Tower of Hanoi

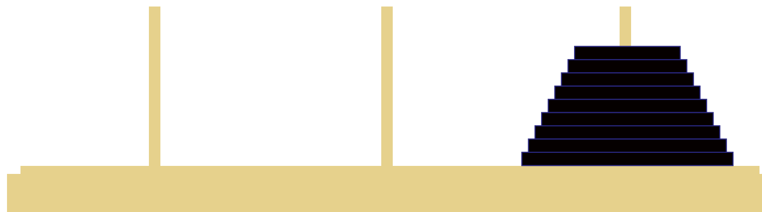
Lecture Notes for CS 5

Delivered on September 17, 2003

The Tower of Hanoi

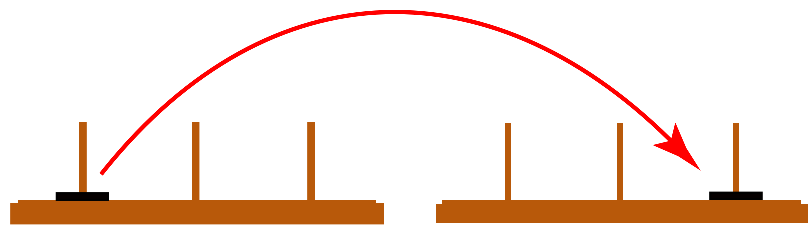
Initial State:

Goal State:



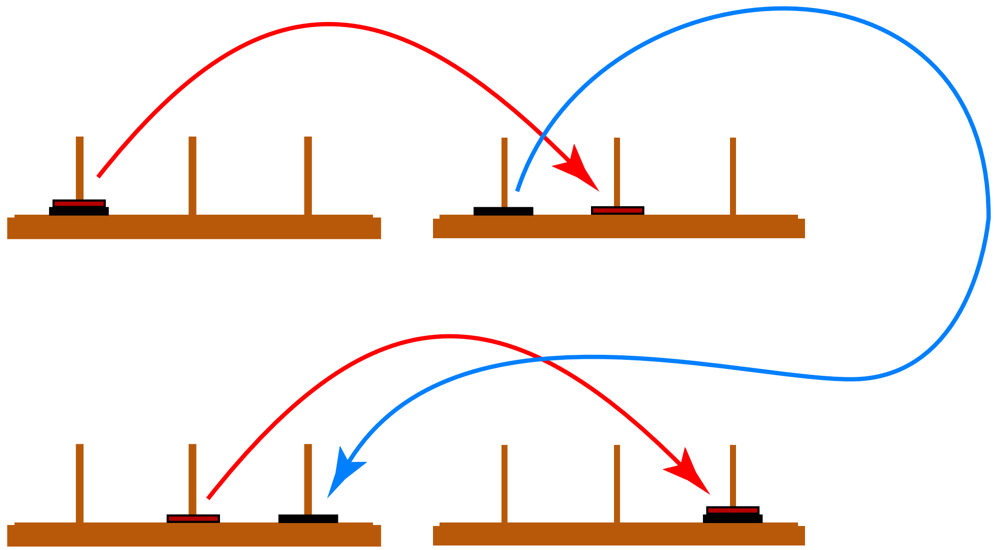
Only one disk can be moved at a time, from one peg to another, such that it never is placed on a disk of smaller diameter.

Tower of Hanoi: One Disk



1 Move.

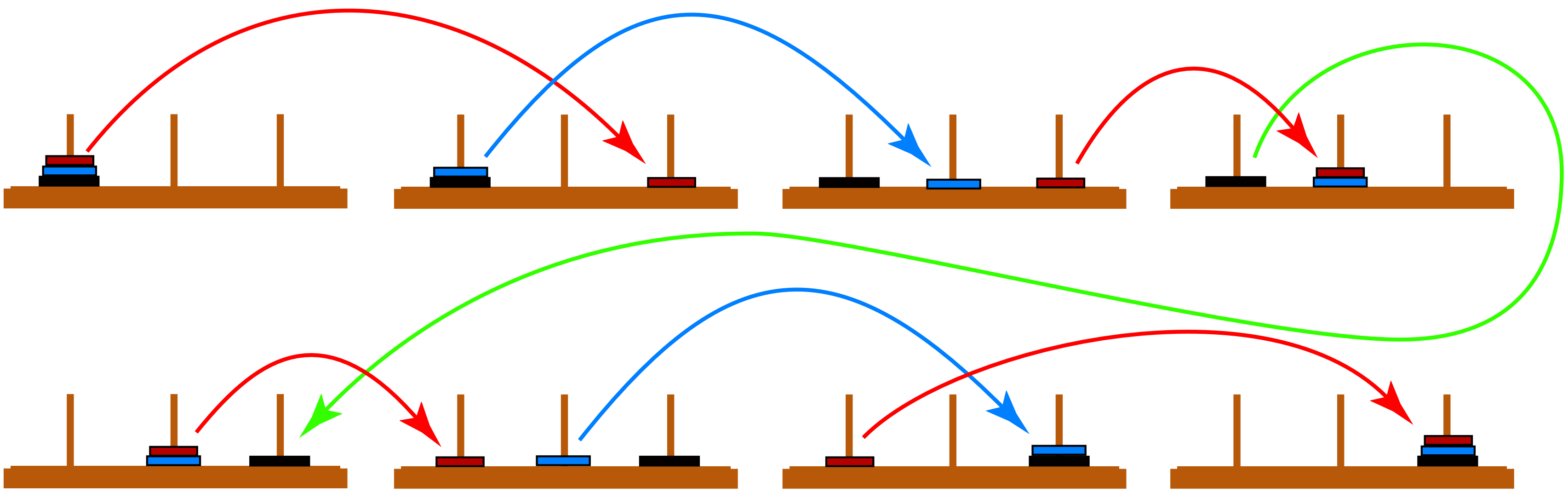
Two Disks



Note that we perform the solution to the one-disk Tower of Hanoi *twice* (once in the top row, and once in the bottom row). Between rows, we move the bottom disk. Thus, we require

$$2 \cdot 1 + 1 = 3 \text{ Moves}$$

Tower of Hanoi: Three Disks



Note that we perform the solution to the two-disk Tower of Hanoi *twice* (once in the top row, and once in the bottom row). Between rows we move the bottom disk. Thus, we require

$$2(2 \cdot 1 + 1) + 1 = 7 \text{ Moves}$$

Tower of Hanoi: What we know so far

Let $M(n)$ denote the minimum number of legal moves required to complete a tower of Hanoi puzzle that has n disks.

n	$M(n)$
1	1
2	3
3	7

Following the pattern, for $n = 4$ we need to solve the three-disk puzzle twice, plus one more operation to move the largest disk. Thus,

$$M(4) = 2 \cdot M(3) + 1 = 2 \cdot 7 + 1 = 15.$$

Similarly, for $n = 5$ disks, we expect that we will need to perform

$$M(5) = 2 \cdot M(4) + 1 = 2 \cdot 15 + 1 = 31.$$

Tower of Hanoi: n Disks

Let $M(n)$ denote the minimum number of legal moves required to complete a tower of Hanoi puzzle that has n disks.

- Before the largest disk (i.e., the n-th disk) can be moved to the rightmost peg, all of the remaining $(n-1)$ disks must moved to the center peg. (These $n-1$ disks must be somewhere, and they can't obstruct the transfer of the largest disk.) This requires $M(n-1)$ legal moves.
- It takes 1 more operation to move the n-th disk to the rightmost peg.
- Finally, another legal sequence of $M(n-1)$ steps is required to move the $n-1$ disks from the center peg, to the rightmost peg.

We thus obtain the *recursion relation*,

$$M(n) = 2M(n-1) + 1.$$

Tower of Hanoi: Solution

With the solution for a single disk

$$M(1) = 1$$

the recursion relation

$$M(n) = 2M(n-1) + 1.$$

defines the solution

$$M(n) = 2^n - 1.$$

In are hierarchy of algorithms, this would be called *exponential* or $\mathcal{O}(2^n)$.

Tower of Hanoi: Practical Consequences

The practical difficulty with exponential algorithms is that they can quickly grow out of hand. (With each additional disk, the minimum number of operations essentially doubles.) N.B., 1 century $\approx 4.5 \times 10^9$ seconds.

n	$2^n - 1$	n	$2^n - 1$	n	$2^n - 1$
8	256	19	524287	30	1073741823
9	511	20	1048575	31	2147483647
10	1027	21	2097151	32	4294967295
11	2047	22	4194303	33	8589934591
12	4095	23	8388607	34	17179869183
13	8191	24	16777215	35	34359738367
14	16383	25	33554431	36	68719476735
15	32767	26	67108863	37	137438953471
16	65635	27	134217727	38	274877906943
17	131071	28	268435455	39	549755813887
18	262143	29	536870911	40	1099511627775

Tower of Hanoi: Strategy

We have seen that the solution to this puzzle is *recursive*:

- In order to move all n disks to the *right* peg, we must first move the top $n-1$ disks to the *center* peg.
- Before this, we must move the top $n-2$ disks to the *right* peg.
- And before this, we must move the top $n-3$ disks to the *center* peg, and so on.

So what should the first move be? Should the top disk be moved to the *center* peg, or to the *right* peg?

Tower of Hanoi: Solution Path

- If the number of disks is *odd*, then the first move should be to transfer the top disk to the *right* peg.
- If the number of disks is *even*, then the first move should be to transfer the top disk to the *center* peg.

The sequence of states that is visited in the course of solving the puzzle is called the *solution path*.

The length of the shortest solution path to the n-disk puzzle is 2^n.

Tower of Hanoi: Legal States

A configuration of disks in the Tower of Hanoi puzzle is said to be a *legal state* if no disk rests on a disk of smaller diameter. That is, the largest disk on each peg must be placed on the bottom, and the remaining disks must be placed in the order of decreasing diameter.

Question: How many legal states are there for n disks placed on *three* pegs?

Tower of Hanoi: Accessible States

A configuration of disks is said to be an *accessible state*, if it can be realized from the initial state after a legal sequence of moves.

Question: How many accessible states are there for n disks placed on *three* pegs?

Tower of Hanoi: Examples

Number of Disks n	Length of Solution Path 2^n	Number of Legal States 3^n	Fraction of Visited States $(2/3)^n$
1	2	3	0.666667
2	4	9	0.444444
3	8	27	0.296296
4	16	81	0.197531
5	32	243	0.131687
6	64	729	0.0877915
7	128	2187	0.0585277
8	256	6561	0.0390184
32	4294967296	1853020188851841	2.32×10^{-6}

State Representation

It's cumbersome to represent a state by an illustration. Instead we will adopt a list notation.

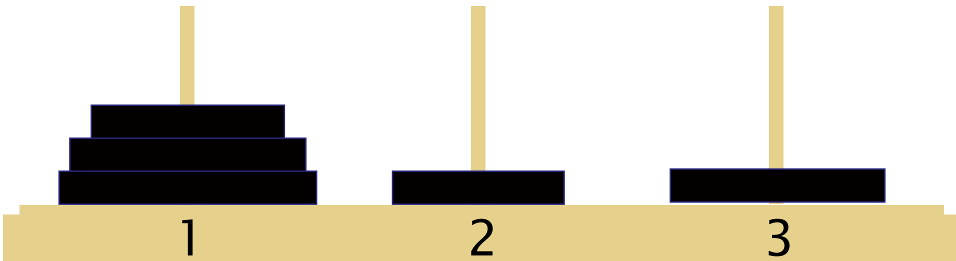
For example, for a *five* disk puzzle, the notation

(2 1 3 1 1)

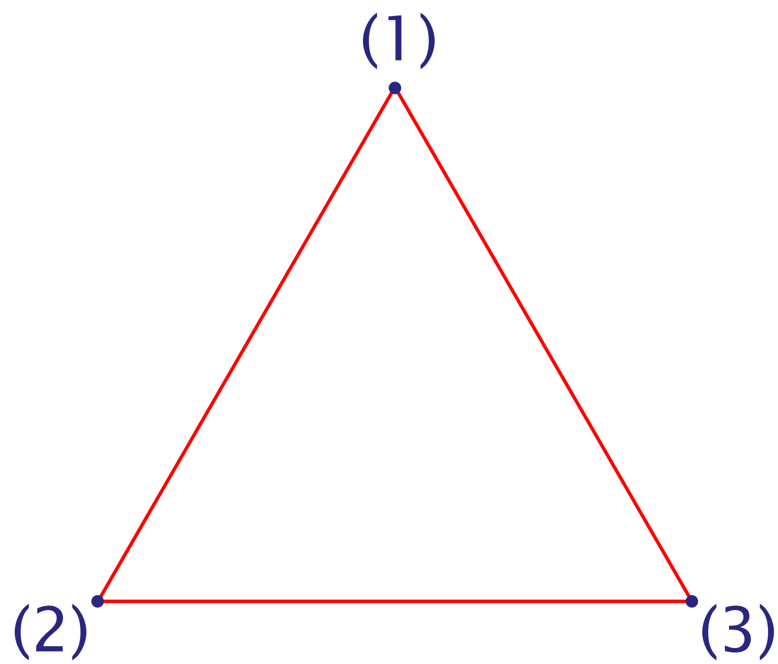
indicates

- The smallest disk is on peg 2 (the center peg).
- The next larger disk is on peg 1 (the left peg).
- The next larger disk is on peg 3 (the right peg).
- The next larger disk is on peg 1.
- The next larger (i.e., largest) disk is on peg 1.

The only legal configuration that describes this is

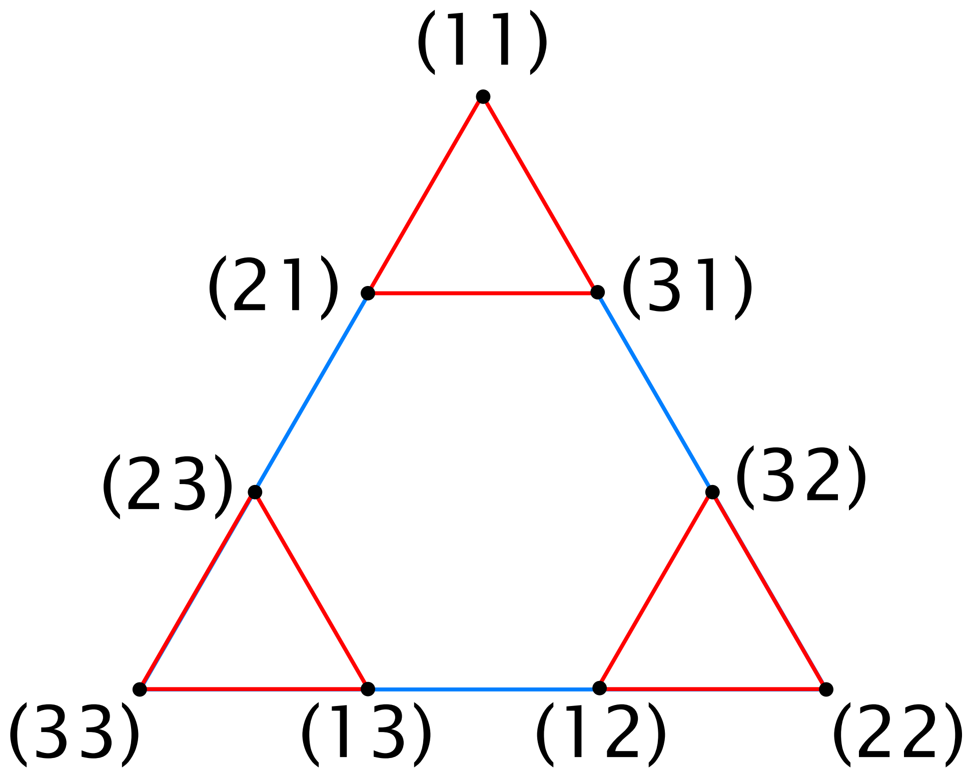


State Graph: 1 disk



Each state is represented by a labeled *vertex*; legal moves are represented by *edges*.

State Graph: 2 disks



State Graph: 3 disks