mathrm{Grass}(m, m+p)$ is odd if and only if

1. $\min(m,p) = 1$ or
2. $\min(m,p) = 2$, $\max(m,p) = 2^k - 1$.

Because when $m = 2 \le p$, $\{p+1, p-1\}$ is a disjoint binary partition if and only if $p = 2^k - 1$, and when $\min(m,p) \ge 3$, all the degrees are even by Corollary 4.6 c).

Corollary 4.7. *$\deg K^1_{m,p}$ is odd if and only if either*

1. $\min(m,p) = 1$ *or*
2. $\min(m,p) = 2$, $\max(m,p) = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_l} - 1$ *with* $n_{i+1} > n_i + 1$, $i = 1, \ldots, l-1$.

Proof. By letting $m + p = 1 + 2^{n_1} + \cdots + 2^{n_l}$ one can easily show that neither of the sets

$$\{m+p-1, m+p-3, m+p-5\},\ \{2(m+p)-1, m+p-3, m+p-5\},$$

$$\{m+p-1, 2(m+p)-3, m+p-5\},\ \{m+p-1, m+p-3, 2(m+p)-5\}$$

can have disjoint exponents in the binary representations of the elements. So the degree is even if $\min(m,p) \ge 3$. Now let $m = 2$, $p > m$ be odd, and

$$p + 1 = 2^{n_1} + \cdots + 2^{n_l}.$$

Then 2^{n_1} appears in both $p+1 = m+p-1$ and $2p+1 = 2(m+p)-3$. So $\deg K^1_{2,p}$ is odd if and only if

$$\{p-1, 2p+3\} = \{2+2^2+\cdots+2^{n_1-1}+2^{n_2}+\cdots+2^{n_l}, 1+2^{n_1+1}+\cdots+2^{n_l+1}\}$$

is a disjoint binary partition; i.e. if and only if $n_{i+1} > n_i + 1$ for $i = 1, \ldots, l-1$. □

Similar results can also be proven for $q > 1$. The combinatorics however becomes very involved. We provide without proof the result for $q = 2$.

COROLLARY 4.8. *$\deg K^2_{m,p}$ is odd if and only if either*

1. $\min(m,p) = 1$ *or*
2. $\min(m,p) = 2$, $\max(m,p) = 8(\frac{4^k-1}{3}) + 1$ *or*
3. $\min(m,p) = 3$, $\max(m,p) = 8$.

5. Corollaries and additional new positive pole placement results. In this section we establish the connection to the classical state space and transfer function formulation of the pole placement problem. We also will derive several results which combine the results derived in Section 3 with some results derived in [17].

Consider a controllable observable linear system

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{5.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ respectively. If a controllable observable dynamic compensator of order q

$$\dot{u} = Fu + Ey, \quad u = Hu + Ky \tag{5.2}$$

is applied to the system, the closed loop system becomes

$$\dot{\begin{pmatrix} x \\ u \end{pmatrix}} = \begin{pmatrix} A+BKC & BH \\ EC & F \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \quad y = Cx. \tag{5.3}$$

So the closed loop characteristic polynomial is

$$\phi(s) = \det \begin{pmatrix} sI - A - BKC & -BH \\ -EC & sI - F \end{pmatrix} \tag{5.4}$$

If $G(s) = C(sI-A)^{-1}B$ and $T(s) = K + H(sI-F)^{-1}E$ are the transfer functions of the system (5.1) and compensator (5.2), respectively and if $G(s) = D(s)^{-1}N(s)$ and $T(s) = T_d^{-1}(s)T_n(s)$ be left coprime fractions such that $\det(sI-A) = \det D(s)$ and $\det(sI-F) = \det T_d(s)$, then $\phi(s)$ can also be written as

$$\phi(s) = \det(sI-A)\det(I-G(s)T(s))\det(sI-F) = \det \begin{pmatrix} D(s) & N(s) \\ T_n(s) & T_d(s) \end{pmatrix}. \tag{5.5}$$

Let $P(s) = (D(s)\ N(s))$ and $C(s) = (T_n(s)\ T_d(s))$. Then $P(s)$ and $C(s)$ can be viewed as autoregressive systems describing the behavior of the plant and the compensator respectively. The combined dynamics is then described through

$$\begin{pmatrix} D(\frac{d}{dt}) & N(\frac{d}{dt}) \\ T_n(\frac{d}{dt}) & T_d(\frac{d}{dt}) \end{pmatrix} \cdot \begin{pmatrix} y \\ -u \end{pmatrix}(t) \;=\; 0. \tag{5.6}$$

The following result combines Theorem 2.15 with [17, Corollary 5.8].

THEOREM 5.1. *Consider a generic set of matrices* $(A,B,C)\in \mathbb{R}^{n(n+m+p)}$ *describing a plant as in* (5.1) *and consider an arbitrary monic polynomial* $\phi(s)\in\mathbb{R}[s]$ *of degree* $n+q$. *If*

$$n\le q(m+p-1)+mp \tag{5.7}$$

then there exists a complex dynamic compensator of the form (5.2) *resulting in the closed loop characteristic polynomial* $\phi(s)$. *If in addition the number* $d(m,p,q)$ *introduced in* (1.5) *is odd then there exists even a real compensator assigning the closed loop characteristic polynomial* $\phi(s)$.

Proof. We only outline the main steps. Using the same argument as in Theorem 2.15 one verifies that $\dim B_p = q(m+p)+mp-n-q-1$ for the generic and strictly proper plant. Theorem 2.14 therefore still applies and the pole placement map is onto if one allows all autoregressive systems. Since the plant is strictly proper a closed loop characteristic polynomial of degree $n+q$ can only be achieved if the compensator is proper. □

The following example illustrates how this theorem can be applied.

EXAMPLE 5.2. Assume the matrices (A,B,C) describe the plant parameters of a generic real 2-input, 9-output plant of McMillan degree n. From the table at the end of Section 3 it follows immediately that there exists a real compensator of degree 1 as long as $n\le 28$. If e.g. $n\le 58$ then it follows that there is a real compensator of degree 4 assigning an arbitrary set of self conjugate closed loop poles.

Combining Theorem 2.15 with [17, Corollary 5.9] one can finally proof:

THEOREM 5.3. *Let* $G(s)$ *be a generic* m*-input,* p*-output proper transfer function of McMillan degree* n *and let* $\phi(s)\in\mathbb{K}[s]$ *be a generic polynomial of degree* $n+q$. *If*

$$n\le q(m+p-1)+mp \tag{5.8}$$

then there exists a proper complex compensator $T(s)$ *of McMillan degree* q *such that the closed loop transfer function*

$$G_T(s):=(I-G(s)T(s))^{-1}G(s)$$

has characteristic polynomial $\phi(s)$. *If in addition the number* $d(m,p,q)$ *introduced in* (1.5) *is odd then there exists even a real transfer function* $T(s)$.

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