Errata and Addenda to *Mathematical Constants*

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June 16, 2005

At this point, there are more additions than errors to report...

1.1. Pythagoras' Constant. A geometric irrationality proof of $\sqrt{2}$ appears in [1] and a curious recursion is studied in [2]. More references on radical denestings include [3, 4, 5, 6].

1.2. The Golden Mean. The cubic irrational $\chi = 1.8392867552...$ is mentioned elsewhere in the literature with regard to iterative functions [7, 8, 9] (the four-numbers game is a special case of what are known as Ducci sequences), geometric constructions [10, 11] and numerical analysis [12]. Another reference on infinite radical expressions is [13]. See [14] for an interesting optimality property of the logarithmic spiral.

1.3. The Natural Logarithmic Base. A proof of the formula

$$\frac{e}{2} = \left(\frac{2}{1}\right)^{\frac{1}{2}} \cdot \left(\frac{2 \cdot 4}{3 \cdot 3}\right)^{\frac{1}{4}} \cdot \left(\frac{4 \cdot 6 \cdot 6 \cdot 8}{5 \cdot 5 \cdot 7 \cdot 7}\right)^{\frac{1}{8}} \cdots$$

appears in [15]; Hurwitzian continued fractions for $e^{1/q}$ and $e^{2/q}$ appear in [16, 17, 18, 19]. Define the following set of integer k-tuples

$$N_k = \left\{ (n_1, n_2, \ldots, n_k) : \sum_{j=1}^{k} \frac{1}{n_j} = 1 \text{ and } 1 \le n_1 < n_2 < \ldots < n_k \right\}.$$

Martin [20] proved that

$$\min_{(n_1,n_2,\ldots,n_k)\in N_k} n_k \sim \frac{e}{e-1} k$$

as $k \to \infty$, but it remains open whether

$$\max_{(n_1,n_2,\ldots,n_k)\in N_k} n_1 \sim \frac{1}{e-1} k.$$

Croot [21] made some progress on the latter: He proved that $n_1 \ge (1 + o(1))k/(e-1)$ for infinitely many values of k, and this bound is best possible. Also, define $f_0(x) = x$ and, for each $n > 0$,

$$f_n(x) = (1 + f_{n-1}(x) - f_{n-1}(0))^{\frac{1}{x}}.$$

This imitates the definition of e, in the sense that the exponent $\to\infty$ and the base $\to 1$ as $x\to 0$. We have $f_1(0)=e=2.718...$,

$$f_2(0)=\exp\left(-\frac{e}{2}\right)=0.257..., \qquad f_3(0)=\exp\left(\frac{11-3e}{24}\exp\left(1-\frac{e}{2}\right)\right)=1.086...$$

and $f_4(0)=0.921...$ (too complicated an expression to include here). Does a pattern develop here?

1.4. Archimedes' Constant. Viète's product

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

has the following close cousin:

$$\frac{2}{L}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}}}\cdot\sqrt{\frac{1}{2}+\frac{\frac{1}{2}}{\sqrt{\frac{1}{2}+\frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}}}}}\cdots$$

where L is the lemniscate constant (pages 420–423). Levin [22, 23] developed analogs of sine and cosine for the curve $x^4+y^4=1$ to prove the latter formula; he also noted that the area enclosed by $x^4+y^4=1$ is $\sqrt{2}L$ and that

$$\frac{2\sqrt{3}}{\pi}=\left(\frac{1}{2}+\sqrt{\frac{1}{2}}\right)\cdot\left(\frac{1}{2}+\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}}}\right)\cdot\left(\frac{1}{2}+\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}}}}\right)\cdots.$$

Additional infinite radical expressions for π appear in [24]; more on the Matiyasevich-Guy formula is covered in [25, 26, 27, 28, 29].

1.5. Euler-Mascheroni Constant. Vacca's series was, in fact, anticipated by Nielsen [30]. An extension was found by Koecher [31]:

$$\gamma=\delta-\frac{1}{2}\sum_{k=2}^{\infty}\frac{(-1)^k}{(k-1)k(k+1)}\left\lfloor\frac{\ln(k)}{\ln(2)}\right\rfloor$$

where $\delta=(1+\alpha)/4=0.6516737881...$ and $\alpha=\sum_{n=1}^{\infty}1/(2^n-1)=1.6066951524...$ is one of the digital search tree constants. Glaisher [32] discovered a similar formula:

$$\gamma=\sum_{n=1}^{\infty}\frac{1}{3^n-1}-2\sum_{k=1}^{\infty}\frac{1}{(3k-1)(3k)(3k+1)}\left\lfloor\frac{\ln(3k)}{\ln(3)}\right\rfloor$$

nearly eighty years earlier. The following series [33] suggest that $\ln(4/\pi)$ is an "alternating Euler constant":

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) \right) = -\int_0^1 \int_0^1 \frac{1-x}{(1-xy)\ln(xy)} dx\, dy,$$

$$\ln\left(\frac{4}{\pi}\right) = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) \right) = -\int_0^1 \int_0^1 \frac{1-x}{(1+xy)\ln(xy)} dx\, dy$$

(see section 1.7 later for more). Sample criteria for the irrationality of γ appear in Sondow [34, 35, 36, 37]. Long ago, Mahler attempted to prove that γ is transcendental; the closest he came to this was to prove the transcendentality of the constant [38, 39]

$$\frac{\pi Y_0(2)}{2J_0(2)} - \gamma$$

where $J_0(x)$ and $Y_0(x)$ are the zeroth Bessel functions of the first and second kinds. (Unfortunately the conclusion cannot be applied to the terms separately!) Diamond [40, 41] proved that, if

$$F_k(n) = \sum \frac{1}{\ln(\nu_1)\ln(\nu_2)\cdots\ln(\nu_k)}$$

where the (finite) sum is over all integer multiplicative compositions $n = \nu_1 \nu_2 \cdots \nu_k$ and each $\nu_j \geq 2$, then

$$\lim_{N\to\infty} \frac{1}{N} \left(1 + \sum_{n=2}^{N} \sum_{k=1}^{\infty} \frac{F_k(n)}{k!} \right) = \exp(\gamma' - \gamma - \ln(\ln(2)) = 1.2429194164...$$

where $\gamma' = 0.4281657248...$ is the analog of Euler's constant when $1/x$ is replaced by $1/(x\ln(x))$ (see Table 1.1). See [42] for a different generalization of γ.

1.6. Apéry's Constant. The famous alternating central binomial series for $\zeta(3)$ dates back at least as far as 1890, appearing as a special case of a formula due to Markov [43, 44, 45]:

$$\sum_{n=0}^{\infty} \frac{1}{(x+n)^3} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^6}{(2n+1)!} \frac{2(x-1)^2 + 6(n+1)(x-1) + 5(n+1)^2}{[x(x+1)\cdots(x+n)]^4}.$$

1.7. Catalan's Constant. Rivoal & Zudilin [46] have proved that there exist infinitely many integers k for which $\beta(2k)$ is irrational, and that at least one of the numbers $\beta(2)$, $\beta(4)$, $\beta(6)$, $\beta(8)$, $\beta(10)$, $\beta(12)$, $\beta(14)$ is irrational. More double integrals (see section 1.5 earlier) include [47, 48, 49, 50]

$$\zeta(3) = -\frac{1}{2}\int_0^1\int_0^1 \frac{\ln(xy)\,dx\,dy}{1-xy}, \qquad G = \frac{1}{8}\int_0^1\int_0^1 \frac{dx\,dy}{(1-xy)\sqrt{x(1-y)}}.$$

Zudilin [49] also found the continued fraction expansion

$$\frac{13}{2G} = 7 + \frac{1040|}{|10699} + \frac{42322176|}{|434871} + \frac{15215850000|}{|4090123} + \cdots.$$

where the partial numerators and partial denominators are generated according to the polynomials $(2n-1)^4(2n)^4(20n^2-48n+29)(20n^2+32n+13)$ and $3520n^6 + 5632n^5 + 2064n^4 - 384n^3 - 156n^2 + 16n + 7$.

1.8. Khintchine-Lévy Constants. If x is a quadratic irrational, then its continued fraction expansion is periodic; hence $\lim_{n\to\infty} M(n,x)$ is easily found and is algebraic. For example, $\lim_{n\to\infty} M(n,\varphi) = 1$, where φ is the Golden mean. We study the set Σ of values $\lim_{n\to\infty} \ln(Q_n)/n$ taken over all quadratic irrationals x in [51]. Additional references include [52, 53].

1.9. Feigenbaum-Coullet-Tresser Constants. Consider the unique solution of $\varphi(x) = T_2[\varphi](x)$ as pictured in Figure 1.6. The Hausdorff dimension D of the Cantor set $\{x_k\}_{k=1}^{\infty} \subseteq [-1,1]$, defined by $x_1 = 1$ and $x_{k+1} = \varphi(x_k)$, is known to satisfy $0.53763 < D < 0.53854$. This set may be regarded as the simplest of all strange attractors [54, 55].

1.11. Chaitin's Constant. Ord & Kieu [56] gave a different Diophantine representation for Ω; apparently Chaitin's equation can be reduced to 2–3 pages in length [57]. A rough sense of the type of equations involved can be gained from [58]. Calude & Stay [59] suggested that the uncomputability of bits of Ω can be recast as a uncertainty principle.

2.1. Hardy-Littlewood Constants. Fix $\varepsilon > 0$. Let $N(x,k)$ denote the number of positive integers $n \le x$ with $\Omega(n) = k$, where k is allowed to grow with x. Nicolas [60] proved that

$$\lim_{x\to\infty} \frac{N(x,k)}{(x/2^k)\ln(x/2^k)} = \frac{1}{4C_{\text{twin}}} = \frac{1}{4}\prod_{p>2}\left(1 + \frac{1}{p(p-2)}\right) = 0.3786950320....$$

under the assumption that $(2+\varepsilon)\ln(\ln(x)) \le k \le \ln(x)/\ln(2)$. More relevant results appear in [61]; see also the next entry. Let $L(x)$ denote the number of positive odd integers $n \le x$ that can be expressed in the form $2^l + p$, where l is a positive integer and p is a prime. Then $0.0868x < L(x) \le 0.49999991x$ for all sufficiently large x.

The lower bound can be improved to $0.2893x$ if the Hardy-Littlewood conjectures in sieve theory are true [62, 63, 64].

2.2. Meissel-Mertens Constants. See [65] for more occurrences of the constants M and M', and [66] for a historical treatment. Higher-order asymptotic series for $\mathrm{E}_n(\omega)$, $\mathrm{Var}_n(\omega)$, $\mathrm{E}_n(\Omega)$ and $\mathrm{Var}_n(\Omega)$ are given in [67]. While $\sum_p 1/p$ is divergent, the following prime series is convergent [68]:

$$\sum_p \left(\frac{1}{p^2}+\frac{1}{p^3}+\frac{1}{p^4}+\cdots\right)=\sum_p \frac{1}{p(p-1)}=0.7731566690....$$

If Q_k denotes the set of positive integers n for which $\Omega(n)-\omega(n)=k$, then $Q_1=\tilde{S}$ and the asymptotic density δ_k satisfies [69, 70, 71]

$$\lim_{k\to\infty} 2^k\delta_k=\frac{1}{4C_{\text{twin}}}=0.3786950320...;$$

the expression $4C_{\text{twin}}$ also appears on pages 86 and 133–134, as well as the preceding entry.

2.3. Landau-Ramanujan Constant. The quantity [72]

$$\prod_{p\equiv 3 \bmod 4}\left(1-\frac{2p}{(p^2+1)(p-1)}\right)=0.6436506796...$$

is evaluated to high precision in [73] via special values of Dirichlet L-series.

2.4. Artin's Constant. Let $\iota(n)=1$ if n is square-free and $\iota(n)=0$ otherwise. Then [74, 75, 76]

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\iota(n)\iota(n+1) &= \prod_p\left(1-\frac{2}{p^2}\right)=0.3226340989...=-1+2(0.6613170494...)\\ &= \frac{6}{\pi^2}\prod_p\left(1-\frac{1}{p^2-1}\right)=\frac{6}{\pi^2}(0.5307118205...),\end{aligned}$$

that is, the Feller-Tornier constant arises with regard to consecutive square-free numbers and to other problems. Also, consider the cardinality $N(X)$ of nontrivial primitive integer vectors (x_0,x_1,x_2,x_3) that fall on Cayley's cubic surface

$$x_0x_1x_2+x_0x_1x_3+x_0x_2x_3+x_1x_2x_3=0$$

and satisfy $|x_j|\le X$ for $0\le j\le 3$. It is known that $N(X)\sim cX(\ln(X))^6$ for some constant $c>0$ [77, 78]; finding c remains an open problem.

2.5. Hafner-Sarnak-McCurley Constant. In the "Added In Press" section (pages 601–602), the asymptotics of coprimality and of square-freeness are discussed for the Gaussian integers and for the Eisenstein-Jacobi integers. More about sums involving $2^{\omega(n)}$ and $2^{-\omega(n)}$ appears in [79]. Also, the asymptotics of $\sum_{n=1}^{N} 3^{\Omega(n)}$, due to Tenenbaum, are mentioned in [67].

2.7. Euler Totient Constants. Let $f(n) = n\varphi(n)^{-1} - e^{\gamma}\ln(\ln(n))$. Nicolas [80] proved that $f(n) > 0$ for infinitely many integers n by the following reasoning. Let P_k denote the product of the first k prime numbers. If the Riemann hypothesis is true, then $f(P_k) > 0$ for all k. If the Riemann hypothesis is false, then $f(P_k) > 0$ for infinitely many k and $f(P_l) \leq 0$ for infinitely many l.

Let $U(n)$ denote the set of values $\leq n$ taken by φ and v_n denote its cardinality; for example [81], $U(15) = \{1, 2, 4, 6, 8, 10, 12\}$ and $v(15) = 7$. Let $\ln_3(x) = \ln(\ln(\ln(n)))$ and $\ln_4(x) = \ln(\ln_3(x))$ for convenience. Ford [82] proved that

$$v(n) = \tfrac{n}{\ln(n)} \exp\left\{C[\ln_3(n) - \ln_4(n)]^2 + D\ln_3(n) - [D + \tfrac{1}{2} - 2C]\ln_4(n) + O(1)\right\}$$

as $n \to \infty$, where

$$C = -\tfrac{1}{2\ln(\rho)} = 0.8178146464...,$$

$$D = 2C\left(1 + \ln(F'(\rho)) - \ln(2C)\right) - \tfrac{3}{2} = 2.1769687435...$$

$$F(x) = \sum_{k=1}^{\infty}\left((k+1)\ln(k+1) - k\ln(k) - 1\right)x^k$$

and $\rho = 0.5425985860...$ is the unique solution on $[0, 1)$ of the equation $F(\rho) = 1$. Also,

$$\lim_{n\to\infty} \frac{1}{v(n)\ln(\ln(n))} \sum_{m\in U(n)} \omega(m) = \frac{1}{1-\rho} = 2.1862634648...$$

which contrasts with a related result of Erdös & Pomerance [83]:

$$\lim_{n\to\infty} \frac{1}{n\ln(\ln(n))^2} \sum_{m=1}^{n} \omega(\varphi(n)) = \frac{1}{2}.$$

These two latter formulas hold as well if ω is replaced by Ω. See [84] for more on Euler's totient.

2.8. Pell-Stevenhagen Constants. The constant P is transcendental via a general theorem on values of modular forms due to Nesterenko [85, 86]. Here is a constant similar to P: The number of positive integers $n \leq N$, for which $2n-1$ is not divisible by $2^p - 1$ for any prime p, is $\sim cN$, where

$$c = \prod_{p}\left(1 - \frac{1}{2^p - 1}\right) = 0.5483008312....$$

A ring-theoretic analog of this statement, plus generalizations, appear in [87].

2.10. Sierpinski's Constant. Sierpinski's formulas for $\hat{S}$ and $\tilde{S}$ contained a few errors: they should be [88, 89, 90, 91, 92, 93]

$$\hat{S} = \gamma + S - \frac{12}{\pi^2}\zeta'(2) + \frac{\ln(2)}{3} - 1 = 1.7710119609... = \frac{\pi}{4}(2.2549224628...),$$

$$\tilde{S} = 2S - \frac{12}{\pi^2}\zeta'(2) + \frac{\ln(2)}{3} - 1 = 2.0166215457... = \frac{1}{4}(8.0664861829...).$$

Also, in the summation formula at the top of page 125, D_n should be D_k.

Define $R(n)$ to be the number of representations of n as a sum of three squares, counting order and sign. Then

$$\sum_{k=1}^{n} R(k) = \frac{4\pi}{3} n^{3/2} + O\left(n^{3/4+\varepsilon}\right)$$

for all $\varepsilon > 0$ and [94]

$$\sum_{k=1}^{n} R(k)^2 = \frac{8\pi^4}{21\zeta(3)} n^2 + O\left(n^{14/9}\right).$$

The former is the same as the number of integer ordered triples falling within the ball of radius $\sqrt{n}$ centered at the origin; an extension of the latter to sums of m squares, when $m > 3$, is also known [94].

2.13. Mills' Constant. Let $q_1 < q_2 < \ldots < q_k$ denote the consecutive prime factors of an integer $n > 1$. Define

$$F(n) = \sum_{j=1}^{k-1} \left(1 - \frac{q_j}{q_{j+1}}\right) = \omega(n) - 1 - \sum_{j=1}^{k-1} \frac{q_j}{q_{j+1}}$$

if $k > 1$ and $F(n) = 0$ if $k = 1$. Erdös & Nicolas [95] demonstrated that there exists a constant $C' = 1.70654185...$ such that, as $n \to \infty$, $F(n) \leq \sqrt{\ln(n)} - C' + o(1)$, with equality holding for infinitely many n. Further, $C' = C + \ln(2) + 1/2$, where [95, 96]

$$C = \sum_{i=1}^{\infty} \left\{ \ln\left(\frac{p_{i+1}}{p_i}\right) - \left(1 - \frac{p_i}{p_{i+1}}\right) \right\} = 0.51339467..., \qquad \sum_{i=1}^{\infty} \left(\frac{p_{i+1}}{p_i} - 1\right)^2 = 1.65310351...,$$

and $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... is the sequence of all primes.

It now seems that $\liminf_{n\to\infty}(p_{n+1} - p_n)/\ln(p_n) = 0$ is a theorem [97], clarifying the uncertainty raised in "Added In Press" (pages 601–602). More about small prime gaps will surely appear soon; research concerning large prime gaps continues as well [98, 99].

2.15. Glaisher-Kinkelin Constant. The two quantities

$$G(\tfrac{1}{2}) = 0.6032442812..., \qquad G(\tfrac{3}{2}) = \sqrt{\pi}G(\tfrac{1}{2}) = 1.0692226492...$$

play a role in a discussion of the limiting behavior of Toeplitz determinants and the Fisher-Hartwig conjecture [100, 101]. Ehrhardt [102] proved Dyson's conjecture regarding the asymptotic expansion of $E(s)$ as $s \to \infty$. In the last paragraph on page 141, the polynomial $q(x)$ should be assumed to have degree n. See [103, 104] for more on the GUE hypothesis.

2.16. Stolarsky-Harboth Constant. Given a positive integer n, define s_1^2 to be the largest square not exceeding n. Then define s_2^2 to be the largest square not exceeding $n - s_1^2$, and so forth. Hence $n = \sum_{j=1}^{r} s_j^2$ for some r. We say that n is a *greedy sum of distinct squares* if $s_1 > s_2 > \ldots > s_r$. Let $A(N)$ be the number of such integers $n < N$, plus one. Montgomery & Vorhauer [105] proved that $A(N)/N$ does not tend to a constant, but instead that there is a continuous function $f(x)$ of period 1 for which

$$\lim_{k\to\infty} \frac{A(4\exp(2^{k+x}))}{4\exp(2^{k+x})} = f(x), \qquad \min_{0\le x\le 1} f(x) = 0.50307... < \max_{0\le x\le 1} f(x) = 0.50964...$$

where k takes on only integer values. This is reminiscent of the behavior discussed for digital sums. Two simple examples, due to Hardy [106, 107] and Elkies [108], involve the series

$$\varphi(x) = \sum_{k=0}^{\infty} x^{2^k}, \qquad \psi(x) = \sum_{k=0}^{\infty} (-1)^k x^{2^k}.$$

As $x \to 1^-$, the asymptotics of $\varphi(x)$ and $\psi(x)$ are complicated by oscillating errors with amplitude

$$\sup_{x\to 1^-} \left| \varphi(x) + \frac{\ln(-\ln(x)) + \gamma}{\ln(2)} - \frac{3}{2} + x \right| = (1.57...) \times 10^{-6},$$

$$\sup_{x\to 1^-} \left| \psi(x) - \frac{1}{6} - \frac{1}{3}x \right| = (2.75...) \times 10^{-3}.$$

The function $\varphi(x)$ also appears in what is known as Catalan's integral (section 1.5.2) for Euler's constant γ.

2.17. Gauss-Kuzmin-Wirsing Constant. The preprint math.NT/9908043 was withdrawn by the author without comment; the bounds we gave for the Hausdorf dimension of real numbers with partial denominators in $\{1, 2\}$ remain unaffected.

2.18. Porter-Hensley Constants. Lhote [109, 110] developed rigorous techniques for computing certain variances to high precision, for example, $4\lambda_1''(2)$. With regard to the binary GCD algorithm, Maze [111] confirmed Brent's functional equation for a certain limiting distribution [112]

$$g(x)=\sum_{k\geq 1}2^{-k}\left(g\left(\frac{1}{1+2^k/x}\right)-g\left(\frac{1}{1+2^k x}\right)\right),\qquad 0\leq x\leq 1$$

as well as the formula

$$2+\frac{1}{\ln(2)}\int_0^1\frac{g(x)}{1-x}dx=\frac{2}{\kappa\ln(2)}=2.8329765709...=\frac{\pi^2(0.3979226811...)}{2\ln(2)}.$$

2.20. Erdös' Reciprocal Sum Constants. A sequence of positive integers $b_1<b_2<\ldots<b_m$ is a B_h-sequence if all h-fold sums $b_{i_1}+b_{i_2}+\cdots+b_{i_h}$, $i_1\leq i_2\leq\ldots\leq i_h$, are distinct. Given n, choose a B_h-sequence $\{b_i\}$ so that $b_m\leq n$ and m is maximal; let $F_h(n)$ be this value of m. It is known that $C_h=\limsup_{n\to\infty}n^{-1/h}F_h(n)$ is finite; we further have [113, 114, 115, 116, 117, 118]

$$C_2=1,\qquad 1\leq C_3\leq(7/2)^{1/3},\qquad 1\leq C_4\leq 7^{1/4}.$$

More generally, a sequence of positive integers $b_1<b_2<\ldots<b_m$ is a $B_{h,g}$-sequence if, for every positive integer k, the equation $x_1+x_2+\cdots+x_h=k$, $x_1\leq x_2\leq\ldots\leq x_h$, has at most g solutions with $x_j=b_{i_j}$ for all j. Defining $F_{h,g}(n)$ and $C_{h,g}$ analogously, we have [118, 119, 120, 121, 122]

$$\tfrac{4\sqrt{7}}{7}\leq C_{2,2}\leq\tfrac{\sqrt{21}}{2},\qquad \tfrac{3\sqrt{2}}{4}g^{1/2}+o(g^{1/2})\leq C_{2,g}\leq\min\left\{\tfrac{7}{2}g-\tfrac{7}{4},\tfrac{17g}{5}\right\}^{1/2}$$

as $g\to\infty$.

2.21. Stieltjes Constants. If $d_k(n)$ denotes the number of sequences x_1, x_2, ..., x_k of positive integers such that $n=x_1x_2\cdots x_k$, then [123, 124, 125]

$$\sum_{n=1}^{N}d_2(n)\sim N\ln(N)+(2\gamma_0-1)N\qquad(d_2\text{ is the divisor function}),$$

$$\sum_{n=1}^{N}d_3(n)\sim\frac{1}{2}N\ln(N)^2+(3\gamma_0-1)N\ln(N)+(-3\gamma_1+3\gamma_0^2-3\gamma_0+1)N,$$

$$\begin{aligned}\sum_{n=1}^{N}d_4(n)\ \sim\ &\frac{1}{6}N\ln(N)^3+\frac{4\gamma_0-1}{2}N\ln(N)^2+(-4\gamma_1+6\gamma_0^2-4\gamma_0+1)N\ln(N)\\&+(2\gamma_2-12\gamma_1\gamma_0+4\gamma_1+4\gamma_0^3-6\gamma_0^2+4\gamma_0-1)N\end{aligned}$$

as $N \to \infty$. More generally, $\sum_{n=1}^{N} d_k(n)$ can be asymptotically expressed as N times a polynomial of degree $k-1$ in $\ln(N)$, which in turn can be described as the residue at $z=1$ of $z^{-1}\zeta(z)^k N^z$. See [67] for an application of $\{\gamma_j\}_{j=0}^{\infty}$ to asymptotic series for $\mathrm{E}_n(\omega)$ and $\mathrm{E}_n(\Omega)$, and [126, 127, 128, 129, 130, 131] for connections to the Riemann hypothesis.

2.25. Cameron's Sum-Free Set Constants. Erdös [132] and Alon & Kleitman [133] showed that any finite set B of positive integers must contain a sum-free subset A such that $|A| > \frac{1}{3}|B|$. See also [134, 135, 136]. The largest constant c such that $|A| > c|B|$ must satisfy $1/3 \leq c < 12/29$, but its exact value is unknown. Using harmonic analysis, Bourgain [137] improved the original inequality to $|A| > \frac{1}{3}(|B|+2)$. Green [138] demonstrated that $s_n = O(2^{n/2})$, but the values $c_o = 6.8...$ and $c_e = 6.0...$ await more precise computation.

2.30. Pisot-Vijayaraghavan-Salem Constants. Compare the sequence $\{(3/2)^n\}$, for which little is known, with the recursion $x_0 = 0$, $x_n = \{x_{n-1} + \ln(3/2)/\ln(2)\}$, for which a musical interpretation exists. If a guitar player touches a vibrating string at a point two-thirds from the end of the string, its fundamental frequency is dampened and a higher overtone is heard instead. This new pitch is a perfect fifth above the original note. It is well-known that the "circle of fifths" never closes, in the sense that 2^{x_n} is never an integer for $n > 0$. Further, the "circle of fifths", in the limit as $n \to \infty$, fills the continuum of pitches spanning the octave [139, 140].

The Collatz function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by

$$f(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases} .$$

Let f^k denote the k^{th} iterate of f. The $3x+1$ conjecture asserts that, given any positive integer n, there exists k such that $f^k(n) = 1$. Let $\sigma(n)$ be the first k such that $f^k(n) < n$, called the *stopping time* of n. If we could demonstrate that every positive integer n has a finite stopping time, then the $3x+1$ conjecture would be proved. Heuristic reasoning [141, 142, 143] provides that the average stopping time over all odd integers $1 \leq n \leq N$ is asymptotically

$$\lim_{N\to\infty} \mathrm{E}_{\mathrm{odd}}(\sigma(n)) = \sum_{j=1}^{\infty} \left\lfloor 1 + \left(1 + \tfrac{\ln(3)}{\ln(2)}\right) j \right\rfloor c_j 2^{-\left\lfloor \frac{\ln(3)}{\ln(2)} j \right\rfloor} = 9.4779555565...$$

where c_j is the number of admissible sequences of order j. Such a sequence $\{a_k\}_{k=1}^{m}$ satisfies $a_k = 3/2$ exactly j times, $a_k = 1/2$ exactly $m-j$ times, $\prod_{k=1}^{m} a_k < 1$ but $\prod_{k=1}^{l} a_k > 1$ for all $1 \leq l < m$ [144]. In contrast, the *total* stopping time $\sigma_\infty(n)$ of n, the first k such that $f^k(n) = 1$, appears to obey

$$\lim_{N\to\infty} \mathrm{E}\left(\frac{\sigma_\infty(n)}{\ln(n)}\right) \sim \frac{2}{2\ln(2) - \ln(3)} = 6.9521189935... = \frac{2}{\ln(10)}(8.0039227796...).$$

2.32. De Bruijn-Newman Constant. Further work regarding Li's criterion, which is equivalent to Riemann's hypothesis and which involves the Stieltjes constants, appears in [126, 127]. A different criterion is due to Matiyasevich [128, 129]; the constant $-\ln(4\pi)+\gamma+2 = 0.0461914179... = 2(0.0230957089...)$ comes out as a special case. See also [130, 131]. As another aside, we mention the unboundedness of $\zeta(1/2+i\,t)$ for $t \in (0,\infty)$, but that a precise order of growth remains open [145, 146, 147]. In contrast, there is a conjecture that [148, 149]

$$\max_{t\in[T,2T]} |\zeta(1+i\,t)| = e^{\gamma}\left(\ln(\ln(T)) + \ln(\ln(\ln(T))) + C + o(1)\right),$$

$$\max_{t\in[T,2T]} \frac{1}{|\zeta(1+i\,t)|} = \frac{6e^{\gamma}}{\pi^2}\left(\ln(\ln(T)) + \ln(\ln(\ln(T))) + C + o(1)\right)$$

as $T \to \infty$, where

$$C = 1 - \ln(2) + \int_0^2 \frac{\ln(I_0(t))}{t^2}dt + \int_2^\infty \frac{\ln(I_0(t)) - t}{t^2}dt = -0.0893...$$

and $I_0(t)$ is the zeroth modified Bessel function. These formulas have implications for $|\zeta(i\,t)|$ and $1/|\zeta(i\,t)|$ as well by the analytic continuation formula.

2.33. Hall-Montgomery Constant. Let ψ be the unique solution on $(0,\pi)$ of the equation $\sin(\psi) - \psi\cos(\psi) = \pi/2$ and define $K = -\cos(\psi) = 0.3286741629....$ Consider any real multiplicative function f whose values are constrained to $[-1,1]$. Hall & Tenenbaum [150] proved that, for some constant $C > 0$,

$$\sum_{n=1}^{N} f(n) \le CN \exp\left\{-K \sum_{p\le N} \frac{1-f(p)}{p}\right\} \qquad \text{for sufficiently large } N,$$

and that, moreover, the constant K is sharp. (The latter summation is over all prime numbers p.) This interesting result is a lemma used in [151]. A table of values of sharp constants K is also given in [150] for the generalized scenario where f is complex, $|f| \le 1$ and, for all primes p, $f(p)$ is constrained to certain elliptical regions in $\mathbb{C}$.

3.3. Landau-Kolmogorov Constants. For $L_2(0,\infty)$, Bradley & Everitt [152] were the first to determine that $C(4,2) = 2.9796339059... = \sqrt{8.8782182137...}$; see also [153, 154, 155]. Ditzian [156] proved that the constants for $L_1(-\infty,\infty)$ are the same as those for $L_\infty(-\infty,\infty)$. Phóng [154] obtained the following best possible inequality in $L_2(0,1)$:

$$\int_0^1 |f'(x)|^2\,dx \le (6.4595240299...)\left(\int_0^1 |f(x)|^2\,dx + \int_0^1 |f''(x)|^2\,dx\right)$$

where the constant is given by $\sec(2\theta)/2$ and θ is the unique zero satisfying $0 < \theta < \pi/4$ of

$$\begin{aligned}&\sin(\theta)^4\left(e^{2\sin(\theta)}-1\right)^2(e^{-2\sin(\theta)}-1)^2+\cos(\theta)^4[2-2\cos(2\cos(\theta))]^2\\&-\cos(2\theta)^4[1+e^{4\sin(\theta)}-2e^{2\sin(\theta)}\cos(2\cos(\theta))][1+e^{-4\sin(\theta)}-2e^{-2\sin(\theta)}\cos(2\cos(\theta))]\\&-2\cos(\theta)^2\sin(\theta)^2[2-2\cos(2\cos(\theta))](1-e^{-2\sin(\theta)})\left(e^{2\sin(\theta)}-1\right).\end{aligned}$$

We wonder about other such additive analogs of Landau-Kolmogorov inequalities.

3.5. Copson-de Bruijn Constant. More relevant material is found in Ackermans [157].

3.6. Sobolev Isoperimetric Constants. In section 3.6.1, $\sqrt{\lambda} = 1$ represents the principal frequency of the sound we hear when a string is plucked; in section 3.6.3, $\sqrt{\lambda} = \theta$ represents likewise when a kettledrum is struck. (The square root was missing in both.) The units of frequency, however, are not compatible between these two examples.

The "rod "constant $500.5639017404... = (4.7300407448...)^4$ appears in [158, 159, 160]. It is the second term in a sequence c_1, c_2, c_3, ... for which $c_1 = \pi^2 = 9.869...$ (in connection with the "string" inequality) and $c_3 = (2\pi)^6 = 61528.908...$; the constant c_4 is the smallest eigenvalue of ODE

$$f^{(viii)}(x) = \lambda\, f(x), \qquad 0 \le x \le 1,$$

$$f(0) = f'(0) = f''(0) = f'''(0) = 0, \qquad f(1) = f'(1) = f''(1) = f'''(1) = 0$$

and evidently is not known. Allied subjects include positive definite Toeplitz matrices and conditioning of certain least squares problems.

More relevant material is found in [161, 162, 163].

3.12. Du Bois Reymond's Constants. The constant $(\pi/\xi)^2$ is equal to the largest eigenvalue of the infinite symmetric matrix $(a_{m,n})_{m\ge1,n\ge1}$ with elements $a_{m,n} = m^{-1}n^{-1} + m^{-2}\delta_{m,n}$, where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$. Boersma [164] employed this fact to give an alternative proof of Szegö's theorem.

3.15. Van der Corput's Constant. We examined only the case in which f is a real twice-continuously differentiable function on the interval $[a, b]$; a generalization to the case where f is n times differentiable, $n \ge 2$, is discussed in [165, 166] with some experimental numerical results for $n = 3$.

3.16. Turán's Power Sum Constants. Recent work appears in [167, 168, 169, 170, 171], to be reported on later.

4.3. Achieser-Krein-Favard Constants. While on the subject of trigonometric polynomials, we mention Littlewood's conjecture [172]. Let $n_1 < n_2 < \ldots < n_k$ be integers and let c_j, $1 \leq j \leq k$, be complex numbers with $|c_j| \geq 1$. Konyagin [173] and McGehee, Pigno & Smith [174] proved that there exists $C > 0$ so that the inequality

$$\int_0^1 \left| \sum_{j=1}^k c_j e^{2\pi i n_j \xi} \right| d\xi \geq C \ln(k)$$

always holds. It is known that the smallest such constant C satisfies $C \leq 4/\pi^2$; Stegeman [175] demonstrated that $C \geq 0.1293$ and Yabuta [176] improved this slightly to $C \geq 0.129590$. What is the true value of C?

4.5. The "One-Ninth" Constant. Zudilin [177] deduced that Λ is transcendental by use of Theorem 4 in [178].

4.7. Berry-Esseen Constant. Significant progress on the asymptotic case (as $\lambda \to 0$) is described in [179, 180, 181]. A different form of the inequality is found in [182].

4.8. Laplace Limit Constant. The quantity $\lambda = 0.6627434193...$ appears in [183] with regard to Plateau's problem for two circular rings dipped in soap solution.

5.4. Golomb-Dickman Constant. Let $P^+(n)$ denote the largest prime factor of n and $P^-(n)$ denote the smallest prime factor of n. We mentioned that

$$\sum_{n=2}^N \ln(P^+(n)) \sim \lambda N \ln(N) - \lambda(1-\gamma)N, \qquad \sum_{n=2}^N \ln(P^-(n)) \sim e^{-\gamma} N \ln(\ln(N)) + cN$$

as $N \to \infty$, but did not give an expression for the constant c. Tenenbaum [184] found that

$$c = e^{-\gamma}(1+\gamma) + \int_1^\infty \frac{\omega(t) - e^{-\gamma}}{t} dt + \sum_p \left\{ e^{-\gamma} \ln\left(1 - \frac{1}{p}\right) + \frac{\ln(p)}{p-1} \prod_{q \leq p} \left(1 - \frac{1}{q}\right) \right\},$$

where the sum over p and product over q are restricted to primes. A numerical evaluation is still open.

The longest tail $L(\varphi)$, given a random mapping $\varphi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, is called the **height** of φ in [185, 186, 187] and satisfies

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{L(\varphi)}{\sqrt{n}} \leq x\right) = \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{k^2 x^2}{2}\right)$$

for fixed $x > 0$. For example,

$$\lim_{n\to\infty} \mathrm{Var}\left(\frac{L(\varphi)}{\sqrt{n}}\right) = \frac{\pi^2}{3} - 2\pi \ln(2)^2.$$

The longest rho-path $R(\varphi)$ is called the **diameter** of φ in [188] and has moments

$$\lim_{n\to\infty} \mathrm{E}\left[\left(\frac{R(\varphi)}{\sqrt{n}}\right)^p\right] = \frac{\sqrt{\pi}p}{2^{p/2}\Gamma((p+1)/2)} \int_0^\infty x^{p-1}\left(1-e^{\mathrm{Ei}(-x)-I(x)}\right) dx$$

for fixed $p > 0$. Complicated formulas for the distribution of the largest tree $P(\varphi)$ also exist [186, 187, 189].

5.5. Kalmár's Composition Constant. See [190] for precise inequalities involving $m(n)$ and $\rho = 1.7286472389....$

5.6. Otter's Tree Enumeration Constants. Higher-order asymptotic series for T_n, t_n and B_n are given in [67]. Also, the asymptotic analysis of series-parallel posets [191] is similar to that of trees. See [192, 193] for more about k-gonal 2-trees, as well as a new formula for α in terms of rational expressions involving e. Finally, the generating function $L(x)$ of leftist trees satisfies a simpler functional equation than previously thought:

$$L(x) = x + L\left(x\, L(x)\right)$$

which involves an unusual nested construction. The radius of convergence $\rho = 0.3637040915... = (2.7494879027...)^{-1}$ of $L(x)$ satisfies

$$\rho\, L'\left(\rho\, L(\rho)\right) = 1$$

and the coefficient of $\rho^{-n}n^{-1/2}$ in the asymptotic expression for L_n is

$$\sqrt{\frac{1}{2\pi\rho^2}\frac{\rho + L(\rho)}{L''\left(\rho\, L(\rho)\right)}} = 0.2503634293... = (0.6883712204...)\rho.$$

The average height of n-leaf leftist trees is asymptotically $(1.81349371...)\sqrt{\pi n}$ and the average depth of vertices belonging to such trees is asymptotically $(0.90674685...)\sqrt{\pi n}$. Nogueira [194] conjectured that the ratio of the two coefficients is exactly 2, but his only evidence is numerical (to over 1000 decimal digits).

5.7. Lengyel's Constant. Constants of the form $\sum_{k=-\infty}^{\infty} 2^{-k^2}$ and $\sum_{k=-\infty}^{\infty} 2^{-(k-1/2)^2}$ appear in [195, 196].

5.9. Pólya's Random Walk Constants. Properties of the gamma function lead to a further simplification [197]:

$$m_3 = \frac{1}{32\pi^3}\left(\sqrt{3}-1\right)\left[\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right]^2$$

Consider a variation in which the drunkard performs a random walk starting from the origin with 2^d equally probable steps, each of the form $(\pm1, \pm1, \ldots, \pm1)$. The number of walks that end at the origin after $2n$ steps is

$$\tilde{U}_{d,0,2n} = \binom{2n}{n}^d$$

and the number of such walks for which $2n$ is the time of *first return* to the origin is $\tilde{V}_{d,0,2n}$, where [198]

$$2^{-n}\tilde{V}_{1,0,2n} = \frac{1}{n2^{2n-1}}\binom{2n-2}{n-1} \sim \frac{1}{2\sqrt{\pi}n^{3/2}},$$

$$2^{-2n}\tilde{V}_{2,0,2n} = \frac{\pi}{n(\ln(n))^2} - 2\pi\frac{\gamma+\pi B}{n(\ln(n))^3} + O\left(\frac{1}{n(\ln(n))^4}\right),$$

$$2^{-3n}\tilde{V}_{3,0,2n} = \frac{1}{\pi^{3/2}C^2n^{3/2}} + O\left(\frac{1}{n^2}\right)$$

as $n \to \infty$, where

$$B = 1 + \sum_{k=1}^{\infty}\left[2^{-4k}\binom{2k}{k}^2 - \frac{1}{\pi k}\right] = 0.8825424006...,$$

$$C = \sum_{k=0}^{\infty} 2^{-6k}\binom{2k}{k}^3 = \frac{1}{4\pi^3}\Gamma\left(\frac{1}{4}\right)^4 = 1.3932039296....$$

5.10. Self-Avoiding Walk Constants. Hueter [199, 200] rigorously proved that $\nu_2 = 3/4$ and that $7/12 \leq \nu_3 \leq 2/3$, $1/2 \leq v_4 \leq 5/8$ (if the mean square end-to-end distance exponents ν_3, v_4 exist; otherwise the bounds apply for

$$\underline{\nu}_d = \liminf_{n\to\infty}\frac{\ln(r_n)}{2\ln(n)}, \qquad \overline{\nu}_d = \limsup_{n\to\infty}\frac{\ln(r_n)}{2\ln(n)}$$

when $d = 3, 4$). She confirmed that the same exponents apply for the mean square radius of gyration s_n for $d = 2, 3, 4$; the results carry over to self-avoiding trails as well [201].

5.12. Hard Square Entropy Constant. McKay [202] observed the following asymptotic behavior:

$$F(n) \sim (1.06608266...)(1.0693545387...)^{2n}(1.5030480824...)^{n^2}$$

based on an analysis of the terms $F(n)$ up to $n = 19$. He emphasized that the form of right hand side is conjectural, even though the data showed quite strong convergence to this form.

5.13. Binary Search Tree Constants. The random permutation model for generating weakly binary trees (given an n-vector of distinct integers, construct T via insertions) does *not* provide equal weighting on the $\binom{2n}{n}/(n+1)$ possible trees. For example, when $n = 3$, the permutations $(2, 1, 3)$ and $(2, 3, 1)$ both give rise to the same tree S, which hence has probability $q(S) = 1/3$ whereas $q(T) = 1/6$ for the other four trees. Fill [198, 203, 204] asked how the numbers $q(T)$ themselves are distributed, for fixed n. If the trees are endowed with the uniform distribution, then

$$\begin{aligned}\frac{-\operatorname{E}\left[\ln(q(T))\right]}{n} &\to \sum_{k=1}^{\infty}\frac{\ln(k)}{(k+1)4^k}\binom{2k}{k}\\ &= -\gamma-\int_0^1\frac{\ln(\ln(1/t))}{\sqrt{1-t}\left(1+\sqrt{1-t}\right)^2}dt = 2.0254384677...\end{aligned}$$

as $n\to\infty$. If, instead, the trees follow the distribution q, then

$$\begin{aligned}\frac{-\operatorname{E}\left[\ln(q(T))\right]}{n} &\to 2\sum_{k=1}^{\infty}\frac{\ln(k)}{(k+1)(k+2)}\\ &= -\gamma-2\int_0^1\frac{((t-2)\ln(1-t)-2t)\ln(\ln(1/t))}{t^3}dt = 1.2035649167....\end{aligned}$$

The maximum value of $-\ln(q(T))$ is $\sim n\ln(n)$ and the minimum value is $\sim c\,n$, where

$$c=\ln(4)+\sum_{k=1}^{\infty}2^{-k}\ln(1-2^{-k}) = 0.9457553021....$$

5.14. Digital Search Tree Constants. The constant Q is transcendental via a general theorem on values of modular forms due to Nesterenko [85, 86]. A correct formula for θ is

$$\theta=\sum_{k=1}^{\infty}\frac{k2^{k(k-1)/2}}{1\cdot 3\cdot 7\cdots(2^k-1)}\sum_{j=1}^{k}\frac{1}{2^j-1} = 7.7431319855...$$

(the exponent $k(k-1)/2$ was mistakenly given as $k+1$ in [205], but the numerical value is correct). The constants α, β and Q^{-1} appear in [206]. Also, α appears in [207] and Q^{-1} appears in [196].

5.15. Optimal Stopping Constants. When discussing the expected rank R_n, we assumed that no applicant would ever refuse a job offer! If each applicant only accepts an offer with known probability p, then [208]

$$\lim_{n\to\infty}R_n=\prod_{i=1}^{\infty}\left(1+\frac{2}{i}\frac{1+pi}{2-p+pi}\right)^{\frac{1}{1+pi}}$$

which is 6.2101994550... in the event that $p=1/2$.

When discussing the full-information problem for Uniform $[0,1]$ variables, we assumed that the number of applicants is known. If instead this itself is a uniformly distributed variable on $\{1, 2, \ldots, n\}$, then for the "nothing but the best objective", the asymptotic probability of success is [209, 210]

$$(1-e^{a})\,\mathrm{Ei}(-a)-(e^{-a}+a\,\mathrm{Ei}(-a))(\gamma+\ln(a)-\mathrm{Ei}(a))=0.4351708055...$$

where $a = 2.1198244098...$ is the unique positive solution of the equation

$$e^{a}(1-\gamma-\ln(a)+\mathrm{Ei}(-a))-(\gamma+\ln(a)-\mathrm{Ei}(a))=1.$$

It is remarkable that these constants occur in other, seemingly unrelated versions of the secretary problem [211, 212, 213].

Suppose that you view successively terms of a sequence X_1, X_2, X_3, ... of independent random variables with a common distribution function F. You know the function F, and as X_k is being viewed, you must either stop the process or continue. If you stop at time k, you receive a payoff $(1/k)\sum_{j=1}^{k} X_j$. Your objective is to maximize the expected payoff. An optimal strategy is to stop at the first k for which $\sum_{j=1}^{k} X_j \geq \alpha_k$, where α_1, α_2, α_3, ... are certain values depending on F. Shepp [214, 215] proved that $\lim_{k\to\infty} \alpha_k/\sqrt{k}$ exists and is independent of F as long as F has zero mean and unit variance; further,

$$\lim_{k\to\infty}\frac{\alpha_k}{\sqrt{k}}=x=0.8399236756...$$

is the unique zero of $2x-\sqrt{2\pi}\,(1-x^2)\exp\left(x^2/2\right)\left(1+\operatorname{erf}(x/\sqrt{2})\right)$.

Also, consider a random binary string $Y_1Y_2Y_3\ldots Y_n$ with $\mathrm{P}(Y_k=1)=1-\mathrm{P}(Y_k=0)$ independent of k and Y_k independent of the other Ys. Let H denote the pattern consisting of the digits

$$\underbrace{1000...0}_{l} \qquad \text{or} \qquad \underbrace{0111...1}_{l}$$

and assume that its probability of occurrence for each k is

$$\mathrm{P}\left(Y_{k+1}Y_{k+2}Y_{k+3}\ldots Y_{k+l}=H\right)=\frac{1}{l}\left(1-\frac{1}{l}\right)^{l-1}\sim\frac{1}{el}=\frac{0.3678794411...}{l}.$$

You observe sequentially the digits Y_1, Y_2, Y_3, ... one at a time. You know the values n and l, and as Y_k is being observed, you must either stop the process or continue. Your objective is to stop at the final appearance of H up to Y_n. Bruss & Louchard [216] determined a strategy that maximizes the probability of meeting this goal. For $n \geq \beta l$, this success probability is

$$\frac{2}{135}e^{-\beta}\left(4-45\beta^{2}+45\beta^{3}\right)=0.6192522709...$$

as $l \to \infty$, where $\beta = 3.4049534663...$ is the largest zero of the cubic $45\beta^3 - 180\beta^2 + 90\beta + 4$. Further, the interval $[0.367..., 0.619...]$ constitutes "typical" asymptotic bounds on success probabilities associated with a wide variety of optimal stopping problems in strings.

5.16. Extreme Value Constants. The median of the Gumbel distribution is $-\ln(\ln(2)) = 0.3665129205....$

5.17. Pattern-Free Word Constants. Recent references include [217, 218, 219].

5.18. Percolation Cluster Density Constants. An integral similar to that for $\kappa_B(p_c)$ on the triangular lattice appears in [220].

5.19. Klarner's Polyomino Constant. The number $\bar{A}(n)$ of row-convex n-ominoes satisfies [221]

$$\bar{A}(n) = 5\bar{A}(n-1) - 7\bar{A}(n-2) + 4\bar{A}(n-3), \qquad n \geq 5,$$

with $\bar{A}(1) = 1$, $\bar{A}(2) = 2$, $\bar{A}(3) = 6$ and $\bar{A}(4) = 19$; hence $\bar{A}(n) \sim u\, v^n$ as $n \to \infty$, where $v = 3.2055694304...$ is the unique real zero of $x^3 - 5x^2 + 7x - 4$ and $u = (41v^2 - 129v + 163)/944 = 0.1809155018....$

5.20. Longest Subsequence Constants. The Sankoff-Mainville conjecture that $\lim_{k\to\infty} \gamma_k k^{1/2} = 2$ was proved by Kiwi, Loebl & Matousek [222]; the constant 2 arises from a connection with the longest increasing subsequence problem. A deeper connection with the Tracy-Widom distribution from random matrix theory has now been confirmed [223]:

$$\mathrm{E}(\lambda_{n,k}) \sim 2k^{-1/2}n + c_1 k^{-1/6} n^{1/3}, \qquad \mathrm{Var}(\lambda_{n,k}) \sim c_0 k^{-1/3} n^{2/3}$$

where $k \to \infty$, $n \to \infty$ in such a way that $n/k^{1/2} \to 0$.

Define $\lambda_{n,k,r}$ to be the length of the longest common subsequence c of a and b subject to the constraint that, if $a_i = b_j$ are paired when forming c, then $|i - j| \leq r$. Define as well $\gamma_{k,r} = \lim_{n\to\infty} \mathrm{E}(\lambda_{n,k,r})/n$. It is not surprising [224] that $\lim_{r\to\infty} \gamma_{k,r} = \gamma_k$. Also, $\gamma_{2,1} = 7/10$, but exact values for $\gamma_{3,1}$, $\gamma_{4,1}$, $\gamma_{2,2}$ and $\gamma_{2,3}$ remain open.

The Tracy-Widom distribution (specifically, $F_{\mathrm{GOE}}(x)$ as described in [225]) seems to play a role in other combinatorial problems [226, 227, 228], although the data is not conclusive.

5.21. k-Satisfiability Constants. On the one hand, the lower bound for $r_c(3)$ was improved to 3.42 in [229] and further improved to 3.52 in [230]. On the other hand, the upper bound 4.506 for $r_c(3)$ in [231] has not been confirmed; the preceding two best upper bounds were 4.596 [232] and 4.571 [233].

5.23. Monomer-Dimer Constants. Friedland & Peled [234] revisited Baxter's computation of A and confirmed that $\ln(A) = 0.66279897....$ They examined the three-dimensional analog, A', of A and found that $0.7652 < \ln(A') < 0.7863$.

5.25. Tutte-Beraha Constants. For any positive integer r, there is a best constant $C(r)$ such that, for each graph of maximum degree $\leq r$, the complex zeros of its chromatic polynomial lie in the disk $|z| \leq C(r)$. Further, $K = \lim_{r\to\infty} C(r)/r$ exists and $K = 7.963906...$ is the smallest number for which

$$\inf_{\alpha>0} \frac{1}{\alpha} \sum_{n=2}^{\infty} e^{\alpha n} K^{-(n-1)} \frac{n^{n-1}}{n!} \leq 1.$$

Sokal [235] proved all of the above, answering questions raised in [236, 237].

6.1. Gauss' Lemniscate Constant. Consider the following game [238]. Players A and B simultaneously choose numbers x and y in the unit interval; B then pays A the amount $|x-y|^{1/2}$. The value of the game (that is, the expected payoff, assuming both players adopt optimal strategies) is $M/2 = 0.59907....$ Also, let $\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_n$ be distinct points in the plane and construct, with these points as centers, squares of side s and of arbitrary orientation that do not overlap. Then

$$s \leq \frac{L}{\sqrt{2}} \left(\frac{\prod\limits_{i=1}^{n} \prod\limits_{j=1}^{n} |\xi_i - \eta_j|}{\prod\limits_{i<j} |\xi_i - \xi_j| \cdot \prod\limits_{i<j} |\eta_i - \eta_j|} \right)^{1/n}$$

and the constant $L/\sqrt{2} = 1.85407...$ is best possible [239].

6.3. Kepler-Bouwkamp Constant. Additional references include [240, 241, 242] and another representation is [243]

$$\rho = \frac{3^{10}\sqrt{3}}{2^7 5^2 7\ 11\,\pi} \exp\left[-\sum_{k=1}^{\infty} \frac{\left(\zeta(2k) - 1 - 2^{-2k} - 3^{-2k}\right) 2^{2k} \left(\lambda(2k) - 1 - 3^{-2k}\right)}{k} \right];$$

the series converges at the same rate as a geometric series with ratio $1/100$. A relevant inequality is [244]

$$\int_0^{\infty} \cos(2x) \prod_{j=1}^{\infty} \cos\left(\frac{x}{j}\right) dx < \frac{\pi}{8}$$

and the difference is less than 10^{-42}!

6.5. Plouffe's Constant. This constant is included in a fascinating mix of ideas by Smith [245], who claims that "angle-doubling" one bit at a time was known centuries ago to Archimedes and was implemented decades ago in binary cordic algorithms (also mentioned in section 5.14). Another constant of interest is $\arctan(\sqrt{2}) = 0.9553166181...$, which is the base angle of a certain isosceles spherical triangle (in fact, the unique non-Euclidean triangle with rational sides and a single right angle).

Chowdhury [246] generalized his earlier work on bitwise XOR sums and the logistic map: A sample new result is

$$\sum_{n=0}^{\infty} \frac{\rho(b_n b_{n-1})}{2^{n+1}} = \frac{1}{4\pi} \oplus \frac{1}{\pi}$$

where $b_n = \cos(2^n)$. The right-hand side is computed merely by shifting the binary expansion of $1/\pi$ two places (to obtain $1/(4\pi)$) and adding modulo two without carries (to find the sum).

6.6. Lehmer's Constant. Rivoal [247] has studied the link between the rational approximations of a positive real number x coming from the continued cotangent representation of x, and the usual convergents that proceed from the regular continued fraction expansion of x.

6.9. Minkowski-Bower Constant. See [248, 249] for a generalization of the Minkowski question mark function.

6.10. Quadratic Recurrence Constants. The sequence $k_{n+1} = (1/n)k_n^2$, where $n \geq 0$, is convergent if and only if

$$|k_0| < \exp\left(\sum_{j=2}^{\infty} \frac{\ln(j)}{2^j}\right) = 1.6616879496....$$

Moreover, the sequence either converges to zero or diverges to infinity [250, 251]. A systematic study of threshold constants like this, over a broad class of quadratic recurrences, has never been attempted. The constant 1.2640847353... and Sylvester's sequence appear in an algebraic-geometric setting [252]. Also, results on Somos' sequences are found in [253, 254].

6.11. Iterated Exponential Constants. Consider the recursion

$$a_1 = 1, \qquad a_n = a_{n-1} \exp\left(\frac{1}{e a_{n-1}}\right)$$

for $n \geq 2$. It is known that [255]

$$a_n = \frac{n}{e} + \frac{\ln(n)}{2e} + \frac{C}{e} + o(1), \qquad (n!)^{1/n} = \frac{n}{e} + \frac{\ln(n)}{2e} + \frac{\ln(\sqrt{2\pi})}{e} + o(1)$$

as $n \to \infty$, where

$$C = e - 1 + \frac{\gamma}{2} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{k - e a_k}{e k a_k} + \sum_{k=1}^{\infty}\left(e a_{k+1} - e a_k - 1 - \frac{1}{2 e a_k}\right) = 1.2905502....$$

Further, $a_n - (n!)^{1/n}$ is strictly increasing and

$$a_n - (n!)^{1/n} \leq \left(C - \ln(\sqrt{2\pi})\right)/e = 0.136708...$$

for all n. The constant is best possible. Putting $b_n = 1/(ea_n)$ yields the recursion $b_n = b_{n-1}\exp(-b_{n-1})$, for which an analogous asymptotic expansion can be written.

The constant $3^{-1}e^{-1/3} = 0.2388437701...$ arises in [256] as a consequence of the formula $-W(-3^{-1}e^{-1/3}) = 1/3$. Note that $-W(-x)$ is the exponential generating function for rooted labeled trees and hence is often called the *tree function*.

7.1. Bloch-Landau Constants. In the definitions of the sets F and G, the functions f need only be analytic on the open unit disk D (in addition to satisfying $f(0) = 0$, $f'(0) = 1$). On the one hand, the weakened hypothesis doesn't affect the values of B, L or A; on the other hand, the weakening is essential for the existence of $f \in G$ such that $m(f) = M$.

The bounds $0.62\pi < A < 0.7728\pi$ were improved by several authors, although they studied the quantity $\tilde{A} = \pi - A$ instead (the omitted area constant). Barnard & Lewis [257] demonstrated that $\tilde{A} \le 0.31\pi$. Barnard & Pearce [258] established that $\tilde{A} \ge 0.240005\pi$, but Banjai & Trefethen [259] subsequently computed that $\tilde{A} = (0.2385813248...)\pi$. It is believed that the earlier estimate was slightly in error. See [260, 261, 262] for related problems.

The spherical analog of Bloch's constant B, corresponding to meromorphic functions f mapping D to the Riemann sphere, was recently determined by Bonk & Eremenko [263]. This constant turns out to be $\arccos(1/3) = 1.2309594173....$ A proof as such gives us hope that someday the planar Bloch-Landau constants will also be exactly known.

More relevant material is found in [163, 264].

7.2. Masser-Gramain Constant. Suppose $f(z)$ is an entire function such that $f^{(k)}(n)$ is an integer for each nonnegative integer n, for each integer $0 \le k \le s-1$. (We have discussed only the case $s = 1$.) The best constant $\theta_s > 0$ for which

$$\limsup_{r\to\infty} \frac{\ln(M_r)}{r} < \theta_s \qquad \text{implies } f \text{ is a polynomial}$$

was proved by Bundschuh & Zudilin [265], building on Gel'fond [266] and Selberg [267], to satisfy

$$s \cdot \frac{\pi}{3} \ge \theta_s > \begin{cases} 0.994077... & \text{if } s = 2, \\ 1.339905... & \text{if } s = 3, \\ 1.674474... & \text{if } s = 4. \end{cases}$$

(Actually they proved much more.) Can a Gaussian integer-valued analog of these integer-valued results be found?

7.5. Hayman Constants. A new upper bound [268] for the Hayman-Korenblum constant $c(2)$ is 0.69472. An update on the Hayman-Wu constant appears in [269].

7.6. Littlewood-Clunie-Pommerenke Constants. The lower limit of summation in the definition of S_2 should be $n = 0$ rather than $n = 1$, that is, the coefficient

b_0 need not be zero. We have sharp bounds $|b_1| \leq 1$, $|b_2| \leq 2/3$, $|b_3| \leq 1/2 + e^{-6}$ [270]. The bounds on γ_k due to Clunie & Pommerenke should be 0.509 and 0.83 [271]; Carleson & Jones' improvement was nonrigorous. While $0.83 = 1 - 0.17$ remains the best established upper bound, the lower bound has been increased to $0.54 = 1 - 0.46$ [272, 273, 274]. Numerical evidence for both the Carleson-Jones conjecture and Brennan's conjecture was found by Kraetzer [275]. Theoretical evidence supporting the latter appears in [276], but a complete proof remains undiscovered. It seems that $\alpha = 1 - \gamma$ is now a theorem [277, 278] whose confirmation is based on the recent work of several researchers [279, 280, 281].

8.1. Geometric Probability Constants. Just as the ratio of a semicircle to its diameter is always $\pi/2$, the ratio of the latus rectum arc of any parabola to its latus rectum is [282, 283]

$$\frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right) = 1.1477935746... = \frac{1}{2}(2.2955871493...)$$

Is it mere coincidence that this constant is so closely related to the quantity $\delta(2)$? The expected distance between two random points on different sides of the unit square is [244]

$$\frac{2 + \sqrt{2} + 5\ln\left(1 + \sqrt{2}\right)}{9} = 0.8690090552...$$

and the expected distance between two random points on different faces of the unit cube is

$$\frac{4 + 17\sqrt{2} - 6\sqrt{3} - 7\pi + 21\ln\left(1 + \sqrt{2}\right) + 21\ln\left(7 + 4\sqrt{3}\right)}{75} = 0.9263900551...$$

Also, the convex hull of random point sets in the unit disk (rather than the unit square) is mentioned in [284].

8.4. Moser's Worm Constant. Relevant progress is described in [285].

8.7. Hermite's Constants. A lattice Λ in $\mathbb{R}^n$ consists of all integer linear combinations of a set of basis vectors $\{e_j\}_{j=1}^n$ for $\mathbb{R}^n$. If the fundamental parallelepiped determined by $\{e_j\}_{j=1}^n$ has Lebesgue measure 1, then Λ is said to be of unit volume. The constants γ_n can be defined via an optimization problem

$$\gamma_n = \max_{\substack{\text{unit volume} \\ \text{lattices } \Lambda}} \min_{\substack{x \in \Lambda, \\ x \neq 0}} \|x\|^2$$

and are listed in Table 8.10. The precise value of the next constant $2 \leq \gamma_9 < 2.1327$ remains open [286, 287, 288], although Cohn & Kumar [289, 290] have recently proved that $\gamma_{24} = 4$. A classical theorem [291, 292, 293] provides that γ_n^n is rational for all

n. It is not known if the sequence γ_1, γ_2, γ_3, $\ldots$ is strictly increasing, or if the ratio γ_n/n tends to a limit as $n \to \infty$. See also [294].

Table 8.10 *Hermite's Constants* γ_n

n	Exact	Decimal	n	Exact	Decimal
1	1	1	5	$8^{1/5}$	1.5157165665...
2	$(4/3)^{1/2}$	1.1547005383...	6	$(64/3)^{1/6}$	1.6653663553...
3	$2^{1/3}$	1.2599210498...	7	$64^{1/7}$	1.8114473285...
4	$4^{1/4}$	1.4142135623...	8	2	2

An arbitrary packing of the plane with disks is called compact if every disk D is tangent to a sequence of disks D_1, D_2, $\ldots$, D_n such that D_i is tangent to D_{i+1} for $i = 1, 2, \ldots, n$ with $D_{n+1} = D_1$. If we pack the plane using disks of radius 1, then the only possible compact packing is the hexagonal lattice packing with density $\pi/\sqrt{12}$. If we pack the plane using disks of radius 1 and $r < 1$ (disks of both sizes must be used), then there are precisely nine values of r for which a compact packing exists. See Table 8.11. For seven of these nine values, it is known that the densest packing is a compact packing; the same is expected to be true for the remaining two values [295, 296, 297].

Table 8.11 *All Nine Values of* $r < 1$ *Which Allow Compact Packings*

Exact (expression or minimal polynomial)	Decimal
$5 - 2\sqrt{6}$	0.1010205144...
$(2\sqrt{3} - 3)/3$	0.1547005383...
$(\sqrt{17} - 3)/4$	0.2807764064...
$x^4 - 28x^3 - 10x^2 + 4x + 1$	0.3491981862...
$9x^4 - 12x^3 - 26x^2 - 12x + 9$	0.3861061048...
$\sqrt{2} - 1$	0.4142135623...
$8x^3 + 3x^2 - 2x - 1$	0.5332964166...
$x^8 - 8x^7 - 44x^6 - 232x^5 - 482x^4 - 24x^3 + 388x^2 - 120x + 9$	0.5451510421...
$x^4 - 10x^2 - 8x + 9$	0.6375559772...

8.19. Circumradius-Inradius Constants. The phrase "Z-admissible" in the caption of Figure 8.22 should be replaced by "Z-allowable".

Table of Constants. The formula corresponding to 0.8427659133... is $(12\ln(2))/\pi^2$ and the formula corresponding to 0.8472130848... is $M/\sqrt{2}$.

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