# Fermat's Last Theorem

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P. Fermat

**Pierre de Fermat(1601-65)** Pierre Fermat's father was a wealthy leather merchant and second consul of Beaumont- de- Lomagne. Pierre had a brother and two sisters and was almost certainly brought up in the town of his birth. Although there is little evidence concerning his school education it must have been at the local Franciscan monastery.

Fermat is best remembered for this work in number theory, in particular for Fermat's Last Theorem. This theorem states that $\boxed{x^n + y^n = z^n}$ has no non-zero integer solutions for x, y and z when $n > 2$. Fermat wrote, in the margin of Bachet's translation of Diophantus's Arithmetica: *I have discovered a truly remarkable proof which this margin is too small to contain.* These marginal notes only became known after Fermat's son Samuel published an edition of Bachet's translation of Diophantus's Arithmetica with his father's notes in 1670. It is now believed that Fermat's proof' was wrong although it is impossible to be completely certain.

Unsuccessful attempts to prove the theorem over a 300 year period led to the discovery of commutative ring theory and a wealth of other mathematical discoveries.

Despite large prizes being offered for a solution, Fermat's Last Theorem remained unsolved. It has the dubious distinction of being the theorem with the largest number of published false proofs. For example over 1000 false proofs were published between 1908 and 1912. The only positive progress seemed to be computing results which merely showed that any counter-example would be very large. Using techniques based on Kummer's work, Fermat's Last Theorem was proved true, with the help of computers, for n up to 4 000 000 by 1993.

A. Wiles

The proof of Fermat's Last Theorem was completed in 1993 by Andrew Wiles, a British mathematician working at Princeton in the USA. Wiles gave a series of three lectures at the Isaac Newton Institute in Cambridge, England the first on Monday 21 June, the second on Tuesday 22 June. In the final lecture on Wednesday 23 June 1993 at around 10:30 in the morning Wiles announced his proof of Fermat's Last Theorem as a corollary to his main results. Having written the theorem on the blackboard he said "I will stop here" and sat down.

This, however, is not the end of the story. On 4 December 1993 Andrew Wiles made a statement in view of the speculation. He said that during the reviewing process a number of problems had emerged, most of which had been resolved. However one problem remains and Wiles essentially withdrew his claim to have a proof.

In March 1994 Faltings, writing in Scientific American, said

> *If it were easy, he would have solved it by now. Strictly speaking, it was not a proof when it was announced.*

In fact, from the beginning of 1994, Wiles began to collaborate with Richard Taylor in an attempt to fill the holes in the proof. However they decided that one of the key steps in the proof, using methods due to Flach, could not be made to work. They tried a new approach with a similar lack of success. In August 1994 Wiles addressed the International Congress of Mathematicians but was no nearer to solving the difficulties.

Taylor suggested a last attempt to extend Flach's method in the way necessary and Wiles, although convinced it would not work, agreed mainly to enable him to convince Taylor that it could never work. Wiles worked on it for about two weeks, then suddenly inspiration struck.

> *In a flash I saw that the thing that stopped it (the extension of Flach's method) working was something that would make another method I had tried previously work.*

On 6 October Wiles sent the new proof to three colleagues including Faltings. All liked the new proof which was essentially simpler than the earlier one. Faltings sent a simplification of part of the proof.

No proof of the complexity of this can easily be guaranteed to be correct, so a very small doubt will remain for some time. However when Taylor lectured at the British Mathematical Colloquium in Edinburgh in April 1995 he gave the impression that no real doubts remained over Fermat's Last Theorem.

## PBS Interview

"I don't believe Fermat had a proof."

**Solving Fermat: Andrew Wiles**

Andrew Wiles devoted much of his entire career to proving Fermat's Last Theorem, the world's most famous mathematical problem. In 1993, he made front-page headlines when he announced a proof of the problem, but this was not the end of the story; an error in his calculation jeopardized his life's work. Andrew Wiles spoke to NOVA and described how he came to terms with the mistake, and eventually went on to achieve his life's ambition.

NOVA: Many great scientific discoveries are the result of obsession, but in your case that obsession has held you since you were a child.

ANDREW WILES: I grew up in Cambridge in England, and my love of mathematics dates from those early childhood days. I loved doing problems in school. I'd take them home and make up new ones of my own. But the best problem I ever found, I found in my local public library. I was just browsing through the section of math books and I found this one book, which was all about one particular problem – Fermat's Last Theorem. This problem had been unsolved by mathematicians for 300 years. It looked so simple, and yet all the great mathematicians in history couldn't solve it. Here was a problem, that I, a ten year old, could understand and I knew from that moment that I would never let it go. I had to solve it.

NOVA: Who was Fermat and what was his Last Theorem?

AW: Fermat was a 17th-century mathematician who wrote a note in the margin of his book stating a particular proposition and claiming to have proved it. His proposition was about an equation which is closely related to Pythagoras' equation. Pythagoras' equation gives you:

$$x^2 + y^2 = z^2.$$

You can ask, what are the whole number solutions to this equation, and you can see that:

$$3^2 + 4^2 = 5^2,$$

and

$$5^2 + 12^2 = 13^2.$$

And if you go on looking then you find more and more such solutions. Fermat then considered the cubed version of this equation: $x^3 + y^3 = z^3$. He raised the question: can you find solutions to the cubed equation? He claimed that there were none. In fact, he claimed that for the general family of equations: $x^n + y^n = z^n$ where $n$ is bigger than 2 it is impossible to find a solution. That's Fermat's Last Theorem.

NOVA: So Fermat said because he could not find any solutions to this equation, then there were no solutions?

AW: He did more than that. Just because we can't find a solution it doesn't mean that there isn't one. Mathematicians aren't satisfied because they know there are no solutions up to four million or four billion, they really want to know that there are no solutions up to infinity. And to do that we need a proof. Fermat said he had a proof. Unfortunately, all he ever wrote down was: "I have a truly marvelous demonstration of this proposition which this margin is too narrow to contain."

NOVA: What do you mean by a proof?

AW: In a mathematical proof you have a line of reasoning consisting of many, many steps, that are almost self-evident. If the proof we write down is really rigorous, then nobody can ever prove it wrong. There are proofs that date back to the Greeks that are still valid today.

NOVA: So the challenge was to rediscover Fermat's proof of the Last Theorem. Why did it become so famous?

AW: Well, some mathematics problems look simple, and you try them for a year or so, and then you try them for a hundred years, and it turns out that they're extremely hard to solve. There's no reason why these problems shouldn't be easy, and yet they turn out to be extremely intricate. The Last Theorem is the most beautiful example of this.

NOVA: But finding a proof has no applications in the real world; it is a purely abstract question. So why have people put so much effort into finding a proof?

AW: Pure mathematicians just love to try unsolved problems – they love a challenge. And as time passed and no proof was found, it became a real challenge. I've read letters in the early 19th century which said that it was an embarrassment to mathematics that the Last Theorem had not been solved. And of course, it's very special because Fermat said that he had a proof.

NOVA: How did you begin looking for the proof?

AW: In my early teens I tried to tackle the problem as I thought Fermat might have tried it. I reckoned that he wouldn't have known much more math than I knew as a teenager. Then when I reached college, I realized that many people had thought about the problem during the 18th and 19th centuries and so I studied those methods. But I still wasn't getting anywhere. Then when I became a researcher, I decided that I should put the problem aside. It's not that I forgot about it – it was always there – but I realized that the only techniques we had to tackle it had been around for 130 years. It didn't seem that these techniques were really getting to the root of the problem. The problem with working on Fermat was that you could spend years getting nowhere. It's fine to work on any problem, so long as it generates interesting mathematics along the way – even if you don't solve it at the end of the day. The definition of a good mathematical problem is the mathematics it generates rather than the problem itself.

NOVA: It seems that the Last Theorem was considered impossible, and that mathematicians could not risk wasting getting nowhere. But then in 1986 everything changed. A breakthrough by Ken Ribet at the University of California at Berkeley linked Fermat's Last Theorem to another unsolved problem, the Taniyama-Shimura conjecture. Can you remember how you reacted to this news?

AW: It was one evening at the end of the summer of 1986 when I was sipping iced tea at the house of a friend. Casually in the middle of a conversation this friend told me that Ken Ribet had proved a link between Taniyama-Shimura and Fermat's

Last Theorem. I was electrified. I knew that moment that the course of my life was changing because this meant that to prove Fermat's Last Theorem all I had to do was to prove the Taniyama-Shimura conjecture. It meant that my childhood dream was now a respectable thing to work on. I just knew that I could never let that go.

NOVA: So, because Taniyama-Shimura was a modern problem, this meant that working on it, and by implication trying to prove Fermat's Last Theorem, was respectable.

AW: Yes. Nobody had any idea how to approach Taniyama-Shimura but at least it was mainstream mathematics. I could try and prove results, which, even if they didn't get the whole thing, would be worthwhile mathematics. So the romance of Fermat, which had held me all my life, was now combined with a problem that was professionally acceptable.

NOVA: At this point you decided to work in complete isolation. You told nobody that you were embarking on a proof of Fermat's Last Theorem. Why was that?

AW: I realized that anything to do with Fermat's Last Theorem generates too much interest. You can't really focus yourself for years unless you have undivided concentration, which too many spectators would have destroyed.

NOVA: But presumably you told your wife what you were doing?

AW: My wife's only known me while I've been working on Fermat. I told her on our honeymoon, just a few days after we got married. My wife had heard of Fermat's Last Theorem, but at that time she had no idea of the romantic significance it had for mathematicians, that it had been such a thorn in our flesh for so many years.

NOVA: On a day-to-day basis, how did you go about constructing your proof?

AW: I used to come up to my study, and start trying to find patterns. I tried doing calculations which explain some little piece of mathematics. I tried to fit it in with some previous broad conceptual understanding of some part of mathematics that would clarify the particular problem I was thinking about. Sometimes that would involve going and looking it up in a book to see how it's done there. Sometimes it was a question of modifying things a bit, doing a little extra calculation. And sometimes I realized that nothing that had ever been done before was any use at all. Then I just had to find something completely new; it's a mystery where that comes from. I carried this problem around in my head basically the whole time. I would wake up with it first thing in the morning, I would be thinking about it all day, and I would be thinking about it when I went to sleep. Without distraction, I would have the same thing going round and round in my mind. The only way I could relax was when I was with my children. Young children simply aren't interested in Fermat. They just want to hear a story and they're not going to let you do anything else.

NOVA: Usually people work in groups and use each other for support. What did you do when you hit a brick wall?

AW: When I got stuck and I didn't know what to do next, I would go out for a walk. I'd often walk down by the lake. Walking has a very good effect in that you're in this state of relaxation, but at the same time you're allowing the sub-conscious to work on you. And often if you have one particular thing buzzing in your mind then you don't need anything to write with or any desk. I'd always have a pencil and paper ready and, if I really had an idea, I'd sit down at a bench and I'd start scribbling away.

NOVA: So for seven years you're pursuing this proof. Presumably there are periods of self-doubt mixed with the periods of success.

AW: Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of – and couldn't exist without – the many months of stumbling around in the dark that proceed them.

NOVA: And during those seven years, you could never be sure of achieving a complete proof.

AW: I really believed that I was on the right track, but that did not mean that I would necessarily reach my goal. It could be that the methods needed to take the next step may simply be beyond present day mathematics. Perhaps the methods I needed to complete the proof would not be invented for a hundred years. So even if I was on the right track, I could be living in the wrong century.

NOVA: Then eventually in 1993, you made the crucial breakthrough.

AW: Yes, it was one morning in late May. My wife, Nada, was out with the children and I was sitting at my desk thinking about the last stage of the proof. I was casually looking at a research paper and there was one sentence that just caught my attention. It mentioned a 19th-century construction, and I suddenly realized that I should be able to use that to complete the proof. I went on into the afternoon and I forgot to go down for lunch, and by about three or four o'clock, I was really convinced that this would solve the last remaining problem. It got to about tea time and I went downstairs and Nada was very surprised that I'd arrived so late. Then I told her I'd solved Fermat's Last Theorem.

NOVA: The New York Times exclaimed "At Last Shout of 'Eureka!' in Age-Old Math Mystery," but unknown to them, and to you, there was an error in your proof. What was the error?

AW: It was an error in a crucial part of the argument, but it was something so subtle that I'd missed it completely until that point. The error is so abstract that it can't really be described in simple terms. Even explaining it to a mathematician would require the mathematician to spend two or three months studying that part of the manuscript in great detail.

NOVA: Eventually, after a year of work, and after inviting the Cambridge mathematician Richard Taylor to work with you on the error, you managed to repair the proof. The question that everybody asks is this; is your proof the same as Fermat's?

AW: There's no chance of that. Fermat couldn't possibly have had this proof. It's 150 pages long. It's a 20th-century proof. It couldn't have been done in the 19th century, let alone the 17th century. The techniques used in this proof just weren't around in Fermat's time.

NOVA: So Fermat's original proof is still out there somewhere.

AW: I don't believe Fermat had a proof. I think he fooled himself into thinking he had a proof. But what has made this problem special for amateurs is that there's a tiny possibility that there does exist an elegant 17th-century proof.

NOVA: So some mathematicians might continue to look for the original proof. What will you do next?

AW: There's no problem that will mean the same to me. Fermat was my childhood passion. There's nothing to replace it. I'll try other problems. I'm sure that some of them will be very hard and I'll have a sense of achievement again, but nothing will mean the same to me. There's no other problem in mathematics that could hold me the way that this one did. There is a sense of melancholy. We've lost something that's been with us for so long, and something that drew a lot of us into mathematics. But perhaps that's always the way with math problems, and we just have to find new ones to capture our attention. People have told me I've taken away their problem – can't I give them something else? I feel some sense of responsibility. I hope that seeing the excitement of solving this problem will make young mathematicians realize that there are lots and lots of other problems in mathematics which are going to be just as challenging in the future.

NOVA: What is the main challenge now?

AW: The greatest problem for mathematicians now is probably the Riemann Hypothesis. But it's not a problem that can be simply stated.

NOVA: And is there any one particular thought that remains with you now that Fermat's Last Theorem has been laid to rest?

AW: Certainly one thing that I've learned is that it is important to pick a problem based on how much you care about it. However impenetrable it seems, if you don't try it, then you can never do it. Always try the problem that matters most to you. I had this rare privilege of being able to pursue in my adult life, what had been my childhood dream. I know it's a rare privilege, but if one can really tackle something in adult life that means that much to you, then it's more rewarding than anything I can imagine.

NOVA: And now that journey is over, there must be a certain sadness?

AW: There is a certain sense of sadness, but at the same time there is this tremendous sense of achievement. There's also a sense of freedom. I was so obsessed by this problem that I was thinking about it all the time – when I woke up in the morning, when I went to sleep at night – and that went on for eight years. That's a long time to think about one thing. That particular odyssey is now over. My mind is now at rest.