INCOMPLETE ELLIPTIC INTEGRALS IN RAMANUJAN'S LOST NOTEBOOK

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Dedicated with appreciation and thanks to Richard Askey on his 65^{th} birthday

Abstract. On pages 51–53 of his lost notebook, S. Ramanujan expressed several integrals of products of Dedekind eta-functions in terms of incomplete elliptic integrals of the first kind. In this paper, we prove these identities using only results found in Ramanujan's notebooks. We then construct several new elliptic integrals of this type using modular identities associated with certain "Hauptmoduls."

1. Introduction

On pages 51–53 in his lost notebook [17], Ramanujan recorded several identities involving integrals of theta-functions and incomplete elliptic integrals of the first kind. We offer here one typical example, proved in Theorem 7.5 below. Let (in Ramanujan's notation) $f(-q) = (q;q)_\infty$. (Detailed notation is given in Section 2. The function f is essentially the Dedekind eta-function; see (2.4).) Let

$$v := v(q) := q\frac{f^3(-q)f^3(-q^{15})}{f^3(-q^3)f^3(-q^5)}. \tag{1.1}$$

Then

$$\int_0^q f(-t)f(-t^3)f(-t^5)f(-t^{15})dt = \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)}^{2\tan^{-1}(1/\sqrt{5})} \frac{d\varphi}{\sqrt{1-\frac{9}{25}\sin^2\varphi}}. \tag{1.2}$$

The reader will immediately realize that these are rather uncommon integrals. Indeed, we have never seen identities like (1.2) in the literature.

In a wonderful paper [13], all of these integral identities were proved by S. Raghavan and S. S. Rangachari. However, in almost all of their proofs, they used results with which Ramanujan would have been unfamiliar. In particular, they relied heavily on results from the theory of modular forms, evidently not known to Ramanujan. For example, for four identities, including (1.2), Raghavan and Rangachari appealed to differential equations

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satisfied by certain quotients of eta-functions, such as (1.1), which can be found in R. Fricke's text [9].

In an effort to discern Ramanujan's methods and to better understand the origins of identities like (1.2), the present authors have devised proofs independent of the theory of modular forms and other ideas with which Ramanujan would have been unfamiliar. In particular, we have relied exclusively on results found in his ordinary notebooks [15] and his lost notebook [17]. It should be emphasized that at the time of the publication of Raghavan and Rangachari's paper [13] a decade ago, many of these results had not yet been proved. Particularly troublesome for us were the aforementioned four differential equations for quotients of eta-functions. To prove these, we used identities for Eisenstein series found in Chapter 21 of Ramanujan's second notebook and several eta-function identities scattered among the unorganized pages of his second notebook [2, Chap. 25]. We have also utilized several results in the lost notebook found on pages in close proximity to the elliptic integral identities.

The authors owe a huge debt to Raghavan and Rangachari's paper [13]. In many cases, we have incorporated large portions of their proofs, while in other instances we have employed different lines of attack. This paper could have been made shorter by referring to their paper for large portions of certain proofs, but considerable readability would have been lost in doing so.

In Section 3, we prove two identities for integrals of theta-functions of forms unlike (1.2). The first proof is virtually the same as that given by Raghavan and Rangachari, while the latter proof is completely different. In Sections 4–6, we prove several integral identities associated with modular equations of degree 5. Here some transformations of incomplete elliptic integrals due to J. Landen and Ramanujan play key roles. In Section 7, several identities of order 15 are established. Here two of the aforementioned differential equations are crucial. Differential equations are also central in Sections 8 and 9, where identities of orders 14 and 35, respectively, are proved.

Since differential equations for quotients of eta-functions are of such paramount importance in proving identities akin to (1.2), we have systematically derived several new differential equations for eta-function quotients in Section 10. We have used two of these new differential equations to derive two new formulas in the spirit of (1.2). In Section 10, we also point out the connection of such integrals with elliptic curves. We plan to return to these matters in a future paper.

2. Preliminary Results

As usual, set, for each nonnegative integer n,

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \qquad |q|<1.$$

Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab|<1.$$

Theta-functions satisfy the very important and useful Jacobi triple product identity [1, p. 35, Entry 19],

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.1}$$

The most important special cases are given by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty(q;q^2)_\infty}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.3}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty$$

$$=: e^{-2\pi iz/24}\eta(z), \qquad q=e^{2\pi iz}, \qquad \operatorname{Im} z>0. \tag{2.4}$$

The product representations in (2.2)–(2.4) are instances of the Jacobi triple product identity (2.1). The function $\eta(z)$, defined in (2.4), is the Dedekind eta-function. It has the transfomation formula

$$\eta(-1/z) = \sqrt{z/i}\,\eta(z). \tag{2.5}$$

The functions φ, ψ, and f in (2.2)–(2.4) can be expressed in terms of the modulus k and the hypergeometric function $z := {}_2F_1(\frac12,\frac12;1;k^2)$. For a catalogue of formulas of this type, see [1, pp. 122–124]. We will need two such formulas in the sequel. If $\alpha=k^2$ and

$$q = \exp\left(\frac{{}_2F_1(\frac12,\frac12;1;1-\alpha)}{{}_2F_1(\frac12,\frac12;1;\alpha)}\right),$$

then

$$\psi(-q) = \sqrt{\tfrac12 z}\,\{\alpha(1-\alpha)/q\}^{1/8} \tag{2.6}$$

and

$$f(-q^2) = \sqrt{z}2^{-1/3}\left\{\alpha(1-\alpha)/q\right\}^{1/12}. \tag{2.7}$$

The Eisenstein series $P(q), Q(q)$, and $R(q)$ are defined by

$$P(q) :=1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{2.8}$$

$$Q(q) :=1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \tag{2.9}$$

and

$$R(q) :=1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}. \tag{2.10}$$

(This is the notation used by Ramanujan in his lost notebook and paper [14], [16, pp. 136–162], but in his ordinary notebooks, P, Q, and R are replaced by L, M, and N, respectively.)

The Rogers–Ramanujan continued fraction $u(q)$ is defined by

$$u := u(q) := \frac{q^{1/5}}{1}\,\frac{}{+}\,\frac{q}{1}\,\frac{}{+}\,\frac{q^2}{1}\,\frac{}{+}\,\frac{q^3}{1}\,\frac{}{+}\cdots, \qquad |q| < 1. \tag{2.11}$$

With $f(-q)$ defined by (2.4), two of the most important properties of $u(q)$ are given by [1, p. 267, eqs. (11.5), (11.6)]

$$\frac{1}{u(q)} - 1 - u(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \tag{2.12}$$

and

$$\frac{1}{u^5(q)} - 11 - u^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{2.13}$$

A common generalization of (2.12) and (2.13) was recorded by Ramanujan in his lost notebook and proved by S. H. Son [18]. Lastly, it can be shown that, with the use of the Rogers–Ramanujan identities [1, p. 79],

$$u(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{2.14}$$

3. Two Simpler Integrals

Theorem 3.1 (p. 51)**.** *Let $P(q), Q(q)$, and $R(q)$ be the Eisenstein series defined by (2.8)–(2.10). Then*

$$\int_{e^{-2\pi}}^{q}\sqrt{Q(t)}\frac{dt}{t} = \log\left(\frac{Q^{3/2}(q) - R(q)}{Q^{3/2}(q) + R(q)}\right).$$

Proof. Following Ramanujan's suggestion, let $z = R^2(t)/Q^3(t)$. Then

$$\frac{1}{z}\frac{dz}{dq} = \frac{2}{R}\frac{dR}{dq} - \frac{3}{Q}\frac{dQ}{dq}. \tag{3.1}$$

Using Ramanujan's differential equations [14, eq. (30)], [17, p. 142], [1, p. 330]

$$q\frac{dR}{dq} = \frac{PR - Q^2}{2} \qquad \text{and} \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3},$$

in (3.1), we find that

$$\frac{q}{z}\frac{dz}{dq} = \frac{R^2 - Q^3}{RQ}. \tag{3.2}$$

Hence, by (3.2),

$$\begin{aligned} q\frac{d}{dq}\log\left(\frac{Q^{3/2} - R}{Q^{3/2} + R}\right) &= q\frac{d}{dq}\log\left(\frac{1 - \sqrt{z}}{1 + \sqrt{z}}\right) \\ &= q\frac{d}{dz}\log\left(\frac{1 - \sqrt{z}}{1 + \sqrt{z}}\right)\frac{dz}{dq} \\ &= \frac{1}{\sqrt{z}(z - 1)}q\frac{dz}{dq} \\ &= \sqrt{Q}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{e^{-2\pi}}^{q} \sqrt{Q(t)}\frac{dt}{t} &= \int_{e^{-2\pi}}^{q} \frac{d}{dt}\log\left(\frac{Q^{3/2} - R}{Q^{3/2} + R}\right) dt \\ &= \log\left(\frac{Q^{3/2}(q) - R(q)}{Q^{3/2}(q) + R(q)}\right) - \log\left(\frac{Q^{3/2}(e^{-2\pi}) - R(e^{-2\pi})}{Q^{3/2}(e^{-2\pi}) + R(e^{-2\pi})}\right). \end{aligned}$$

But it is well-known that $R(e^{-2\pi}) = 0$ [8, p. 88], and so Theorem 3.1 follows. □

Theorem 3.2 (p. 53)**.** *Let $u(q)$ denote the Rogers–Ramanujan continued fraction, defined by (2.11), and set $v = u(q^2)$. Recall that $\psi(q)$ is defined by (2.3). Then*

$$\frac{8}{5}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \log(u^2v^3) + \sqrt{5}\log\left(\frac{1 + (\sqrt{5} - 2)uv^2}{1 - (\sqrt{5} + 2)uv^2}\right). \tag{3.3}$$

Proof. Let $k := k(q) := uv^2$. Then from page 53 of Ramanujan's lost notebook [17], or from page 326 of his second notebook [3, pp. 12–13],

$$u^5 = k\left(\frac{1 - k}{1 + k}\right)^2 \qquad \text{and} \qquad v^5 = k^2\left(\frac{1 + k}{1 - k}\right). \tag{3.4}$$

(See also S.–Y. Kang's paper [11].) It follows that

$$\log(u^2v^3) = \frac{1}{5}\log\left(k^8\frac{1-k}{1+k}\right). \tag{3.5}$$

If we set $\epsilon = (\sqrt{5}+1)/2$, we readily find that $\epsilon^3 = \sqrt{5}+2$ and $\epsilon^{-3} = \sqrt{5}-2$. Then, with the use of (3.5), we see that (3.3) is equivalent to the equality

$$\frac{8}{5}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \frac{1}{5}\log\left(k^8\frac{1-k}{1+k}\right) + \sqrt{5}\log\left(\frac{1+\epsilon^{-3}k}{1-\epsilon^3 k}\right). \tag{3.6}$$

Now from Entry 9(vi) in Chapter 19 of Ramanujan's second notebook [1, p. 258],

$$\frac{\psi^5(q)}{\psi(q^5)} = 25q^2\psi(q)\psi^3(q^5) + 1 - 5q\frac{d}{dq}\log\frac{f(q^2,q^3)}{f(q,q^4)}. \tag{3.7}$$

By the Jacobi triple product identity (2.1),

$$\begin{aligned}\frac{f(q^2,q^3)}{f(q,q^4)} &= \frac{(-q^2;q^5)_\infty(-q^3;q^5)_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty}\\ &= \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^2;q^{10})_\infty(q^8;q^{10})_\infty}\\ &= q^{1/5}\frac{u(q)}{v(q)}, \qquad (3.8)\end{aligned}$$

by (2.14). Using (3.8) in (3.7), we find that

$$\begin{aligned}\frac{8}{5}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} &= 40\int q\psi(q)\psi^3(q^5)dq + \int\frac{8}{5q}dq - 8\int\frac{d}{dq}\log\left(q^{1/5}u/v\right)dq\\ &= 40\int q\psi(q)\psi^3(q^5)dq - 8\log(u/v)\\ &= 40\int q\psi(q)\psi^3(q^5)dq + \frac{8}{5}\log k - \frac{24}{5}\log\frac{1-k}{1+k}, \qquad (3.9)\end{aligned}$$

where (3.4) has been employed. Comparing (3.9) with (3.6), we now see that it suffices to prove that

$$8\int q\psi(q)\psi^3(q^5)dq = \log\frac{1-k}{1+k} + \frac{1}{\sqrt{5}}\log\left(\frac{1+\epsilon^{-3}k}{1-\epsilon^3 k}\right). \tag{3.10}$$

Upon differentiation of both sides of (3.10) and simplification, we find that (3.10) is equivalent to

$$q\psi(q)\psi^3(q^5) = \frac{k(q)k'(q)}{(1-k^2(q))(1-4k(q)-k^2(q))}. \tag{3.11}$$

We now prove (3.11). By (3.4) again,

$$\frac{v}{u^2} = \frac{1+k}{1-k}. \tag{3.12}$$

Taking the logarithmic derivative of both sides of (3.12), we find that

$$\frac{k'(q)}{1-k^2(q)}=\frac{1}{2}\frac{v'(q)}{v(q)}-\frac{u'(q)}{u(q)}. \tag{3.13}$$

By the logarithmic differentiation of (2.14),

$$\frac{u'(q)}{u(q)}=\frac{1}{5q}-\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^{n-1}}{1-q^n}$$

and

$$\frac{v'(q)}{v(q)}=2\left(\frac{1}{5q}-\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^{2n-1}}{1-q^{2n}}\right),$$

where $\left(\frac{n}{5}\right)$ denotes the Legendre symbol. Using these derivatives in (3.13), we see that

$$\frac{k'(q)}{1-k^2(q)}=\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^{n-1}}{1-q^{2n}}. \tag{3.14}$$

However, from Entry 8(i) in Chapter 19 of Ramanujan's second notebook [1, p. 249],

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^{2n}}=q\psi^3(q)\psi(q^5)-5q^2\psi(q)\psi^3(q^5),$$

so that, by (3.14),

$$\frac{k'(q)}{1-k^2(q)}=\psi^3(q)\psi(q^5)-5q\psi(q)\psi^3(q^5). \tag{3.15}$$

From page 56 in Ramanujan's lost notebook [17],

$$\frac{\psi^2(q)}{q\psi^2(q^5)}=\frac{1-k^2(q)}{k(q)}+1, \tag{3.16}$$

which has been proved by Kang [11, Thm. 4.2]. Putting (3.16) in (3.15), we deduce that

$$\frac{k'(q)}{1-k^2(q)}=\left(\frac{1-k^2(q)}{k(q)}-4\right)q\psi(q)\psi^3(q^5). \tag{3.17}$$

It is easily seen that (3.17) is equivalent to (3.11), and so the proof of (3.3) is complete. □

4. Elliptic Integrals of Order 5 (I)

Theorem 4.1 (p. 52)**.** *With* $f(-q), \psi(q)$, *and* $u(q)$ *defined by (2.4), (2.3), and (2.11), respectively, and with* $\epsilon = (\sqrt{5}+1)/2$,

(4.1)

$$5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = 2\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}}$$

$$= \int_0^{2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right)} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}} \tag{4.2}$$

$$= \sqrt{5}\int_0^{2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)\right)} \frac{d\varphi}{\sqrt{1-\epsilon 5^{-1/2}\sin^2\varphi}}. \tag{4.3}$$

To prove (4.1), we need the following lemma.

Lemma 4.2. *Let* $u(q)$ *be defined by (2.11). Then*

$$u'(q) = \frac{u(q)}{5q}\frac{f^5(-q)}{f(-q^5)}.$$

Proof. By (2.14) and the Jacobi triple product identity (2.1),

$$u(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

By logarithmic differentiation and the use of Entry 9(v) in Chapter 19 of Ramanujan's second notebook [1, p. 258],

$$\begin{aligned}\frac{u'(q)}{u(q)} &= \frac{1}{5q} + \frac{d}{dq}\log\frac{f(-q,-q^4)}{f(-q^2,-q^3)}\\ &= \frac{1}{5q} + \frac{1}{5q}\left(-1+\frac{f^5(-q)}{f(-q^5)}\right) = \frac{1}{5q}\frac{f^5(-q)}{f(-q^5)},\end{aligned}$$

which completes the proof. □

Proof of (4.1). Let

$$\cos^2\varphi = \epsilon^5 u^5(t). \tag{4.4}$$

If $t=0$, then $\varphi = \pi/2$; if $t = q$, then $\varphi = \cos^{-1}\left((\epsilon u)^{5/2}\right)$. Upon differentiation and the use of Lemma 4.2,

$$2\cos\varphi(-\sin\varphi)\frac{d\varphi}{dt} = 5\epsilon^5 u^4(t)u'(t)$$

$$= \epsilon^5\frac{u^5(t)}{t}\frac{f^5(-t)}{f(-t^5)} = \cos^2\varphi\frac{f^5(-t)}{tf(-t^5)}. \tag{4.5}$$

Hence, by (4.5), (2.13), and (4.4),

$$
\begin{aligned}
&5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt\\
=&5^{3/4}\int_{\pi/2}^{\cos^{-1}\left((\epsilon u)^{5/2}\right)} \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}\frac{-2tf(-t^5)}{f^5(-t)}\frac{\sin\varphi}{\cos\varphi}d\varphi\\
=&2\cdot 5^{3/4}\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \sqrt{t}\frac{f^3(-t^5)}{f^3(-t)}\frac{\sin\varphi}{\cos\varphi}d\varphi\\
=&2\cdot 5^{3/4}\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \frac{1}{\sqrt{1/u^5(t)-11-u^5(t)}}\frac{\sin\varphi}{\cos\varphi}d\varphi\\
=&2\cdot 5^{3/4}\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \frac{\sin\varphi}{\sqrt{\epsilon^5-11\cos^2\varphi-\epsilon^{-5}\cos^4\varphi}}d\varphi. \qquad (4.6)
\end{aligned}
$$

Since $\epsilon^{\pm 5}=(5\sqrt{5}\pm 11)/2$,

$$
\begin{aligned}
\epsilon^5-11\cos^2\varphi-\epsilon^{-5}\cos^4\varphi =&\epsilon^5-11(1-\sin^2\varphi)-\epsilon^{-5}\cos^4\varphi\\
=&\epsilon^{-5}+11\sin^2\varphi-\epsilon^{-5}\cos^4\varphi\\
=&\epsilon^{-5}(1-\cos^2\varphi)(1+\cos^2\varphi)+11\sin^2\varphi\\
=&\epsilon^{-5}\sin^2\varphi(2-\sin^2\varphi)+11\sin^2\varphi\\
=&\sin^2\varphi(2\epsilon^{-5}+11-\epsilon^{-5}\sin^2\varphi)\\
=&\sin^2\varphi(5\sqrt{5}-\epsilon^{-5}\sin^2\varphi)\\
=&5\sqrt{5}\sin^2\varphi(1-\epsilon^{-5}5^{-3/2}\sin^2\varphi).
\end{aligned}
$$

Thus, from (4.6),

$$
5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = 2\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2}\frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}},
$$

which is (4.1). □

To prove (4.2), we need two transformations for incomplete elliptic integrals found in Chapter 17 of Ramanujan's second notebook [1, pp. 105–106, Entries 7(ii), (vi)].

Lemma 4.3. *If* $\tan\gamma=\sqrt{1-x}\ \tan\alpha$, *then*

$$
\int_0^\alpha \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}=\int_0^\gamma\frac{d\varphi}{\sqrt{1-x\cos^2\varphi}}. \qquad (4.7)
$$

If $\cot\alpha\ \tan(\beta/2)=\sqrt{1-x\sin^2\alpha}$, *then*

$$
2\int_0^\alpha \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}=\int_0^\beta\frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}. \qquad (4.8)
$$

Proof of (4.2). In (4.7), replace φ by $\pi/2-\varphi$ and combine the result with (4.8) to deduce that

$$\int_0^{\beta}\frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}=2\int_{\pi/2-\gamma}^{\pi/2}\frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}, \tag{4.9}$$

provided that
(i) $\cot\alpha\ \tan(\beta/2)=\sqrt{1-x\sin^2\alpha}$,
(ii) $\tan\gamma=\sqrt{1-x}\ \tan\alpha$.

Examining (4.1) and (4.2), we see that we want to set $x=\epsilon^{-5}5^{-3/2}$ and $\gamma=\frac{\pi}{2}-\cos^{-1}\left((\epsilon u)^{5/2}\right)$. We also see that, to prove (4.2), we will need to show that (i) and (ii) imply that

$$\beta=2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right). \tag{4.10}$$

Since $\epsilon^{\pm5}=(5\sqrt{5}\pm11)/2$, a short calculation gives

$$1-\epsilon^{-5}5^{-3/2}=\epsilon^5 5^{-3/2}.$$

Thus, from (ii) and elementary trigonometry,

$$\begin{aligned}\tan\alpha=&\frac{1}{\sqrt{1-\epsilon^{-5}5^{-3/2}}}\cot\left(\cos^{-1}(\epsilon u)^{5/2}\right)\\ =&\epsilon^{-5/2}5^{3/4}\frac{(\epsilon u)^{5/2}}{\sqrt{1-(\epsilon u)^5}}=\frac{5^{3/4}u^{5/2}}{\sqrt{1-(\epsilon u)^5}}.\end{aligned} \tag{4.11}$$

Thus, by (i),

$$\tan(\beta/2)=\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\alpha}\frac{5^{3/4}u^{5/2}}{\sqrt{1-(\epsilon u)^5}}. \tag{4.12}$$

From (4.11) and elementary trigonometry,

$$x\ \sin^2\alpha=\frac{\epsilon^{-5}u^5}{1+\epsilon^{-5}u^5}.$$

Using this in (4.12), we deduce that

$$\begin{aligned}\tan(\beta/2)=&\sqrt{1-\frac{\epsilon^{-5}u^5}{1+\epsilon^{-5}u^5}}\frac{5^{3/4}u^{5/2}}{\sqrt{1-(\epsilon u)^5}}\\ =&\frac{5^{3/4}u^{5/2}}{\sqrt{(1+\epsilon^{-5}u^5)(1-\epsilon^5u^5)}}\\ =&\frac{5^{3/4}u^{5/2}}{\sqrt{1-11u^5-u^{10}}}\\ =&\frac{5^{3/4}}{\sqrt{1/u^5-11-u^5}}\\ =&5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q),\end{aligned}$$

by (2.13). Clearly, the last equality is equivalent to (4.10), and so the proof of (4.2) is complete. □

For the proof of (4.3), we need another transformation for incomplete elliptic integrals.

Lemma 4.4. *If* $0 < p < 1$ *and*

$$\tan\left(\frac{1}{2}(A-B)\right) = \frac{1-p}{1+2p}\tan B, \tag{4.13}$$

then

$$(1+2p)\int_0^A \frac{d\varphi}{\sqrt{1-p^3\left(\frac{2+p}{1+2p}\right)\sin^2\varphi}} = 3\int_0^B \frac{d\varphi}{\sqrt{1-p\left(\frac{2+p}{1+2p}\right)^3\sin^2\varphi}}.$$

This lemma is Entry 6(iv) in Chapter 19 in Ramanujan's second notebook and is a consequence of a Theorem of Jacobi; see [1, pp. 238–241] for a proof.

Proof of (4.3). We apply Lemma 4.4 with

$$p = \frac{1}{\epsilon^2\sqrt{5}},$$

where $\epsilon = (\sqrt{5}+1)/2$. Then

$$1+2p = \frac{3}{\sqrt{5}} \qquad \text{and} \qquad 2+p = \frac{3\epsilon}{\sqrt{5}}, \tag{4.14}$$

and so

$$p^3\left(\frac{2+p}{1+2p}\right) = \epsilon^{-5}5^{-3/2} \qquad \text{and} \qquad p\left(\frac{2+p}{1+2p}\right)^3 = \frac{\epsilon}{\sqrt{5}}.$$

If we substitute these quantities in Lemma 4.4, and if we set

$$A = 2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right) \tag{4.15}$$

and

$$B = 2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)\right), \tag{4.16}$$

we shall be finished with the proof of (4.3) if we can prove (4.13).

Using the subtraction formula for the tangent function, (4.15), and (4.16), we deduce that

$$\tan\left(\frac{1}{2}(A-B)\right) = \frac{5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q) - 5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)}{1+5q\dfrac{f^3(-q^5)\psi(q^5)}{f^3(-q)\psi(q)}}. \tag{4.17}$$

It will be convenient to use some results from the lost notebook proved by Kang [10]. Set

$$t = q^{1/6}\frac{(-q^5;q^5)_\infty}{(-q;q)_\infty} \qquad \text{and} \qquad s = \frac{\varphi(-q)}{\varphi(-q^5)}, \tag{4.18}$$

where $\varphi(q)$ is defined by (2.2). Then

$$\frac{f(-q)}{q^{1/6}f(-q^5)}=\frac{s}{t} \qquad \text{and} \qquad \frac{\psi(q)}{\sqrt{q}\psi(q^5)}=\frac{s}{t^3}. \tag{4.19}$$

Employing (4.19) in (4.17), we readily deduce that

$$5^{-1/4}\tan\left(\frac{1}{2}(A-B)\right)=\frac{\sqrt{5}t^3s-t^3s^3}{s^4+5t^6}. \tag{4.20}$$

Next, a simple calculation shows that

$$1-p=\frac{3}{\epsilon\sqrt{5}}. \tag{4.21}$$

Hence, by (4.14), (4.21), (4.16), and the double angle formula,

$$\begin{aligned}5^{-1/4}\frac{1-p}{1+2p}\tan B =&5^{-1/4}\epsilon^{-1}\tan B\\ =&5^{-1/4}\epsilon^{-1}\tan\left(2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)\right)\right)\\ =&\frac{2\epsilon^{-1}\sqrt{q}\psi(q^5)/\psi(q)}{1-\sqrt{5}\ q\psi^2(q^5)/\psi^2(q)}\\ =&\frac{2\epsilon^{-1}t^3s}{s^2-\sqrt{5}\ t^6}.\end{aligned} \tag{4.22}$$

Comparing (4.20) and (4.22), in view of (4.13), we must prove that

$$\frac{2\epsilon^{-1}t^3s}{s^2-\sqrt{5}\ t^6}=\frac{\sqrt{5}\ t^3s-t^3s^3}{s^4+5t^6}.$$

After considerable simplification, the last equality is seen to be equivalent to

$$s^4+5t^6=s^2+s^2t^6. \tag{4.23}$$

Now, from (4.18) and (2.3), we find that

$$t=t(q)=\frac{q^{1/6}\psi(q^5)f(-q^2)}{\psi(q)f(-q^{10})}.$$

Replacing q by $-q$ and employing (2.6) and (2.7), we find that

$$t^6(-q)=-q\left(\frac{\psi(-q^5)f(-q^2)}{\psi(-q)f(-q^{10})}\right)^6=-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4},$$

where β has degree 5 over α. On the other hand, from (4.18),

$$s(-q)=\frac{\varphi(q)}{\varphi(q^5)}=:\sqrt{m},$$

where m is the multiplier of degree 5. Hence, replacing q by $-q$ in (4.23), we see that this equality is equivalent to

$$m^2-5\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}=m-m\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}. \tag{4.24}$$

Using formulas for m and $5/m$ given in Entry 13(xii) of Chapter 19 in Ramanujan's second notebook [1, pp. 281–282], namely,

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}$$

and

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4},$$

we may easily verify that (4.24) does hold to complete the proof. ☐

5. Elliptic Integrals of Order 5 (II)

Theorem 5.1 (p. 52)**.** *As before, let* $\epsilon = (\sqrt{5}+1)/2$, *and let* $u(q)$ *and* $f(-q)$ *be defined by (2.11) and (2.4), respectively. Then*

(5.1)
$$5^{-3/4}\int_0^q \frac{f^5(-t)}{\sqrt{f(-t^{1/5})f(-t^5)}}\frac{dt}{t^{9/10}} = 2\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2} \frac{d\varphi}{\sqrt{1-\epsilon^{-1}5^{-1/2}\sin^2\varphi}}$$

$$= \int_0^{2\tan^{-1}\left(5^{1/4}q^{1/10}\sqrt{f(-q^5)/f(-q^{1/5})}\right)} \frac{d\varphi}{\sqrt{1-\epsilon^{-1}5^{-1/2}\sin^2\varphi}} \tag{5.2}$$

(5.3)
$$= \frac{1}{\sqrt{5}}\int_0^{2\tan^{-1}\left(5^{3/4}q^{1/10}\left(\frac{f(-q^{1/5})+q^{1/5}f(-q^5)}{f(-q^{1/5})+5q^{1/5}f(-q^5)}\right)\sqrt{\frac{f(-q^5)}{f(-q^{1/5})}}\right)} \frac{d\varphi}{\sqrt{1-\epsilon^5 5^{-3/2}\sin^2\varphi}}.$$

Proof of (5.1). Let

$$\cos^2\varphi = \epsilon u(t). \tag{5.4}$$

Thus, if $t = 0$, then $\varphi = \pi/2$; if $t = q$, then $\varphi = \cos^{-1}\left(\sqrt{\epsilon u}\right)$. Upon differentiation and the use of Lemma 4.2,

$$2\cos\varphi(-\sin\varphi)\frac{d\varphi}{dt} = \epsilon u'(t) = \epsilon\frac{u(t)}{5t}\frac{f^5(-t)}{f(-t^5)}. \tag{5.5}$$

Therefore, by (5.5), (2.12), and (5.4),

$$\begin{aligned}
&5^{-3/4}\int_0^q \frac{f^5(-t)}{\sqrt{f(-t^{1/5})f(-t^5)}}\frac{dt}{t^{9/10}}\\
=&2\cdot 5^{1/4}\int_{\cos^{-1}(\sqrt{\epsilon u})}^{\pi/2}\sqrt{\frac{t^{1/5}f(-t^5)}{f(-t^{1/5})}}\frac{\sin\varphi\ \cos\varphi}{\epsilon u(t)}d\varphi\\
=&2\cdot 5^{1/4}\int_{\cos^{-1}(\sqrt{\epsilon u})}^{\pi/2}\frac{1}{\sqrt{1/u(t)-1-u(t)}}\frac{\sin\varphi}{\cos\varphi}d\varphi\\
=&2\cdot 5^{1/4}\int_{\cos^{-1}(\sqrt{\epsilon u})}^{\pi/2}\frac{\sin\varphi}{\sqrt{\epsilon-\cos^2\varphi-\epsilon^{-1}\cos^4\varphi}}d\varphi. \qquad (5.6)
\end{aligned}$$

Now,

$$\begin{aligned}
\epsilon-\cos^2\varphi-\epsilon^{-1}\cos^4\varphi =&\epsilon-(1-\sin^2\varphi)-\epsilon^{-1}\cos^4\varphi\\
=&\epsilon^{-1}+\sin^2\varphi-\epsilon^{-1}\cos^4\varphi\\
=&\epsilon^{-1}(1-\cos^2\varphi)(1+\cos^2\varphi)+\sin^2\varphi\\
=&\epsilon^{-1}\sin^2\varphi(2-\sin^2\varphi)+\sin^2\varphi\\
=&\sin^2\varphi(2\epsilon^{-1}-\epsilon^{-1}\sin^2\varphi+1)\\
=&\sin^2\varphi(\sqrt{5}-\epsilon^{-1}\sin^2\varphi).
\end{aligned}$$

Using this calculation in (5.6), we find that

$$5^{-3/4}\int_0^q \frac{f^5(-t)}{\sqrt{f(-t^{1/5})f(-t^5)}}\frac{dt}{t^{9/10}} = 2\cdot 5^{1/4}\int_{\cos^{-1}(\sqrt{\epsilon u})}^{\pi/2}\frac{d\varphi}{\sqrt{\sqrt{5}-\epsilon^{-1}\sin^2\varphi}},$$

from which (5.1) is immediate. □

Proof of (5.2). The proof is similar to that of (4.2). We begin with (4.9), set $x=\epsilon^{-1}5^{-1/2}$, and put $\gamma=\frac{\pi}{2}-\cos^{-1}\left(\sqrt{\epsilon u}\right)$. Thus,

$$\tan\gamma=\cot\left(\cos^{-1}\left(\sqrt{\epsilon u}\right)\right)=\sqrt{\frac{\epsilon u}{1-\epsilon u}}. \qquad (5.7)$$

As with the proof of (4.2), we want to show that conditions (i) and (ii) imply that

$$\beta=2\tan^{-1}\left(5^{1/4}q^{1/10}\sqrt{f(-q^5)/f(-q^{1/5})}\right). \qquad (5.8)$$

From condition (ii) and (5.7),

$$\tan\alpha=\frac{\tan\gamma}{\sqrt{1-\epsilon^{-1}5^{-1/2}}}=\sqrt{\frac{\sqrt{5}\ u}{1-\epsilon u}} \qquad (5.9)$$

and

$$\sin^2\alpha=\frac{\tan^2\alpha}{1+\tan^2\alpha}=\frac{\sqrt{5}\ u}{1+\epsilon^{-1}u}. \qquad (5.10)$$

Using (5.9) and (5.10) in conjunction with condition (ii), we arrive at

$$\begin{aligned}\tan(\beta/2) =&(\tan\alpha)\sqrt{1-x\ \sin^2\alpha} = \sqrt{\frac{\sqrt{5}\ u}{1-\epsilon u}}\sqrt{1-\frac{\epsilon^{-1}u}{1+\epsilon^{-1}u}}\\ =&5^{1/4}\sqrt{\frac{1}{1/u-1-u}} = 5^{1/4}q^{1/10}\sqrt{\frac{f(-q^5)}{f(-q^{1/5})}}.\end{aligned}$$

Hence, (5.8) follows, and so the proof of (5.2) is finished. □

To prove (5.3), we need another transformation for incomplete elliptic integrals from Chapter 19 in Ramanujan's second notebook [1, p. 238, Entry 6(iii)].

Lemma 5.2. *If*

$$\tan\left(\frac{1}{2}(\alpha+\beta)\right) = (1+p)\tan\alpha,$$

where $0<p<1$, *then*

$$(1+2p)\int_0^\alpha \frac{d\varphi}{\sqrt{1-p^3\left(\frac{2+p}{1+2p}\right)\sin^2\varphi}} = \int_0^\beta \frac{d\varphi}{\sqrt{1-p\left(\frac{2+p}{1+2p}\right)^3\sin^2\varphi}}.$$

Proof of (5.3). We apply Lemma 5.2 with $p=1/\epsilon$. Thus, $1+2p=\sqrt{5}$ and $2+p=\epsilon^2$. Hence,

$$p^3\frac{2+p}{1+2p} = \frac{1}{\epsilon\sqrt{5}} \qquad \text{and} \qquad p\left(\frac{2+p}{1+2p}\right)^3 = \frac{\epsilon^5}{5\sqrt{5}}.$$

We abbreviate notation by setting

$$A=f(-q^{1/5}), \qquad B=f(-q^5), \qquad \text{and} \qquad C=q^{1/5}.$$

Put

$$\alpha = 2\tan^{-1}\left(5^{1/4}\sqrt{\frac{CB}{A}}\right)$$

and

$$\beta = 2\tan^{-1}\left(5^{3/4}\frac{A+CB}{A+5CB}\sqrt{\frac{CB}{A}}\right).$$

Examining (5.2) and (5.3) in relation to Lemma 5.2, we see that we will be finished with the proof if we can show that

$$\tan\left(\frac{1}{2}(\alpha+\beta)\right) = \epsilon\tan\alpha. \tag{5.11}$$

First, by the addition formula for the tangent function,

$$\begin{aligned}
&\tan\left(\frac{1}{2}(\alpha+\beta)\right)\\
&=\tan\left(\tan^{-1}\left(5^{1/4}\sqrt{\frac{CB}{A}}\right)+\tan^{-1}\left(5^{3/4}\frac{A+CB}{A+5CB}\sqrt{\frac{CB}{A}}\right)\right)\\
&=\frac{5^{1/4}\sqrt{\frac{CB}{A}}+5^{3/4}\frac{A+CB}{A+5CB}\sqrt{\frac{CB}{A}}}{1-5\frac{CB}{A}\frac{A+CB}{A+5CB}}\\
&=\frac{5^{1/4}\left(A+5CB+\sqrt{5}(A+CB)\right)\sqrt{ABC}}{A^2-5B^2C^2}. \qquad (5.12)
\end{aligned}$$

On the other hand, by the double angle formula for the tangent function,

$$\epsilon\tan\alpha=\epsilon\tan\left(2\tan^{-1}\left(5^{1/4}\sqrt{\frac{BC}{A}}\right)\right)=\frac{2\epsilon 5^{1/4}\sqrt{ABC}}{A-\sqrt{5}\,CB}. \qquad (5.13)$$

Comparing (5.12) and (5.13), we are required to prove that

$$\frac{2\epsilon}{A-\sqrt{5}\,CB}=\frac{A+5CB+\sqrt{5}(A+CB)}{A^2-5B^2C^2}.$$

This can be established by elementary algebra, and so the proof is complete. □

6. Elliptic Integrals of Order 5 (III)

Theorem 6.1 (p. 52). *Recall that Ramanujan's continued fraction $u(q)$ is defined by (2.11). Then there exists a constant C such that*

(6.1)
$$u^5+u^{-5}=\frac{1}{2\sqrt{q}}\frac{f^3(-q)}{f^3(-q^5)}\left(C+\int_q^1\frac{f^8(-t)}{f^4(-t^5)}\frac{dt}{t^{3/2}}+125\int_0^q\frac{f^8(-t^5)}{f^4(-t)}\sqrt{t}dt\right).$$

To prove Theorem 6.1, we need to establish a differential equation for a certain quotient of eta-functions.

Lemma 6.2. *Let*

$$\lambda:=\lambda(q):=q\frac{f^6(-q^5)}{f^6(-q)}. \qquad (6.2)$$

Then

$$q\frac{d}{dq}(\lambda(q))=\sqrt{q}f^2(-q)f^2(-q^5)\sqrt{125\lambda^3+22\lambda^2+\lambda}. \qquad (6.3)$$

Proof. By logarithmic differentiation,

$$\frac{1}{\lambda}\frac{d\lambda}{dq}=\frac{1}{q}-30\sum_{n=1}^{\infty}\frac{nq^{5n-1}}{1-q^{5n}}+6\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^{n}}.$$

We now apply Entry 4(i) in Chapter 21 of Ramanujan's second notebook [1, p. 463]. Accordingly,

$$\begin{aligned}\frac{q}{\lambda}\frac{d\lambda}{dq}&=\frac{\sqrt{f^{12}(-q)+22qf^{6}(-q)f^{6}(-q^5)+125q^2f^{12}(-q^5)}}{f(-q)f(-q^5)}\\&=\sqrt{q}f^2(-q)f^2(-q^5)\sqrt{\frac{1}{\lambda}+22+125\lambda},\end{aligned}$$

or

$$q\frac{d\lambda}{dq}=\sqrt{q}f^2(-q)f^2(-q^5)\sqrt{\lambda+22\lambda^2+125\lambda^3},$$

and the proof is complete. □

Proof of Theorem 6.1. From (2.13), in the notation (6.2),

$$\frac{1}{u^5}-11-u^5=\frac{1}{\lambda}. \tag{6.4}$$

Considering (6.4) as a quadratic equation in $x:=u^{-5}$, we find upon solving it that

$$2x=2u^{-5}=\frac{1}{\lambda}\left(11\lambda+1+\sqrt{125\lambda^2+22\lambda+1}\right).$$

The other root of this quadratic equation is easily seen to be

$$-2u^{5}=\frac{1}{\lambda}\left(11\lambda+1-\sqrt{125\lambda^2+22\lambda+1}\right).$$

Hence,

$$u^5+u^{-5}=\frac{1}{\lambda}\sqrt{125\lambda^2+22\lambda+1}.$$

Thus,

$$G(q):=2\sqrt{\lambda}(u^5+u^{-5})=2\sqrt{125\lambda+22+\frac{1}{\lambda}}. \tag{6.5}$$

Thus, by (6.5) and (6.3),

$$\begin{aligned}\frac{dG}{dq}&=\frac{125-1/\lambda^2}{\sqrt{125\lambda+22+1/\lambda}}\frac{d\lambda}{dq}\\&=\frac{125-1/\lambda^2}{\sqrt{125\lambda+22+1/\lambda}}\frac{f^2(-q)f^2(-q^5)}{\sqrt{q}}\sqrt{125\lambda^3+22\lambda^2+\lambda}\\&=\frac{125\lambda-1/\lambda}{\sqrt{q}}f^2(-q)f^2(-q^5)\\&=125\sqrt{q}\frac{f^8(-q^5)}{f^4(-q)}-\frac{f^8(-q)}{q^{3/2}f^4(-q^5)},\end{aligned}$$

upon the use of (6.2) again.

Thus, for any q_0 such that $0 < q_0 < 1$,

$$G(q) - G(q_0) = \int_{q_0}^{q} \frac{dG}{dt} dt = 125 \int_{q_0}^{q} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t} dt - \int_{q_0}^{q} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}},$$

or, by (6.5) and (6.2),

$$\begin{aligned} &u^5 + u^{-5} \\ &= \frac{f^3(-q)}{2\sqrt{q} f^3(-q^5)} \left(G(q_0) + 125 \int_{q_0}^{q} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t} dt - \int_{q_0}^{q} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}} \right) \\ &= \frac{f^3(-q)}{2\sqrt{q} f^3(-q^5)} \Bigg(G(q_0) + 125 \int_{0}^{q} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t} dt - 125 \int_{0}^{q_0} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t} dt \\ &\qquad + \int_{q}^{1} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}} - \int_{q_0}^{1} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}} \Bigg) \\ &= \frac{f^3(-q)}{2\sqrt{q} f^3(-q^5)} \left(C + 125 \int_{0}^{q} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t} dt + \int_{q}^{1} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}} \right), \end{aligned}$$

where

$$C = G(q_0) - 125 \int_{0}^{q_0} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t} dt - \int_{q_0}^{1} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}}. \tag{6.6}$$

Thus, we have completed the proof of Theorem 6.1 and have furthermore shown that the constant C is given by (6.6). □

Now set $q_0 = e^{-2\pi/\theta}$. Ramanujan calculated $G(e^{-2\pi/\theta})$ for three values of θ.

Theorem 6.3 (p. 52)**.** *We have*

(i) $$G(e^{-2\pi/\sqrt{5}}) = 4 \left(\frac{\sqrt{5}+1}{2} \right)^{5/2},$$

(ii) $$G(e^{-2\pi}) = G(e^{-2\pi/5}) = 6 \cdot 5^{1/4}(3 + \sqrt{5}).$$

Ramanujan erroneously claimed that

$$G(e^{-2\pi}) = G(e^{-2\pi/5}) = 16 \cdot 5^{-1/4}(2 + \sqrt{5}).$$

Raghavan and Rangachari [13] used a different method to prove Theorem 6.3.

Proof. To prove (i), we need to evaluate

$$\lambda(e^{-2\pi/\sqrt{5}}) = e^{-2\pi/\sqrt{5}} \frac{f^6(-e^{-2\pi\sqrt{5}})}{f^6(-e^{-2\pi/\sqrt{5}})} =: L(\sqrt{5}). \tag{6.7}$$

To evaluate $L(\sqrt{5})$, we will use [5, Theorem 2.3(i)]. Let G_n denote the Ramanujan–Weber class invariant. Set

$$V' = \frac{G_{25n}}{G_n} \tag{6.8}$$

and

$$A' = e^{2\pi\sqrt{n}/6}\frac{f(-e^{-2\pi\sqrt{n}})}{f(-e^{-10\pi\sqrt{n}})}. \tag{6.9}$$

Then

$$\frac{A'^2}{\sqrt{5}V'} - \frac{\sqrt{5}V'}{A'^2} = \frac{1}{\sqrt{5}}(V'^3 - V'^{-3}). \tag{6.10}$$

Let $n = 1/5$. It is well known that $G_n = G_{1/n}$, and so, from (6.8), $V' = 1$. Thus, from (6.10),

$$\frac{A'^2}{\sqrt{5}} - \frac{\sqrt{5}}{A'^2} = 0.$$

Hence, $A'^2 = \sqrt{5}$, and since, from (6.7) and (6.10), $L(\sqrt{5}) = A'^{-6}$, we conclude that $L(\sqrt{5}) = 5^{-3/2}$. Thus, from (6.5),

$$\begin{aligned} G(e^{-2\pi/\sqrt{5}}) &= 2\sqrt{125\cdot 5^{-3/2} + 22 + 5^{3/2}} \\ &= 2\sqrt{2}(5^{3/2} + 11)^{1/2} \\ &= 4\left(\frac{\sqrt{5}+1}{2}\right)^{5/2}. \end{aligned}$$

We now prove (ii). First, by (6.2), (2.4), and (2.5),

$$125\lambda(e^{-2\pi/5}) = 125\frac{\eta^6(i)}{\eta^6(i/5)} = \frac{\eta^6(i)}{\eta^6(5i)} = \frac{1}{\lambda(e^{-2\pi})}.$$

Hence, by (6.5),

$$\begin{aligned} G(e^{-2\pi/5}) &= 2\sqrt{125\lambda(e^{-2\pi/5}) + 22 + 1/\lambda(e^{-2\pi/5})} \\ &= 2\sqrt{1/\lambda(e^{-2\pi}) + 22 + 125\lambda(e^{-2\pi})} = G(e^{-2\pi}). \end{aligned}$$

Thus, it suffices to evaluate $G(e^{-2\pi/5})$, and so we need to determine

$$\lambda(e^{-2\pi/5}) = e^{-2\pi/5}\frac{f^6(-e^{-2\pi})}{f^6(-e^{-2\pi/5})} =: L(5). \tag{6.11}$$

Again, we shall employ (6.10).

Set $n = 1/25$. Then, by (6.8) and the relation $G_n = G_{1/n}$,

$$V' = \frac{G_1}{G_{1/25}} = \frac{1}{G_{25}} = \frac{\sqrt{5}-1}{2}. \tag{6.12}$$

(See, e.g. [3, p. 190] for the value of G_{25}). Set $\epsilon = (\sqrt{5}+1)/2$. Then from (6.10) and (6.12),

$$\frac{\epsilon A'^2}{\sqrt{5}} - \frac{\sqrt{5}}{\epsilon A'^2} = -\frac{4}{\sqrt{5}},$$

from which we easily find that $A'^2 = \epsilon^{-1}$. Hence, from (6.9) and (6.11), $L(5) = A'^{-6} = \epsilon^3$. Lastly, we then conclude from (6.5) that

$$\begin{aligned} G(e^{-2\pi/5}) &= 2\sqrt{125\epsilon^3 + 22 + \epsilon^{-3}} \\ &= 2\sqrt{270 + 126\sqrt{5}} \\ &= 6 \cdot 5^{1/4}\sqrt{14 + 6\sqrt{5}} \\ &= 6 \cdot 5^{1/4}(3 + \sqrt{5}), \end{aligned}$$

which completes the proof of (ii). □

Ramanujan claimed that

$$\begin{aligned} C =& 5^{3/4}\left(-\pi + 4\int_0^{\pi/2} \sqrt{1 - \epsilon^{-5}5^{-3/2}\sin^2\varphi}\, d\varphi \right. \\ & \left. -2\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \epsilon^{-5}5^{-3/2}\sin^2\varphi}}\right), \end{aligned} \tag{6.13}$$

which is quite different from (6.6). Now in Theorem 4.1, let q tend to 1. Then u tends to ϵ^{-1}, and so $\cos^{-1}((\epsilon u)^{5/2})$ tends to $\cos^{-1} 1 = 0$. Thus, (4.1) yields

$$5^{3/4}\int_0^1 \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = 2\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \epsilon^{-5}5^{-3/2}\sin^2\varphi}}.$$

Thus, one of the integrals in (6.13) can be identified as an integral of eta-functions. But this is the only progress we have made in identifying (6.13) with (6.6).

Numerically, (6.6) and (6.13) do not agree. First, by (6.13),

$$C = 5^{3/4}(-\pi + 4 \cdot 1.56762\cdots - 2 \cdot 1.57398\ldots) = -0.06377\ldots. \tag{6.14}$$

To calculate C via (6.6), we set $q_0 = e^{-2\pi}$ and use Theorem 6.3. Accordingly,

$$C = 6 \cdot 5^{1/4}(3 + \sqrt{5}) - 250\pi \int_1^\infty \frac{\eta^8(5ix)}{\eta^4(ix)}dx - 2\pi \int_0^1 \frac{\eta^8(ix)}{\eta^4(5ix)}dx. \tag{6.15}$$

We used *Mathematica* to calculate the integrals in (6.15) and found that

$$\begin{aligned} C =& 46.978487\cdots - 250\pi \cdot 8.60104\cdots \times 10^{-6} - 2\pi \cdot 5.81407\ldots \\ =& 10.44085\ldots. \end{aligned} \tag{6.16}$$

Thus, (6.14) and (6.16) show that Ramanujan's claim (6.13) is erroneous. Nonetheless, we are haunted by the possibility that a corrected version of (6.13) exists, for Ramanujan very rarely made a serious error.

7. Elliptic Integrals of Order 15

The entries in this and the following two sections depend upon remarkable differential equations satisfied by certain quotients of eta-functions. For the first series of results, that quotient is defined by

$$v := v(q) := q\left(\frac{f(-q)f(-q^{15})}{f(-q^3)f(-q^5)}\right)^3 \tag{7.1}$$

We need three ancillary lemmas. The first and third are found in Ramanujan's notebooks [15].

Lemma 7.1. *Let v be defined by (7.1), and let*

$$R = \frac{1}{q}\left(\frac{f(-q)f(-q^5)}{f(-q^3)f(-q^{15})}\right)^2.$$

Then

$$R + 5 + \frac{9}{R} = \frac{1}{v} - v.$$

For a proof of Lemma 7.1, see Berndt's book [2, p. 221, Entry 62].

Lemma 7.2. *Let R be given above, and let*

$$P = \frac{1}{q}\left(\frac{f(-q)}{f(-q^5)}\right)^6 \qquad \text{and} \qquad Q = \frac{1}{q^3}\left(\frac{f(-q^3)}{f(-q^{15})}\right)^6.$$

Then

$$P + \frac{125}{P} = R - 4 + \frac{135}{R} + \frac{486}{R^2} + \frac{729}{R^3}$$

and

$$Q + \frac{125}{Q} = R^3 + 6R^2 + 15R - 4 + \frac{9}{R}.$$

Proof. From Berndt's book [2, p. 223, Entry 63; p. 226, Entry 64], we have, respectively,

$$\sqrt{PQ} + \frac{125}{\sqrt{PQ}} = \sqrt{K^2+4}(K-9), \tag{7.2}$$

and

$$\sqrt{PQ} - \frac{125}{\sqrt{PQ}} = (K-4)\sqrt{(K-11)(K+1)}, \tag{7.3}$$

where

$$K = \frac{1}{v} - v, \tag{7.4}$$

where v is given by (7.1). (The forms of Entries 63 and 64 in [2] are slightly different from those in (7.2) and (7.3), respectively, but their equivalences are easily demonstrated by elementary algebra.) Multiplying (7.2) by $(\frac{1}{v}+v)$ and (7.3) by K in (7.4), we deduce that

$$P+\frac{125}{P}+Q+\frac{125}{Q}=(K^2+4)(K-9) \tag{7.5}$$

and

$$-P-\frac{125}{P}+Q+\frac{125}{Q}=K(K-4)\sqrt{(K-11)(K+1)}. \tag{7.6}$$

From Lemma 7.1, we know that

$$K=R+5+\frac{9}{R}. \tag{7.7}$$

Hence, from (7.5), (7.6), and (7.7), we deduce that

$$P+\frac{125}{P}+Q+\frac{125}{Q}=R^3+6R^2+16R-8+\frac{144}{R}+\frac{486}{R^2}+\frac{729}{R^3} \tag{7.8}$$

and

$$-P-\frac{125}{P}+Q+\frac{125}{Q}=R^3+6R^2+14R-\frac{126}{R}-\frac{486}{R^2}-\frac{729}{R^3}. \tag{7.9}$$

Solving (7.8) and (7.9) yields Lemma 7.2. □

Lemma 7.3. *We have*

$$1+6\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}} = \sqrt{\frac{f^{12}(-q)+22qf^6(-q)f^6(-q^5)+125q^2f^{12}(-q^5)}{f^2(-q)f^2(-q^5)}}.$$

For a proof of Lemma 7.3, see Berndt's book [1, p. 463, Entry 4].

Lemma 7.4. *Let v be defined by (7.1). Then*

$$\frac{dv}{dq}=f(-q)f(-q^3)f(-q^5)f(-q^{15})\sqrt{1-10v-13v^2+10v^3+v^4}.$$

Proof. From the definition (7.1) of v, we find that

$$\begin{aligned}
(7.10)\qquad \frac{1}{v}\frac{dv}{dq} &= \frac{d\log v}{dq} = \frac{d\log\{q^{3/2}\frac{f^3(-q^{15})}{f^3(-q^3)}\}}{dq} + \frac{d\log\{q^{-1/2}\frac{f^3(-q)}{f^3(-q^5)}\}}{dq}\\
&= \frac{3}{2q} + 9\sum_{n=1}^{\infty}\frac{nq^{3n-1}}{1-q^{3n}} - 45\sum_{n=1}^{\infty}\frac{nq^{15n-1}}{1-q^{15n}}\\
&\quad - \frac{1}{2q} + 15\sum_{n=1}^{\infty}\frac{nq^{5n-1}}{1-q^{5n}} - 3\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^{n}}\\
&= \frac{3}{2q}\sqrt{\frac{f^{12}(-q^3)+22q^3f^6(-q^3)f^6(-q^{15})+125q^6f^{12}(-q^{15})}{f^2(-q^3)f^2(-q^{15})}}\\
&\quad - \frac{1}{2q}\sqrt{\frac{f^{12}(-q)+22qf^6(-q)f^6(-q^5)+125q^2f^{12}(-q^5)}{f^2(-q)f^2(-q^5)}},
\end{aligned}$$

by Lemma 7.3. Simplifying (7.10) by using the definitions of P, Q, and R from Lemmas 7.1 and 7.2, as well as Lemmas 7.2, and 7.1 themselves, we find that

$$\begin{aligned}
\frac{dv}{dq} &= vf(-q)f(-q^3)f(-q^5)f(-q^{15})\left(\frac{3}{2}q^{1/2}\frac{f(-q^3)f(-q^{15})}{f(-q)f(-q^5)}\right.\\
&\times\left.\sqrt{Q+22+\frac{125}{Q}} - \frac{1}{2}q^{-1/2}\frac{f(-q)f(-q^5)}{f(-q^{15})f(-q^3)}\sqrt{P+22+\frac{125}{P}}\right)\\
&= vf(-q)f(-q^3)f(-q^5)f(-q^{15})\left(\frac{3}{2}\frac{1}{\sqrt{R}}\sqrt{Q+22+\frac{125}{Q}}\right.\\
&\left.-\frac{1}{2}\sqrt{R}\sqrt{P+22+\frac{125}{P}}\right)\\
&= vf(-q)f(-q^3)f(-q^5)f(-q^{15})\left(\frac{3}{2}\frac{1}{\sqrt{R}}\sqrt{R^3+6R^2+15R+18+\frac{9}{R}}\right.\\
&\left.-\frac{1}{2}\sqrt{R}\sqrt{R+18+\frac{135}{R}+\frac{486}{R^2}+\frac{729}{R^3}}\right)\\
&= vf(-q)f(-q^3)f(-q^5)f(-q^{15})\left(\frac{3}{2}\sqrt{\left(R+3+\frac{3}{R}\right)^2}\right.\\
&\left.-\frac{1}{2}\sqrt{\left(R+9+\frac{27}{R}\right)^2}\right)
\end{aligned}$$

$$\begin{aligned}
=&vf(-q)f(-q^3)f(-q^5)f(-q^{15})\left(\sqrt{R}-\frac{3}{\sqrt{R}}\right)\left(\sqrt{R}+\frac{3}{\sqrt{R}}\right)\\
=&vf(-q)f(-q^3)f(-q^5)f(-q^{15})\sqrt{\frac{1}{v}-v-11}\sqrt{\frac{1}{v}-v+1}\\
=&f(-q)f(-q^3)f(-q^5)f(-q^{15})\sqrt{1-10v-13v^2+10v^3+v^4}.
\end{aligned}$$

This completes the proof. □

Theorem 7.5 (p. 51)**.** *Let v be defined by (7.1), and let $\epsilon=(\sqrt{5}+1)/2$. Then*

(7.11)
$$\int_0^q f(-t)f(-t^3)f(-t^5)f(-t^{15})dt=\frac{1}{5}\int_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)}^{2\tan^{-1}(1/\sqrt{5})}\frac{d\varphi}{\sqrt{1-\frac{9}{25}\sin^2\varphi}}$$

$$=\frac{1}{9}\int_{2\tan^{-1}\left(\frac{1-v\epsilon^{-3}}{1+v\epsilon^3}\sqrt{\frac{(1+v\epsilon)(1-v\epsilon^5)}{(1-v\epsilon^{-1})(1+v\epsilon^{-5})}}\right)}^{\pi/2}\frac{d\varphi}{\sqrt{1-\frac{1}{81}\sin^2\varphi}} \tag{7.12}$$

$$=\frac{1}{4}\int_{\tan^{-1}\left((3-\sqrt{5})\sqrt{\frac{(1-v\epsilon^{-1})(1-v\epsilon^5)}{(1+v\epsilon)(1+v\epsilon^{-5})}}\right)}^{\tan^{-1}(3-\sqrt{5})}\frac{d\varphi}{\sqrt{1-\frac{15}{16}\sin^2\varphi}}. \tag{7.13}$$

Proof of (1). Let

$$\tan(\varphi/2)=\sqrt{\frac{1-11v(t)-v^2(t)}{5(1+v(t)-v^2(t))}}. \tag{7.14}$$

Clearly,
(7.15)
$$\varphi(0)=2\tan^{-1}\left(\frac{1}{\sqrt{5}}\right)\qquad\text{and}\qquad\varphi(q)=2\tan^{-1}\sqrt{\frac{1-11v(q)-v^2(q)}{5(1+v(q)-v^2(q))}}.$$

Differentiating both sides of (7.14) with respect to t, we find, after a modest calculation, that

$$\tan(\varphi/2)\sec^2(\varphi/2)\frac{d\varphi}{dt}=-\frac{12(1+v^2(t))}{5(1+v(t)-v^2(t))^2}\frac{dv}{dt}. \tag{7.16}$$

From (7.14) and elementary trigonometry, with the argument t deleted for brevity,

$$\tan(\varphi/2)\sec^2(\varphi/2)=\frac{6(1-v-v^2)\sqrt{1-11v-v^2}}{(5(1+v-v^2))^{3/2}}. \tag{7.17}$$

From (7.16) and (7.17), it follows that

$$\begin{aligned} \frac{d\varphi/dt}{dv/dt} &= -\frac{2\sqrt{5}(1+v^2)}{(1-v-v^2)\sqrt{(1-11v-v^2)(1+v-v^2)}} \\ &= -\frac{2\sqrt{5}(1+v^2)}{(1-v-v^2)\sqrt{1-10v-13v^2+10v^3+v^4}}. \end{aligned} \tag{7.18}$$

From further elementary trigonometry,

$$\sin^2\varphi = 4\sin^2(\varphi/2)\cos^2(\varphi/2) = \frac{5(1-11v-v^2)(1+v-v^2)}{9(1-v-v^2)^2},$$

and so

$$1-\frac{9}{25}\sin^2\varphi = \frac{4(1+v^2)^2}{5(1-v-v^2)^2}. \tag{7.19}$$

From (7.18) and (7.19), we deduce that

$$\frac{d\varphi/dt}{dv/dt} = -\frac{5\sqrt{1-\frac{9}{25}\sin^2\varphi}}{\sqrt{1-10v-13v^2+10v^3+v^4}}. \tag{7.20}$$

Using (7.15) and (7.20), we find that, with $v = v(q)$,

$$\begin{aligned} &\int_0^q f(-t)f(-t^3)f(-t^5)f(-t^{15})dt \\ &\qquad = \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)}^{2\tan^{-1}(1/\sqrt{5})} f(-t)f(-t^3)f(-t^5)f(-t^{15}) \\ &\qquad\quad \times \frac{\sqrt{1-10v-13v^2+10v^3+v^4}}{\dfrac{dv}{dt}\sqrt{1-\frac{9}{25}\sin^2\varphi}}d\varphi. \end{aligned}$$

Invoking Lemma 7.4, we complete the proof of (7.11). □

Lemma 7.6 (First version of Landen's transformation)**.** *If* $0 \le \alpha, \beta \le \pi/2$, $0 < x < 1$, *and* $\sin(2\beta-\alpha) = x\sin\alpha$, *then*

$$\int_0^\alpha \frac{d\varphi}{\sqrt{1-x^2\sin^2\varphi}} = \frac{2}{1+x}\int_0^\beta \frac{d\varphi}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\varphi}}.$$

Lemma 7.6 can be found as Entry 7(xiii) in Chapter 17 in Ramanujan's second notebook [1, p. 113]. If we replace x by $(1-\sqrt{1-x^2})/(1+\sqrt{1-x^2})$ in Lemma 7.6 and interchange the roles of α and β, we obtain the following second version of Landen's transformation.

Lemma 7.7 (Second version of Landen's transformation)**.** *If* $0 \le \alpha, \beta \le \pi/2$, $0 < x < 1$, *and* $\tan(\beta-\alpha) = \sqrt{1-x^2}\tan\alpha$, *then*

$$\int_0^\alpha \frac{d\varphi}{\sqrt{1-x^2\sin^2\varphi}} = \frac{1}{1+\sqrt{1-x^2}}\int_0^\beta \frac{d\varphi}{\sqrt{1-\left(\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)^2\sin^2\varphi}}.$$

Proof of (7.12). We apply Lemma 7.7 with $x = \frac{3}{5}$. Then $\tan(\beta-\alpha) = \frac{4}{5}\tan\alpha$. Suppose that

$$\alpha = 2\tan^{-1}\sqrt{\frac{1-11v(q)-v^2(q)}{5(1+v(q)-v^2(q))}}. \tag{7.21}$$

If $q=0$, then $\alpha = 2\tan^{-1}(1/\sqrt{5})$. In comparing (7.11) and (7.12), we must prove that, with the agrument q deleted for brevity,

$$\tan(\beta/2) = \frac{(1-v\epsilon^{-3})}{(1+v\epsilon^3)}\sqrt{\frac{(1+v\epsilon)(1-v\epsilon^5)}{(1-v\epsilon^{-1})(1+v\epsilon^{-5})}}, \tag{7.22}$$

for if $q=0$, then $\beta=\pi/2$.

Set $t_1 = \tan(\alpha/2)$ and $t_2 = \tan\left((\beta-\alpha)/2\right)$. Then

$$\frac{2t_2}{1-t_2^2} = \tan(\beta-\alpha) = \tfrac{4}{5}\tan\alpha = \frac{8t_1}{5(1-t_1^2)}.$$

If we consider the extremal equality as a quadratic equation in t_2, a routine calculation gives

$$t_2 = -\frac{5(1-t_1^2)}{8t_1} + \frac{1}{2}\sqrt{\frac{25(1-t_1^2)^2}{16t_1^2}+4}, \tag{7.23}$$

since $t_2>0$. Using (7.21) and the definition of t_1, we find that

$$1-t_1^2 = \frac{4(1+4v-v^2)}{5(1+v-v^2)} \tag{7.24}$$

and

$$\frac{25(1-t_1^2)^2}{16t_1^2}+4 = \frac{9(1+v^2)^2}{(1+v-v^2)(1-11v-v^2)}, \tag{7.25}$$

after a lengthy calculation. Employing (7.21), (7.24), and (7.25) in (7.23), we conclude that

$$\begin{aligned} t_2 &= -\frac{\sqrt{5}(1+4v-v^2)+3(1+v^2)}{2\sqrt{(1+v-v^2)(1-11v-v^2)}} \\ &= \frac{\epsilon^2(\epsilon-v)(\epsilon^{-5}-v)}{\sqrt{(1-v\epsilon^{-1})(1+v\epsilon)(1-v\epsilon^5)(1+v\epsilon^{-5})}} \\ &= \epsilon^{-2}\sqrt{\frac{(1-v\epsilon^{-1})(1-v\epsilon^5)}{(1+v\epsilon)(1+v\epsilon^{-5})}}. \end{aligned} \tag{7.26}$$

Hence, by (7.21) and (7.26),

$$
\begin{aligned}
\tan(\beta/2) = \tan\left(\alpha/2 + (\beta-\alpha)/2\right) &= \frac{t_1+t_2}{1-t_1t_2} \\
&= \frac{\sqrt{\dfrac{(1-v\epsilon^5)(1+v\epsilon^{-5})}{5(1-v\epsilon^{-1})(1+v\epsilon)}} + \epsilon^{-2}\sqrt{\dfrac{(1-v\epsilon^{-1})(1-v\epsilon^5)}{(1+v\epsilon)(1+v\epsilon^{-5})}}}{1-\dfrac{\epsilon^{-2}(1-v\epsilon^5)}{\sqrt{5}(1+v\epsilon)}} \\
&= \sqrt{\frac{(1-v\epsilon^5)(1+v\epsilon)}{(1-v\epsilon^{-1})(1+v\epsilon^{-5})}}\left(\frac{(1+v\epsilon^{-5})+\epsilon^{-2}\sqrt{5}(1-v\epsilon^{-1})}{\sqrt{5}(1+v\epsilon)-\epsilon^{-2}(1-v\epsilon^5)}\right) \\
&= \sqrt{\frac{(1-v\epsilon^5)(1+v\epsilon)}{(1-v\epsilon^{-1})(1+v\epsilon^{-5})}}\,\frac{(\frac{3}{2}\sqrt{5}-\frac{3}{2})(1-v\epsilon^{-3})}{(\frac{3}{2}\sqrt{5}-\frac{3}{2})(1+v\epsilon^{3})}.
\end{aligned}
$$

Thus, (7.22) has been established, and the proof of (7.12) is complete. □

Proof of (7.13). We apply Lemma 7.6 with $x = \frac{3}{5}$, and let α be given by (7.21). Comparing (7.11) and (7.13), we see that it suffices to prove that, with the argument q deleted for brevity,

$$
t := \tan\beta = 2\epsilon^{-2}\sqrt{\frac{(1-v\epsilon^{-1})(1-v\epsilon^5)}{(1+v\epsilon)(1+v\epsilon^{-5})}}, \tag{7.27}
$$

for if $q = 0$, then $t = 2\epsilon^{-2} = 3-\sqrt{5}$.

Now the hypothesis $\sin(2\beta-\alpha) = \frac{3}{5}\sin\alpha$ in Lemma 7.6, by the addition formula for the sine function and the double angle formulas for both the sine and cosine functions, easily translates to the condition

$$
\tan\alpha = \frac{\sin(2\beta)}{\frac{3}{5}+\cos(2\beta)} = \frac{5\tan\beta}{4-\tan^2\beta}. \tag{7.28}
$$

Using (7.28), (7.27), and (7.21), we have

$$
\begin{aligned}
\frac{5t}{4-t^2} = \tan\alpha &= \frac{2\tan(\alpha/2)}{1-\tan^2(\alpha/2)} \\
&= \frac{\sqrt{5(1-11v-v^2)(1+v-v^2)}}{2(1+4v-v^2)}. \qquad (7.29)
\end{aligned}
$$

Considering (7.29) as a quadratic equation in t, we solve it to deduce that

$$
\begin{aligned}
t = \frac{1}{2}\Bigg(&-\frac{2\sqrt{5}(1+4v-v^2)}{\sqrt{(1-11v-v^2)(1+v-v^2)}} \\
&+\sqrt{\frac{20(1+4v-v^2)^2}{(1-11v-v^2)(1+v-v^2)}+16}\Bigg),
\end{aligned}
$$

since $t > 0$. Simplifying, we find that

$$t = -\frac{\sqrt{5}(1+4v-v^2)}{\sqrt{(1-11v-v^2)(1+v-v^2)}} + \frac{3(1+v^2)}{\sqrt{(1-11v-v^2)(1+v-v^2)}}$$
$$= 2\epsilon^{-2}\sqrt{\frac{(1-v\epsilon^{-1})(1-v\epsilon^5)}{(1+v\epsilon)(1-v\epsilon^{-5})}},$$

by identically the same calculation that we used in (7.26). Thus, (7.27) has been proved, and the proof of (7.13) is complete. □

For the remainder of this section, set

$$v = q\left(\frac{f(-q^3)f(-q^{15})}{f(-q)f(-q^5)}\right)^2. \tag{7.30}$$

Theorem 7.8 (p. 53)**.** *If v is defined by (7.30), then*

$$\int_0^q f(-t)f(-t^3)f(-t^5)f(-t^{15})dt = \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1-3v}{\sqrt{5}(1+3v)}\right)}^{2\tan^{-1}(1/\sqrt{5})} \frac{d\varphi}{\sqrt{1-\frac{9}{25}\sin^2\varphi}}. \tag{7.31}$$

Proof. Because of the conflict in notation between (7.1) and (7.30), for this proof only, we set

$$u := q\left(\frac{f(-q)f(-q^{15})}{f(-q^3)f(-q^5)}\right)^3. \tag{7.32}$$

In the notation (7.30) and (7.32), Lemma 7.1 takes the form

$$\frac{1}{v} + 5 + 9v = \frac{1}{u} - u.$$

By using the previous equality, we can easily verify that

$$\frac{1-3v}{1+3v} = \sqrt{\frac{1-11u-u^2}{1+u-u^2}}.$$

Thus, (7.31) follows immediately from (7.1). □

8. Elliptic Integrals of Order 14

As in the previous section, the primary theorem in the present section depends upon a first order differential equation satisfied by a certain quotient of eta-functions and established through a series of lemmas. Let

$$v := v(q) := q\left(\frac{f(-q)f(-q^{14})}{f(-q^2)f(-q^7)}\right)^4. \tag{8.1}$$

Lemma 8.1. *If v is defined by (8.1) and*

$$R = \frac{1}{q}\left(\frac{f(-q)f(-q^7)}{f(-q^2)f(-q^{14})}\right)^3, \tag{8.2}$$

then

$$R + 7 + \frac{8}{R} = v + \frac{1}{v}. \tag{8.3}$$

Lemma 8.1 is a reformulation of Entry 19(ix) in Chapter 19 of Ramanujan's second notebook [1, p. 315]. To see this, replace q by $-q$ in the definitions of (8.1) and (8.2). Then use Entries 12(i), (iii) in Chapter 17 of the second notebook [1, p. 124] to convert (8.3) into a modular equation, which is readily seen to be the same as Entry 19(ix).

Lemma 8.2. *Let*

$$P = \frac{1}{q}\left(\frac{f(-q)}{f(-q^7)}\right)^4 \qquad \textit{and} \qquad Q = \frac{1}{q^2}\left(\frac{f(-q^2)}{f(-q^{14})}\right)^4. \tag{8.4}$$

Then

$$P + \frac{49}{P} = R - 1 + \frac{48}{R} + \frac{64}{R^2} \tag{8.5}$$

and

$$Q + \frac{49}{Q} = R^2 + 6R - 1 + \frac{8}{R}. \tag{8.6}$$

Proof. In the notation (8.4), Ramanujan discovered the eta-function identity [2, p. 209, Entry 55]

$$\begin{aligned}\sqrt{PQ} + \frac{49}{\sqrt{PQ}} &= v^{3/2} - 8v^{1/2} - 8v^{-1/2} + v^{-3/2} \\ &= \left(\frac{1}{\sqrt{v}} + \sqrt{v}\right)^3 - 11\left(\frac{1}{\sqrt{v}} + \sqrt{v}\right) \\ &= K(K^2 - 11), \end{aligned} \tag{8.7}$$

where

$$K = \frac{1}{\sqrt{v}} + \sqrt{v}. \tag{8.8}$$

Letting $c = K(K^2 - 11)$ and solving (8.7) for $\sqrt{PQ}$, we find that

$$\sqrt{PQ} = \frac{c + \sqrt{c^2 - 196}}{2},$$

where the correct root was found by an examination of $\sqrt{PQ}$ in a neighborhood of $q=0$. A brief calculation now gives

$$\begin{aligned}\sqrt{PQ}-\frac{49}{\sqrt{PQ}} &= \sqrt{c^2-196}\\ &= \sqrt{K^6-22K^4+121K^2-196}\\ &= \sqrt{(K^2-4)(K^4-18K^2+49)}.\end{aligned}\tag{8.9}$$

Multiplying (8.7) by K (given by (8.8)) and (8.9) by $\frac{1}{\sqrt{v}}-\sqrt{v}=\sqrt{K^2-4}$, and using (8.3), we deduce that, respectively,

$$\begin{aligned}P+\frac{49}{P}+Q+\frac{49}{Q} &= K^2(K^2-11)\\ &= \left(R+\frac{8}{R}+9\right)\left(R+\frac{8}{R}-2\right)\\ &= \frac{(R+8)(R+1)(R^2-2R+8)}{R^2}\end{aligned}\tag{8.10}$$

and

$$\begin{aligned}&-P-\frac{49}{P}+Q+\frac{49}{Q}\\ &= (K^2-4)\sqrt{K^4-18K^2+49}\\ &= \left(R+\frac{8}{R}+5\right)\sqrt{\left(R+\frac{8}{R}+9\right)^2-18\left(R+\frac{8}{R}+9\right)+49}\\ &= \frac{(R^2+5R+8)(R^2-8)}{R^2}.\end{aligned}\tag{8.11}$$

Solving (8.10) and (8.11), we deduce (8.5) and (8.6). □

Lemma 8.3. *We have*

$$1+4\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-28\sum_{k=1}^{\infty}\frac{kq^{7k}}{1-q^{7k}} = \left\{\frac{f^8(-q)+13qf^4(-q)f^4(-q^7)+49q^2f^8(-q^7)}{f(-q)f(-q^7)}\right\}^{2/3}.$$

Lemma 8.3 is part of Entry 5(i) in Chapter 21 of Ramanujan's second notebook [1, p. 467].

Lemma 8.4. *If v is defined by (8.1), then*

$$\frac{dv}{dq}=f(-q)f(-q^2)f(-q^7)f(-q^{14})\sqrt{1-14v+19v^2-14v^3+v^4}.$$

Proof. From the definition (8.1) of v, Lemma 8.3, (8.4), (8.2), and Lemma 8.2,

(8.12)

$$\begin{aligned}
\frac{q}{v}\frac{dv}{dq} =& q\frac{d\log v}{dq} = q\frac{d\log\{q^2\frac{f^4(-q^{14})}{f^4(-q^2)}\}}{dq} + q\frac{d\log\{q^{-1}\frac{f^4(-q)}{f^4(-q^7)}\}}{dq}\\
=& q\left\{\frac{2}{q} + 8\sum_{n=1}^{\infty}\frac{nq^{2n-1}}{1-q^{2n}} - 56\sum_{n=1}^{\infty}\frac{nq^{14n-1}}{1-q^{14n}}\right\}\\
&+ q\left\{-\frac{1}{q} + 28\sum_{n=1}^{\infty}\frac{nq^{7n-1}}{1-q^{7n}} - 4\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^{n}}\right\}\\
=& 2\left(\frac{f^8(-q^2)+13q^2f^4(-q^2)f^4(-q^{14})+49q^4f^8(-q^{14})}{f(-q^2)f(-q^{14})}\right)^{2/3}\\
&- \left(\frac{f^8(-q)+13qf^4(-q)f^4(-q^7)+49q^2f^8(-q^7)}{f(-q)f(-q^7)}\right)^{2/3}\\
=& qf(-q)f(-q^2)f(-q^7)f(-q^{14})\left\{2q^{1/3}\frac{f(-q^2)f(-q^{14})}{f(-q)f(-q^7)}\right.\\
&\times\left.\left(Q+\frac{49}{Q}+13\right)^{2/3} - q^{-1/3}\frac{f(-q)f(-q^7)}{f(-q^2)f(-q^{14})}\left(P+\frac{49}{P}+13\right)^{2/3}\right\}\\
=& qf(-q)f(-q^2)f(-q^7)f(-q^{14})\left\{\frac{2}{R^{1/3}}\left(R^2+6R+12+\frac{8}{R}\right)^{2/3}\right.\\
&\left. - R^{1/3}\left(R+12+\frac{48}{R}+\frac{64}{R^2}\right)^{2/3}\right\}\\
=& qf(-q)f(-q^2)f(-q^7)f(-q^{14})\left\{\frac{2}{R^{1/3}}\left(\frac{(R+2)^6}{R^2}\right)^{1/3}\right.\\
&\left. - R^{1/3}\left(\frac{(R+4)^6}{R^4}\right)^{1/3}\right\}\\
=& qf(-q)f(-q^2)f(-q^7)f(-q^{14})\left(R-\frac{8}{R}\right).
\end{aligned}$$

Now, by using Lemma 8.1, we can easily verify that

$$\begin{aligned}
\left(R-\frac{8}{R}\right)^2 &= \left(R+\frac{8}{R}\right)^2 + 14\left(R+\frac{8}{R}\right) + 49 - 14\left(R+7+\frac{8}{R}\right) + 17\\
&= \left(R+7+\frac{8}{R}\right)^2 - 14\left(R+7+\frac{8}{R}\right) + 17
\end{aligned}$$

$$=\left(v+\frac{1}{v}\right)^2-14\left(v+\frac{1}{v}\right)+17$$

$$=\frac{1-14v+19v^2-14v^3+v^4}{v^2}. \tag{8.13}$$

Taking the square roots of both sides of (8.13) and substituting in (8.12), we complete the proof. □

Theorem 8.5 (p. 51)**.** *If v is defined by (8.1) and if*

$$c=\frac{\sqrt{13+16\sqrt{2}}}{7},$$

then

$$\int_0^q f(-t)f(-t^2)f(-t^7)f(-t^{14})dt=\frac{1}{\sqrt{8\sqrt{2}}}\int_{\cos^{-1}\left(c\frac{1+v}{1-v}\right)}^{\cos^{-1}c}\frac{d\varphi}{\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\varphi}}. \tag{8.14}$$

Proof. Let

$$\cos\varphi=c\frac{1+v(t)}{1-v(t)}, \tag{8.15}$$

so that at $t=0,q$ we obtain the upper and lower limits, respectively, in the integral on the right side of (8.14). Differentiating (8.15), we find that

$$-\sin\varphi\frac{d\varphi}{dt}=\frac{2c}{(1-v(t))^2}\frac{dv}{dt}. \tag{8.16}$$

By elementary trigonometry,

$$\sin\varphi=\frac{\sqrt{(1-v)^2-c^2(1+v)^2}}{1-v}. \tag{8.17}$$

Putting (8.17) in (8.16), we arrive at

$$\frac{d\varphi/dt}{dv/dt}=-\frac{2c}{(1-v)\sqrt{(1-v)^2-c^2(1+v)^2}}. \tag{8.18}$$

Next, by (8.17),

$$1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\varphi=\frac{32\sqrt{2}(1-v)^2-(16\sqrt{2}-13)\left\{(1-v)^2-c^2(1+v)^2\right\}}{32\sqrt{2}(1-v)^2}$$

$$=\frac{(16\sqrt{2}+13)(1-v)^2+7(1+v)^2}{32\sqrt{2}(1-v)^2}. \tag{8.19}$$

Thus, by (8.18) and (8.19),

$$\begin{aligned}
&\frac{dv/dt}{d\varphi/dt}\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\varphi}\\
&=-\frac{\sqrt{(1-v)^2-c^2(1+v)^2}}{2c}\frac{\sqrt{(16\sqrt{2}+13)(1-v)^2+7(1+v)^2}}{2\sqrt{8\sqrt{2}}}\\
&=-\frac{\sqrt{49(1-v)^2-(16\sqrt{2}+13)(1+v)^2}\sqrt{(16\sqrt{2}+13)(1-v)^2+7(1+v)^2}}{4\sqrt{16\sqrt{2}+13}\sqrt{8\sqrt{2}}}\\
&=-\frac{\sqrt{1-14v+19v^2-14v^3+v^4}}{\sqrt{8\sqrt{2}}},
\end{aligned}$$

after a calculation via *Mathematica*. Thus,

$$\begin{aligned}
&\int_0^q f(-t)f(-t^2)f(-t^7)f(-t^{14})dt\\
=&\frac{1}{\sqrt{8\sqrt{2}}}\int_{\cos^{-1}\left(c\frac{1+v}{1-v}\right)}^{\cos^{-1}c} f(-t)f(-t^2)f(-t^7)f(-t^{14})\\
&\times\frac{\sqrt{1-14v+19v^2-14v^3+v^4}}{\frac{dv}{dt}\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\varphi}}d\varphi\\
=&\frac{1}{\sqrt{8\sqrt{2}}}\int_{\cos^{-1}\left(c\frac{1+v}{1-v}\right)}^{\cos^{-1}c}\frac{d\varphi}{\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\varphi}},
\end{aligned}$$

upon the use of Lemma 8.4 □

9. An Elliptic Integral of Order 35

To avoid square roots, we have modestly reformulated Ramanujan's integral equality (Theorem 9.5 below). Throughout this section, set

$$v:=v(q):=q\frac{f(-q)f(-q^{35})}{f(-q^5)f(-q^7)}. \tag{9.1}$$

(Ramanujan defined v by the square of the right side of (9.1).) Ramanujan's theorem depends upon a differential equation for v which we prove through a series of lemmas.

Lemma 9.1. *Let*

$$R=\frac{f(-q)f(-q^5)}{q^{3/2}f(-q^7)f(-q^{35})}.$$

Then

$$R^2-5+\frac{49}{R^2}=\frac{1}{v^3}-5\frac{1}{v^2}-5v^2-v^3.$$

Lemma 9.1 can be found on page 303 of Ramanujan's second notebook [15]; a proof is given in [2, pp. 236–242].

Lemma 9.2. *Let*

$$P = \frac{f(-q)}{q^{1/6} f(-q^5)} \quad \textit{and} \quad Q = \frac{f(-q^7)}{q^{7/6} f(-q^{35})}.$$

Then

$$(PQ)^3 + \frac{125}{(PQ)^3} = \left(\frac{1}{v^4} - v^4\right) - 7\left(\frac{1}{v^3} + v^3\right) + 7\left(\frac{1}{v^2} - v^2\right) + 14\left(\frac{1}{v} + v\right).$$

This eta-function identity is not found in Ramanujan's ordinary notebooks [15], but it is recorded in his lost notebook [17, p. 55]. Berndt's paper [4] contains a proof.

Lemma 9.3. *We have*

$$1 + 6\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} - 30\sum_{k=1}^{\infty} \frac{kq^{5k}}{1-q^{5k}}$$
$$= \sqrt{\frac{f^{12}(-q) + 22q f^6(-q) f^6(-q^5) + 125q^2 f^{12}(-q^5)}{f^2(-q) f^2(-q^5)}}.$$

For a proof of this result from Chapter 21 of Ramanujan's second notebook, see Berndt's book [1, p. 463, Entry 4(i)].

Lemma 9.4. *If v is defined by (9.1), then*

$$v\frac{dv}{dq} = q f(-q) f(-q^5) f(-q^7) f(-q^{35}) \sqrt{(1+v-v^2)(1-5v-9v^3-5v^5-v^6)}.$$

Proof. Set

$$K = \frac{1}{v} - v. \tag{9.2}$$

Then Lemma 9.2 can be reformulated as

$$(PQ)^3 + \frac{125}{(PQ)^3} = (K^3 - 7K^2 + 9K + 7)\sqrt{K^2+4}. \tag{9.3}$$

Considering (9.3) as a quadratic equation in $(PQ)^3$, we solve it. Then after a tedious, but elementary, calculation, we find that

$$(PQ)^3 - \frac{125}{(PQ)^3} = (K-1)(K-4)\sqrt{(K+1)(K^3 - 5K^2 + 3K - 19)}. \tag{9.4}$$

Now multiply both sides of (9.3) by

$$\frac{1}{v^3} + v^3 = (K^2+1)\sqrt{K^2+4}$$

and both sides of (9.4) by

$$\frac{1}{v^3} - v^3 = K(K^2+3),$$

and use the observation $v = P/Q$ to deduce that, respectively,

$$P^6 + \frac{125}{P^6} + Q^6 + \frac{125}{Q^6} = U_1 \tag{9.5}$$

and

$$-P^6 - \frac{125}{P^6} + Q^6 + \frac{125}{Q^6} = U_2, \tag{9.6}$$

where

$$U_1 := (K^2+4)(K^2+1)(K^3-7K^2+9K+7) \tag{9.7}$$

and

$$U_2 := K(K-1)(K-4)(K^2+3)\sqrt{(K+1)(K^3-5K^2+3K-19)}. \tag{9.8}$$

Solving (9.5) and (9.6), we deduce that

$$P^6 + \frac{125}{P^6} = \frac{1}{2}\left(U_1 - U_2\right) \tag{9.9}$$

and

$$Q^6 + \frac{125}{Q^6} = \frac{1}{2}\left(U_1 + U_2\right). \tag{9.10}$$

Using the definition of K in (9.2), we can rewrite Lemma 9.1 in the form

$$R^2 + \frac{49}{R^2} = K^3 - 5K^2 + 3K - 5. \tag{9.11}$$

Considering (9.11) as a quadratic equation in R^2, we solve it and find that

$$R^2 = \frac{1}{2}\left(V_1 + V_2\right) \qquad \text{and} \qquad \frac{49}{R^2} = \frac{1}{2}\left(V_1 - V_2\right), \tag{9.12}$$

where

$$V_1 := K^3 - 5K^2 + 3K - 5 \tag{9.13}$$

and

$$V_2 := (K-3)\sqrt{(K+1)(K^3-5K^2+3K-19)}. \tag{9.14}$$

Now, by Lemma 9.3, we find that

$$
\begin{aligned}
(9.15)\quad \frac{1}{v}\frac{dv}{dq} =&\frac{d\log v}{dq} = \frac{d\log\{q^{7/6}\frac{f(-q^{35})}{f(-q^7)}\}}{dq} + \frac{d\log\{q^{-1/6}\frac{f(-q)}{f(-q^5)}\}}{dq}\\
=&\frac{7}{6q} + 7\sum_{n=1}^{\infty}\frac{nq^{7n-1}}{1-q^{7n}} - 35\sum_{n=1}^{\infty}\frac{nq^{35n-1}}{1-q^{35n}}\\
&-\frac{1}{6q} + 5\sum_{n=1}^{\infty}\frac{nq^{5n-1}}{1-q^{5n}} - \sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^{n}}\\
=&\frac{7}{6q}\sqrt{\frac{f^{12}(-q^7)+22q^7f^6(-q^7)f^6(-q^{35})+125q^{14}f^{12}(-q^{35})}{f^2(-q^7)f^2(-q^{35})}}\\
&-\frac{1}{6q}\sqrt{\frac{f^{12}(-q)+22qf^6(-q)f^6(-q^5)+125q^2f^{12}(-q^5)}{f^2(-q)f^2(-q^5)}}\\
=&qf(-q)f(-q^5)f(-q^7)f(-q^{35})\\
&\times\left(-\frac{7}{6R}\sqrt{Q^6+\frac{125}{Q^6}+22} + \frac{R}{6}\sqrt{P^6+\frac{125}{P^6}+22}\right).
\end{aligned}
$$

where we have used the definitions of P, Q, and R in Lemmas 9.2 and 9.1.

Squaring both sides of (9.15) and simplifying with the use of (9.12), (9.9), (9.10), (9.7), (9.8), (9.13), and (9.14), we find that

$$
\begin{aligned}
(9.16)\quad &\left(\frac{1}{qf(-q)f(-q^5)f(-q^7)f(-q^{35})}\frac{dv}{dq}\right)^2\\
&=\frac{v^2}{36}\Bigg\{\frac{49}{R^2}\left(Q^6+\frac{125}{Q^6}+22\right) + R^2\left(P^6+\frac{125}{P^6}+22\right)\\
&\qquad -14\sqrt{\left(Q^6+\frac{125}{Q^6}+22\right)\left(P^6+\frac{125}{P^6}+22\right)}\Bigg\}\\
&=\frac{v^2}{36}\Bigg\{\frac{(V_1-V_2)}{2}\left(\frac{U_1+U_2}{2}+22\right) + \frac{(V_1+V_2)}{2}\left(\frac{U_1-U_2}{2}+22\right)\\
&\qquad -14\sqrt{\left(\frac{U_1}{2}+22\right)^2-\frac{U_2^2}{4}}\Bigg\}\\
&=\frac{v^2}{36}\left\{\frac{1}{2}(U_1V_1-U_2V_2)+22V_1-14\sqrt{\left(\frac{U_1}{2}+22\right)^2-\frac{U_2^2}{4}}\right\}\\
&=v^2(K^4-4K^3-2K^2-16K-19),
\end{aligned}
$$

where the last step involves a considerable amount of algebra. Lastly, by (9.2), we substitute $K=\frac{1}{v}-v$ into (9.16). Upon simplification, factorization, and taking the square roots of both sides, we complete the proof. □

Theorem 9.5 (p. 53). *If v is defined by (9.1), then*

$$\int_0^q t\ f(-t)f(-t^5)f(-t^7)f(-t^{35})dt$$
$$=\int_0^v \frac{t\ dt}{\sqrt{(1+t-t^2)(1-5t-9t^3-5t^5-t^6)}}.$$

Proof. Let $v(t)$ be defined by (9.1). Then the limits $t=0,q$ are transformed into $0, v=v(q)$, respectively. Thus,

$$\int_0^q t\ f(-t)f(-t^5)f(-t^7)f(-t^{35})dt$$
$$=\int_0^{v(q)} \frac{t\ f(-t)f(-t^5)f(-t^7)f(-t^{35})}{dv/dt}dv$$
$$=\int_0^{v(q)} \frac{v\ dv}{\sqrt{(1+v-v^2)(1-5v-9v^3-5v^5-v^6)}},$$

upon the employment of Lemma 9.4. Thus, the proof is complete. □

10. Constructions of new incomplete elliptic integral identities

It is clear from the previous sections that Ramanujan's incomplete elliptic integrals arise from differential equations satisfied by quotients of eta-functions. In this section, we first list three differential equations and derive their corresponding elliptic integrals of order 6. We will then briefly describe why such differential equations exist and construct several new examples, as well as their corresponding incomplete elliptic integrals.

Theorem 10.1. *Let*

$$v := q\frac{f^4(-q^3)f^4(-q^6)}{f^4(-q)f^4(-q^2)}.$$

Then

$$\frac{dv}{dq} = q^{-1/2}f(-q)f(-q^2)f(-q^3)f(-q^6)\sqrt{81v^3+14v^2+v}.$$

To prove Theorem 10.1, we need the following lemmas:

Lemma 10.2. *Let*

$$P := q^{-1/6}\frac{f^2(-q)}{f^2(-q^3)}, \quad Q := q^{-1/3}\frac{f^2(-q^2)}{f^2(-q^6)},$$
$$R := Q/P, \quad \text{and} \quad S := q^{-1/6}\frac{f(-q)f(-q^3)}{f(-q^2)f(-q^6)}.$$

Then [2, p. 204, Entry 51]

$$PQ + \frac{9}{PQ} = R^3 + \frac{1}{R^3} \tag{10.1}$$

and [2, p. 205, Entry 52]

$$PQ - \frac{9}{PQ} = S^3 - \frac{8}{S^3}. \tag{10.2}$$

Lemma 10.3.

$$\left(Q^3 + \frac{27}{Q^3}\right)^2 = \left(\frac{S^6+4}{S^2}\right)^3, \tag{10.3}$$

$$\left(P^3 + \frac{27}{P^3}\right)^2 = \left(\frac{S^6+16}{S^4}\right)^3. \tag{10.4}$$

Lemma 10.3 appears to be new. We will present the proof of this lemma using ideas illustrated in Sections 7–9 and Lemma 10.2.

Proof. By cubing both sides of (10.1), we deduce that

$$(PQ)^3 + \frac{9^3}{(PQ)^3} + 27\left(PQ + \frac{9}{PQ}\right) = \left(R^3 + \frac{1}{R^3}\right)^3.$$

This implies that

$$(PQ)^3 + \frac{9^3}{(PQ)^3} = \left(R^3 + \frac{1}{R^3}\right)^3 - 27\left(R^3 + \frac{1}{R^3}\right), \tag{10.5}$$

by (10.1). Similarly, by cubing (10.2) and simplifying, we deduce that

$$(PQ)^3 - \frac{9^3}{(PQ)^3} = \left(S^3 - \frac{8}{S^3}\right)^3 + 27\left(S^3 - \frac{8}{S^3}\right). \tag{10.6}$$

Multiplying (10.5) by $\left(R^3 + 1/R^3\right)$ and (10.6) by $\left(R^3 - 1/R^3\right)$, we find that

$$Q^6 + \frac{9^3}{Q^6} + P^6 + \frac{9^3}{P^6} = \left(R^3 + \frac{1}{R^3}\right)^4 - 27\left(R^3 + \frac{1}{R^3}\right)^2 \tag{10.7}$$

and

$$Q^6 + \frac{9^3}{Q^6} - P^6 - \frac{9^3}{P^6} = \left(R^3 - \frac{1}{R^3}\right)\left\{\left(S^3 - \frac{8}{S^3}\right)^3 + 27\left(S^3 - \frac{8}{S^3}\right)\right\}. \tag{10.8}$$

Next, from (10.1) and (10.2), we have

$$R^3 + \frac{1}{R^3} = PQ + \frac{9}{PQ} = \sqrt{\left(S^3 - \frac{8}{S^3}\right)^2 + 36}.$$

Therefore,

$$\left(R^3 + \frac{1}{R^3}\right)^2 = S^6 + \frac{64}{S^6} + 20. \tag{10.9}$$

From (10.9), we also obtain

$$R^3 - \frac{1}{R^3} = S^3 + \frac{8}{S^3}. \tag{10.10}$$

Substituting (10.9) and (10.10) into (10.7) and (10.8), we find that

$$Q^6 + \frac{9^3}{Q^6} + P^6 + \frac{9^3}{P^6} = \left(S^6 + \frac{64}{S^6} + 20\right)^2 - 27\left(S^6 + \frac{64}{S^6} + 20\right) \tag{10.11}$$

and

$$Q^6 + \frac{9^3}{Q^6} - P^6 - \frac{9^3}{P^6} = \left(S^3 + \frac{8}{S^3}\right)\left\{\left(S^3 - \frac{8}{S^3}\right)^3 + 27\left(S^3 - \frac{8}{S^3}\right)\right\}. \tag{10.12}$$

Adding (10.11) and (10.12), we have

$$Q^6 + \frac{9^3}{Q^6} = S^{12} + 12S^6 - 6 + \frac{64}{S^6},$$

which implies (10.3). Similarly, by subtracting (10.12) from (10.11), we have

$$P^6 + \frac{9^3}{P^6} = S^6 - 6 + \frac{768}{S^6} + \frac{4096}{S^{12}},$$

which implies (10.4). This completes the proof of Lemma 10.3 □

Lemma 10.4. [1, p. 460, Entry 3(i)]

$$1 + 12\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}} = \left(\frac{f^{12}(-q) + 27qf^{12}(-q^3)}{f^3(-q)f^3(-q^3)}\right)^{2/3}.$$

Proof of Theorem 10.1. By logarithmically differentiating v, we deduce that

$$\begin{aligned}\frac{1}{v}\frac{dv}{dq} =& \frac{1}{3q}\left(1 + 12\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}} + 2\right.\\ &\left. + 24\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} - 72\sum_{n=1}^{\infty}\frac{nq^{6n}}{1-q^{6n}}\right).\end{aligned}$$

By Lemmas 10.2, 10.3, and 10.4, we may rewrite the last equality in the form

$$\begin{aligned}\frac{1}{v}\frac{dv}{dq} =& \frac{1}{3q^{1/2}}f(-q)f(-q^2)f(-q^3)f(-q^6)\\ &\times\left\{S\left(P^3 + \frac{27}{P^3}\right)^{2/3} + \frac{2}{S}\left(Q^3 + \frac{27}{Q^3}\right)^{2/3}\right\} \qquad (10.13)\\ =& \frac{1}{q^{1/2}}f(-q)f(-q^2)f(-q^3)f(-q^6)\left(S^3 + \frac{8}{S^3}\right).\end{aligned}$$

Now, it is clear that $\sqrt{v} = 1/(PQ)$ and hence, by (10.2), we deduce that

$$\frac{dv}{dq} = \frac{1}{q^{1/2}} f(-q)f(-q^2)f(-q^3)f(-q^6)\sqrt{81v^3 + 14v^2 + v}.$$

□

Now, from (10.1), we observe that if $v_1 = R^{-6}$ then

$$\frac{1}{\sqrt{v}} + 9\sqrt{v} = \frac{1}{\sqrt{v_1}} + \sqrt{v_1}. \tag{10.14}$$

This implies that

$$\frac{1}{v}\left(\frac{1}{\sqrt{v}} - 9\sqrt{v}\right)\frac{dv}{dq} = \frac{1}{v_1}\left(\frac{1}{\sqrt{v_1}} - \sqrt{v_1}\right)\frac{dv_1}{dq}.$$

From (10.14), we find that

$$\frac{1}{\sqrt{v}} - 9\sqrt{v} = \sqrt{\left(\frac{1}{\sqrt{v_1}} + \sqrt{v_1}\right)^2 - 36} = \sqrt{\frac{1}{v_1} + v_1 - 34}.$$

The last two equalities imply that

$$\begin{aligned}\frac{1}{v_1}\frac{dv_1}{dq} &= \sqrt{\frac{1}{v_1} + v_1 - 34}\left(\frac{1}{\sqrt{v_1}} - \sqrt{v_1}\right)^{-1}\frac{1}{v}\frac{dv}{dq}\\ &= \frac{1}{q^{1/2}} f(-q)f(-q^2)f(-q^3)f(-q^6)\sqrt{\frac{1}{v_1} + v_1 - 34},\end{aligned}$$

by (10.10) and (10.13). Hence, we deduce the following differential equation.

Theorem 10.5. *Let*

$$v_1 := q\frac{f^{12}(-q)f^{12}(-q^6)}{f^{12}(-q^2)f^{12}(-q^3)}.$$

Then

$$\frac{dv_1}{dq} = q^{-1/2} f(-q)f(-q^2)f(-q^3)f(-q^6)\sqrt{v_1^3 - 34v_1^2 + v_1}.$$

Recall that $\sqrt{v} = 1/(PQ)$, and define $v_2 = 1/S^6$. Then (10.2) takes the form

$$\frac{1}{\sqrt{v}} - 9\sqrt{v} = \frac{1}{\sqrt{v_2}} - 8\sqrt{v_2},$$

which implies that

$$\left(\frac{1}{\sqrt{v}} + 9\sqrt{v}\right)\frac{1}{v}\frac{dv}{dq} = \left(\frac{1}{\sqrt{v_2}} + 8\sqrt{v_2}\right)\frac{1}{v_2}\frac{dv_2}{dq}.$$

By the previous two equalities, (10.2), and (10.13), we find that

$$\begin{aligned}\frac{1}{v_2}\frac{dv}{dq} &= \frac{1}{v}\frac{dv}{dq}\left(\frac{1}{\sqrt{v}} + 9\sqrt{v}\right)\left(\frac{1}{\sqrt{v_2}} + 8\sqrt{v_2}\right)^{-1}\\ &= q^{-1/2} f(-q)f(-q^2)f(-q^3)f(-q^6)\sqrt{\frac{1}{v_2} + 64v_2 + 20}.\end{aligned}$$

Hence we obtain the following differential equation.

Theorem 10.6. *Recall that*

$$v_2 := q\frac{f^6(-q^2)f^6(-q^6)}{f^6(-q)f^6(-q^3)}. \tag{10.15}$$

Then

$$\frac{dv_2}{dq} = q^{-1/2}f(-q)f(-q^2)f(-q^3)f(-q^6)\sqrt{v_2(4v_2+1)(16v_2+1)}.$$

Using Theorem 10.6, we now derive an identity associated with an incomplete elliptic integral of order 6.

Theorem 10.7. *If v_2 is defined by* (10.15)*, then*

$$\begin{aligned}\int_0^q \frac{1}{\sqrt{t}}f(-t)f(-t^2)f(-t^3)f(-t^6)\,dt &= \frac{1}{2}\int_{\sin^{-1}\sqrt{\frac{1}{4v_2+1}}}^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-\frac{3}{4}\sin^2\varphi}}\\ &= \frac{1}{2}\int_0^{\cot^{-1}\frac{1}{4\sqrt{v_2}}} \frac{d\varphi}{\sqrt{1-\frac{3}{4}\sin^2\varphi}}.\end{aligned} \tag{10.16}$$

We remark here that the upper limit of the second integral may be expressed in terms of v or v_1 by using (10.1) and (10.10), namely,

$$\begin{aligned}\frac{1}{\sqrt{v_2}} &= \frac{1}{2}\left(\sqrt{\frac{1}{v}+81v+18}+\sqrt{\frac{1}{v}+81v+14}\right)\\ &= \frac{1}{2}\left(\sqrt{\frac{1}{v_1}+64v_1+20}+\sqrt{\frac{1}{v_1}+64v_1+16}\right).\end{aligned}$$

This yields incomplete elliptic integrals associated with v or v_1. Of course, these integrals are by no means unique representations of the left hand side of Theorem 10.7 in terms of v and v_1.

Proof of Theorem 10.7. By Theorem 10.6, we find that

$$\int_0^q \frac{1}{\sqrt{t}}f(-t)f(-t^2)f(-t^3)f(-t^6)\,dt = \int_0^{v_2(q)} \frac{dv_2}{\sqrt{v_2(4v_2+1)(16v_2+1)}}. \tag{10.17}$$

Set

$$\sin^2\varphi = \frac{1}{4v_2+1}. \tag{10.18}$$

Then a routine calculation gives

$$d\varphi = -\frac{1}{\sqrt{v_2}(4v_2+1)}$$

and

$$\sqrt{1-\frac{3}{4}\sin^2\varphi} = \sqrt{\frac{16v_2+1}{4(4v_2+1)}},$$

which yield

$$\int_0^{v_2(q)} \frac{dv_2}{\sqrt{v_2(4v_2+1)(16v_2+1)}} = \frac{1}{2}\int_{\sin^{-1}\sqrt{\frac{1}{4v_2+1}}}^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-\frac{3}{4}\sin^2\varphi}}. \tag{10.19}$$

Combining (10.17) and (10.19), we deduce the first equality in (10.16).

The second integral follows from the transformation formula [1, p. 106]

$$\int_0^{\alpha} \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}} = \int_{\beta}^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}},$$

where

$$\cot\alpha\cot\beta = \sqrt{1-x}.$$

□

The key to the proof of Theorem 10.7 is the substitution (10.18). We briefly describe here how we arrive at this substitution. Consider the equation

$$y^2 = x(4x+1)(16x+1)$$

for an elliptic curve E. By substituting $x_1 = 4x+1$, we may rewrite the equation in the form

$$y^2 = x_1(x_1-1)(x_1-\tfrac{3}{4}).$$

Next, by setting $y_2 = yx_2^3$ and $x_2^2 = 1/x_1$ [12, pp. 42–43], we obtain the Legendre form of the elliptic curve E, namely,

$$y_1^2 = (1-x_2^2)(1-\tfrac{3}{4}x_2^2).$$

Our substitution (10.18) is obtained by letting

$$\sin^2\varphi = x_2^2 = \frac{1}{x_1} = \frac{1}{4x+1}.$$

We now describe how one can construct differential equations analogous to that of Theorem 10.1. The quotients of eta-products which appear in Ramanujan's integrals happen to be Hauptmoduls associated with discrete groups of genus zero of the form $\Gamma_0(N) + W_p$, where $p|N$ and W_p is an Atkin-Lehner involution of $\Gamma_0(N)$ (see [7] for more details). Suppose v is the Hauptmodul associated with a discrete group of genus zero Γ. Then the derivative of v with respect to q is a modular form of weight 2 under Γ. To construct a differential equation associated with v, we search for another modular form of weight 2 under Γ for which the quotient $w^{-1}\frac{dv}{dq}$ is invariant under Γ. Since every modular function invariant under Γ can be expressed as a rational function of v, we can easily determine the relation between the two modular forms. Using this method, we derive the following differential equations.

Theorem 10.8. *Let*

$$v = q\frac{f^4(-q^2)f^4(-q^{10})}{f^4(-q)f^4(-q^5)}.$$

Then

$$\frac{dv}{dq} = q^{-1/2}f^2(-q)f^2(-q^5)\sqrt{v(4v+1)(16v^2+12v+1)}.$$

Theorem 10.9. *Let*

$$v = q\frac{f^3(-q^2)f^3(-q^{14})}{f^3(-q)f^3(-q^7)}.$$

Then

$$\frac{dv}{dq} = f(-q)f(-q^2)f(-q^7)f(-q^{14})\sqrt{(8v+1)(v+1)(8v^2+5v+1)}.$$

Theorem 10.10. *Let*

$$v = q\frac{f^2(-q^2)f^2(-q^{22})}{f^2(-q)f^2(-q^{11})}.$$

Then

$$v^2\frac{dv}{dq} = qf^2(-q^2)f^2(-q^{22})\sqrt{(4v^3+8v^2+4v+1)(16v^3+16v^2+8v+1)}.$$

Theorem 10.11. *Let*

$$v = q\frac{f(-q^3)f(-q^{33})}{f(-q)f(-q^{11})}.$$

Then

$$\begin{aligned} v^2\frac{dv}{dq} =& q^4f^2(-q^3)f^2(-q^{33}) \\ &\times \sqrt{(3v^2+v+1)(27v^6+63v^5+84v^4+59v^3+28v^2+7v+1)}. \end{aligned}$$

We now use two of these differential equations to derive two new identities for incomplete elliptic integrals.

Theorem 10.12. *If*

$$v = q\frac{f^4(-q^2)f^4(-q^{10})}{f^4(-q)f^4(-q^5)},$$

then

(10.20)

$$\int_0^q \frac{1}{\sqrt{t}}f^2(-t)f^2(-t^5)\,dt = 5^{-1/4}\int_0^{\sin^{-1}\sqrt{\frac{4\sqrt{5}v}{(6+2\sqrt{5})v+1}}} \frac{d\varphi}{\sqrt{1-\frac{1+\sqrt{5}}{2\sqrt{5}}\sin^2\varphi}}.$$

Proof. We first transform the differential equation in Theorem 10.8 to a differential equation involving $\sigma := 1/v$, namely,

$$\frac{d\sigma}{dq} = -q^{-1/2}f^2(-q)f^2(-q^5)\sqrt{(\sigma+4)(\sigma^2+12\sigma+16)}.$$

Using the method described after the proof of Theorem 10.7, we then derive the substitution

$$\sin^2\varphi = \frac{4\sqrt{5}}{\sigma + (6+2\sqrt{5})} = \frac{4\sqrt{5}v}{(6+2\sqrt{5})v+1}. \tag{10.21}$$

It follows that

$$\frac{d\varphi}{d\sigma} = -\frac{5^{1/4}}{(\sigma+6+2\sqrt{5})\sqrt{\sigma+6-2\sqrt{5}}}.$$

Hence,

$$\begin{aligned}\int_0^q \frac{1}{\sqrt{t}} f^2(-t)f^2(-t^5)\,dt &= -\int_\infty^{1/q} \frac{d\sigma}{\sqrt{(\sigma+4)(\sigma^2+12\sigma+16)}} \\ &= 5^{-1/4}\int_0^{\sin^{-1}\sqrt{\frac{4\sqrt{5}v}{(6+2\sqrt{5})v+1}}} \sqrt{\frac{\sigma+6+2\sqrt{5}}{\sigma+4}}\,d\varphi.\end{aligned}$$

Upon simplification with the use of (10.21), we obtain (10.20). □

We next describe how we derive an integral identity from Theorem 10.9. First, we derive a substitution similar to that of Ramanujan. Our aim is to convert the differential expression

$$\frac{dx}{\sqrt{(8x+1)(x+1)(8x^2+5x+1)}} \quad \text{to} \quad \frac{Rdt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Using the method outlined in [6, pp. 311–312], we use the substitution

$$x = \frac{1}{2\sqrt{2}}\frac{1+s}{1-s}$$

and obtain

$$\begin{aligned}&\frac{dx}{\sqrt{(8x+1)(x+1)(8x^2+5x+1)}} \\ &= \frac{2\sqrt{2}}{\sqrt{(8+9\sqrt{2})(8+5\sqrt{2})}}\frac{ds}{\sqrt{(1-ns^2)(1+ms^2)}},\end{aligned} \tag{10.22}$$

where

$$n = \frac{9\sqrt{2}-8}{9\sqrt{2}+8} \quad \text{and} \quad m = \frac{8-5\sqrt{2}}{8+5\sqrt{2}}.$$

Next, we use the substitution [6, p. 316]

$$t^2 = 1 - ns^2,$$

and simplifying, we find that

$$\frac{2\sqrt{2}}{\sqrt{(8+9\sqrt{2})(8+5\sqrt{2})}}\frac{ds}{\sqrt{(1-ns^2)(1+ms^2)}} = -2^{-7/4}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{10.23}$$

where $k^2 = \frac{32-13\sqrt{2}}{64}$. Hence, we should use the substitution

$$\sin^2\varphi = t^2 = 1 - ns^2 = 1 - n\left(\frac{2\sqrt{2}x-1}{2\sqrt{2}x+1}\right)^2, \tag{10.24}$$

or

$$\cos\varphi = \sqrt{n}\frac{1-2\sqrt{2}x}{1+2\sqrt{2}x}. \tag{10.25}$$

Using (10.22), (10.23), and (10.24), we deduce that

$$\frac{dx}{\sqrt{(8x+1)(x+1)(8x^2+5x+1)}} = -2^{-7/4}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}. \tag{10.26}$$

Using Theorem 10.9, (10.25), and (10.26), we can deduce the following theorem with no difficulty.

Theorem 10.13. *Let*

$$v = q\frac{f^3(-q^2)f^3(-q^{14})}{f^3(-q)f^3(-q^7)} \quad \textit{and} \quad c = \frac{9-4\sqrt{2}}{7}.$$

Then

$$\begin{aligned}&\int_0^q f(-t)f(-t^2)f(-t^7)f(-t^{14})\,dt\\ =&2^{-7/4}\int_{\cos^{-1}\{c\frac{1-2\sqrt{2}x}{1+2\sqrt{2}x+1}\}}^{\cos^{-1}c} \frac{d\varphi}{\sqrt{1-\frac{32-13\sqrt{2}}{64}\sin^2\varphi}}.\end{aligned}$$

The integral above is clearly an analogue of (8.14). It is not clear how one can obtain this integral from (8.14) and vice versa using the modular equation found in [1, p. Entry 19(ix)] or [7, Entry 3.6].

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