Hexagonal lattice directed site animals

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Site animals on the directed square and triangular lattices are known to have quadratic, algebraic generating functions. Attempts to find the corresponding generating function for directed hexagonal lattice animals have been unsuccessful. Here we show that the generating function is almost certainly not algebraic, nor differentiably finite, nor constructible differentiably algebraic.

1 Introduction

A directed lattice animal $\mathcal{A}$ is a finite *connected* set of vertices on an acyclic lattice such that all vertices $p \in \mathcal{A}$ are either the (unique) origin vertex or one (lattice) step in one of the preferred directions from some other vertex in $\mathcal{A}$. The *area* of the animal is the number of vertices in $\mathcal{A}$. A *neighbour* of $\mathcal{A}$ is a vertex that does not belong to $\mathcal{A}$, but is connected by an edge to a vertex of $\mathcal{A}$. The *site perimeter* of $\mathcal{A}$ is the number of its neighbours. As usual, animals are defined up to a translation on an infinite, periodic lattice. As well as their intrinsic interest, they are closely related to lattice models of cell growth in general, and percolation in particular.

For the square lattice, only steps in the $+x$ and $+y$ directions are permitted, while on the triangular lattice only steps in the three directions with positive x component are permitted. For these two lattices, the problem of enumerating the number of site animals was first studied on strips of finite width [1–2]. Subsequently Dhar et al. [3] solved the problem completely, obtaining the generating function for both lattices empirically. Later Dhar [4] showed the problem to be equivalent to the Baxter hard-square problem. This equivalence was then [5] proved by a bijection to a class of one-dimensional walks, and later [6] proved by a bijection to asymmetric trees. Similarly, Viennot [7] established the result for the triangular case.

For the square lattice, the generating function $f_s(x)$ satisfies

$$(f_s^2(x) + f_s(x))(1 - 3x) - x = 0$$

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from which one readily obtains

$$f_s(x) = \frac{1}{2}(\sqrt{\frac{1+x}{1-3x}} - 1).$$

The corresponding result for the triangular lattice is even simpler, being

$$f_t(x) = \frac{1}{2}(\frac{1}{\sqrt{1-4x}} - 1).$$

Corresponding results for bond animals are not known. Neither is the solution for site animals on the hexagonal, Kagomé or other Archimedean lattices known. In [9] a family of so-called *strange* lattices was proposed, for which the exact generating function for site animals was empirically obtained. These results were subsequently proved [8,10]. The strange lattices are a type of decorated square lattice, but bear a superficial resemblance to the hexagonal lattice.

From the above expressions for the generating functions, it is clear that the number of site-animals with n-sites grows asymptotically as $\mu^n/\sqrt{n}$, where $\mu = 3$ for the square lattice and $\mu = 4$ for the triangular lattice animals. For hexagonal lattice animals, an enumeration [3] and subsequent analysis of all animals up to 48 sites led to the conclusion that the asymptotic form was unchanged, and to the estimate $\mu = 2.0252 \pm 0.0005$. In [9] we presented a 99 term series, and an analysis which gave the more precise estimate $\mu = 2.025131 \pm 0.000005$. However we failed to find an expression for the generating function.

More recently, a detailed study of two-dimensional directed animals has been given [10]. In that paper Bousquet-Mélou considers the two-variable generating function $f(x,t)$ where the second variable t is conjugate to the perimeter. In terms of these two-variable generating functions, it is shown [10] that

$$f_h(x,t) = xt + f_s(x^2, t(1+x)),$$

where $f_h(x,t)$ is the honeycomb lattice site and area generating function. Hence the site generating function that we require is

$$f_h(x,1) = x + f_s(x^2, (1+x)).$$

As the two-variable generating function for the square lattice is not known, this doesn't provide the required solution! Note however that from the known solution for the square lattice generating function we have the result that

$$f_h(x, 1/(1+x)) = \frac{x}{1+x} + f_s(x^2, 1) = \frac{x}{1+x} + f_s(x^2).$$

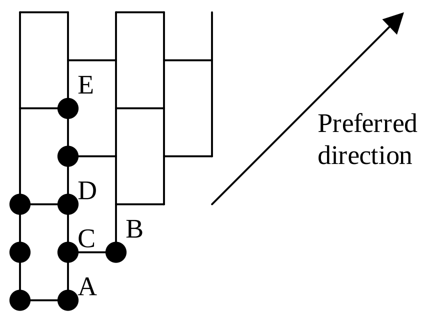

Figure 1: Directed hexagonal lattice

Thus we see that the full two-variable hexagonal lattice generating function is known along one curve of its domain. This is reminiscent of the susceptibility of the anisotropic two-dimensional Ising model on the triangular lattice, which is solvable along the disorder line.

Another parameter that can enter the problem arises if we look closely at the various cases that can occur at an occupied vertex of a site animal. For the hexagonal lattice, there are two types of site, and each site can be *supported* in a variety of ways. Imagine a directed animal on the honeycomb lattice rendered as a brickwork lattice (see Figure 1). Sites are allowed to be to the right or to the left of an occupied site, subject to the restriction that sites can only be supported on the left if there is a bond to their left.

We define two types of sites : those that have a support to the left, and those that do not. In the figure, sites A, B and D all have supports to the left; no other sites do. Note that C does not have support to the left, as there is no bond to the left, and E has no support to the left as the site to the left is unoccupied.

In this paper we have generalised the problem of hexagonal site animals by enumerating not only by area, but also by the number of vertices supported only one particular way. The result is a two-variable generating function

$$F_h(x,s) = \sum_{m,n} a_{m,n} x^m s^n, \qquad (1)$$

where $a_{m,n}$ is the number of site animals of area m and with n sites supported one particular way. Summation over m gives

$$F_h(x,s) = \sum_n H_n(x) s^n, \tag{2}$$

where $H_n(x)$ is the generating function for all animals with exactly n sites supported in one particular way. By empirically obtaining the first few functions $H_n(x)$ we are able to observe a pattern unfolding which we recognise as characteristic of what we call a *D-unsolvable* problem [11–12]. This concept is fully discussed in the next section.

2 D-unsolvability

Some of the most famous results in mathematics involve a proof of the intrinsic unsolvability of certain problems. Some, such as 'trisecting an angle' go back to antiquity, while others, such as the lack of integer solutions to the equation $x^n + y^n = z^n$ for $n > 2$ have only quite recently been acceptably proved [27]. In mathematical physics and combinatorics such results about the solvability or otherwise of problems are largely unknown. We have [11–12] taken the first steps in filling this gap by presenting and developing what is essentially a numerical method that provides, at worst, strong evidence that a problem has no solution within a large class of functions, including algebraic, differentiably finite (D-finite) [26,24] and a sub-class [18] of differentiably algebraic functions, called *constructible differentiably algebraic* (CDA). Since most of the special functions of mathematical physics — in terms of which most known solutions are given — are differentiably finite, this exclusion renders the problem unsolvable within this class. We use the term *D-unsolvable* to mean that the problem has no solution within the class of D-finite functions as well as the sub-class of differentiably algebraic functions described above.

In fact, the exclusion is wider than this, as we show that the solutions possess a natural boundary on the unit circle in an appropriately defined complex plane. We show that this property implies that the appropriately defined two-variable generating function cannot be either an algebraic, D-finite nor CDA function. Indeed, the exclusion goes further than this, — though we have no simple way to describe this excluded class.

It may be worthwhile to recall the definitions of these classes of functions, and their hierarchy. Algebraic functions are a subset of both the set of D-finite and of CDA functions, while both D-finite and CDA functions are subsets of the set of differentiably algebraic functions. However some D-finite functions are not CDA functions, and conversely, some CDA functions are not D-finite.

Let $\mathbb{K}$ be a commutative field. A series $f(z) \in \mathbb{K}[[z]]$ is said to be *differentiably finite* if there exists an integer k and polynomials $P_0(z), \cdots, P_k(z)$ with coefficients in $\mathbb{K}$ such that $P_k(z)$ is not the null polynomial and

$$P_0(z)f(z) + P_1(z)f'(z) + \cdots + P_k(z)f^{(k)}(z) = 0.$$

A series $f(z) \in \mathbb{K}[[z]]$ is said to be *differentiably algebraic* if there exists an integer k and a polynomial P in $k+2$ variables with coefficients in $\mathbb{K}$, such that

$$P(z, f(z), f'(z), \cdots, f^{(k)}(z)) = 0.$$

A series $f(z) \in \mathbb{K}[[z]]$ is said to be *constructible differentiably algebraic* if there exists both series $f_1(z), f_2(z), \cdots, f_k(z)$ with $f = f_1$, and polynomials $P_1, P_2, \cdots, P_k$ in k-variables, with coefficients in $\mathbb{K}$, such that

$$\begin{aligned} f_1' &= P_1(f_1, f_2, \cdots, f_k), \\ f_2' &= P_2(f_1, f_2, \cdots, f_k), \\ &\cdots \\ f_k' &= P_k(f_1, f_2, \cdots, f_k). \end{aligned} \tag{3}$$

Differentiably finite functions in several variables are discussed in [24], while CDA functions in several variables are discussed in [19].

The method which we shall describe and which can, in favourable circumstances, be sharpened into a formal proof, has been applied to a wide variety of problems in both statistical mechanics and combinatorics. A review is given in [12]. Typically, the solution of the problem will require the calculation of the graph generating function in terms of some parameter, such as perimeter, area, number of bonds or sites. A key first step is to *anisotropise* the generating function. The anisotropisation is often non-unique, in which case some thought needs to be given to the appropriate operation. For example, if counting graphs by the number of bonds on, say, an underlying square lattice, one distinguishes between horizontal and vertical bonds. In this way, one can construct a two-variable generating function, $G(x,y) = \sum_{m,n} g_{m,n} x^m y^n$ where $g_{m,n}$ denotes the number of graphs with m horizontal and n vertical bonds.

An important requirement to bear in mind is that the anisotropisation should not make the solution more complicated. For example, if studying an Ising model in zero field, the appropriate anisotropisation is to distinguish between coupling constants in different lattice directions. Generalising to temperature and magnetic field, while producing a two-variable generating function, is a poor choice as the solution will clearly be much more complicated. (Furthermore, it is not an anisotropisation of a single variable — it is the addition of a second, independent, variable.)

Returning to the two variable generating function introduced above, if we sum over one of the variables, we may write

$$G(x,y) = \sum_{m,n} g_{m,n} x^m y^n = \sum_n H_n(x) y^n \tag{4}$$

where $H_n(x)$ is the generating function for the relevant graphs with n vertical bonds. It has been observed in all the problems so far studied, that the functions $H_n(x)$ are rational, with denominator zeros lying on the unit circle in the complex x plane.

In some cases one finds only a small finite number (typically one or two) of denominator zeros on the unit circle. Loosely speaking, this is the hallmark of a solvable problem. If, as is often observed, the denominator zeros become dense on the unit circle as n increases, so that in the limit a natural boundary is formed, then this is the hallmark of a D-unsolvable problem.

The significance of this observation is substantial. It follows that, as n increases, the denominators of $H_n(x)$ contain zeros given by steadily higher roots of unity. Hence the structure of the functions $H_n(x)$ is that of a rational function whose poles all lie on the unit circle in the complex x-plane, such that the poles become dense on the unit circle as n gets large. This behaviour of the functions $H_n(x)$ implies that $G(x,y)$ as a formal power series in y with coefficients in $\mathbb{C}(x)$ (a) has a natural boundary (b) is neither algebraic nor D-finite, nor CDA. Further, provided that $G(x,c)$ is well-defined for a given complex value c, then, in the absence of miraculous cancellations, it follows that $G(x,c)$ also is neither D-finite nor CDA.

Of course, we are primarily interested in the solution of the *isotropic* case, when $x = y$, and it is clear that the anisotropic case can behave quite differently from the isotropic case. This is most easily seen by construction. Consider the function

$$f(x,y) = f_1(x,y) + (x-y) f_2(x,y), \tag{5}$$

where $f_1(x,y)$ is D-finite and $f_2(x,y)$ is not. Clearly, the function $f(x,y)$ is not D-finite, while $f(x,x)$ is D-finite. However, in all the cases we have studied where the solutions are known, the effect of anisotropisation *does not* change the analytic structure of the solution. Rather, it simply moves singularities around in the complex plane, at most causing the bifurcation of a real singularity into a complex pair. Further, for unsolved problems, numerical procedures indicate that similar behaviour prevails. Nevertheless, this remains an observation, rather than an established fact, and, strictly speaking, should be established for each new problem.

If we now ask what functions *do* display the type of behaviour we have just observed — a build up of singularities on the unit circle in the complex plane, then the most obvious candidate that displays this behaviour is the q-generalisation of the standard functions of mathematical physics, or possibly a modular function, when expressed in terms of its original expansion variable z or $q = \exp(2\pi i z)$ [23]. This behaviour has been seen in a number of solutions already, such as the solution of the hard hexagon model [15], certain interacting walk models [25] and some polygon models [16].

That being said, not all problems with a small number of denominator zeros have been solved, while some D-unsolvable problems have been solved. In the former case however we believe that it is only a matter of time before a solution is found for these problems, while in the latter case the solutions have usually been expressed in terms of q-generalisations of the standard functions, which are of course not D-finite, or modular functions.

As an example, the generating function for the number of parallelogram polygons given in terms of the area (q), horizontal semi-perimeter (x) and vertical semi-perimeter (y) is [17]

$$G(x,y,q) = y\frac{J_1}{J_0} \text{ where} \tag{6}$$

$$J_1(x,y,q) = \sum_{n\geq 1} \frac{(-1)^{n-1} x^n q^{\binom{n+1}{2}}}{(q)_{n-1}(yq)_n} \text{ and} \tag{7}$$

$$J_0(x,y,q) = \sum_{n\geq 0} \frac{(-1)^{n} x^n q^{\binom{n+1}{2}}}{(q)_{n}(yq)_n}. \tag{8}$$

In this case, it is clear that if we look at $G(x,1,q)$ in the complex q-plane with x held fixed, the solution possesses a natural boundary on the unit circle.

Based on these observations, we have suggested that the following method is a particularly useful first step in the study of such problems. One first anisotropises the problem, then one computes enough terms in the generating function to be able to construct the first few functions H_n, and one then studies the denominator pattern. If it appears that the zeros are becoming dense on the unit circle, one has reason to suspect that the problem is D-unsolvable. If on the other hand there are only one or two zeros, one is in an excellent position to seek the solution in terms of the standard functions of mathematical physics — loosely defined as those described in [13]. In some cases one may be able to *prove* that the observed denominator pattern persists. In that case, one has proved the observed results.

The construction of the functions H_n deserves some explanation. At very low order this can often be done exactly, by combinatorial arguments based on the allowed graphs. Beyond this, our method is to generate the coefficients in the expansion, *assume* the function $H_n(x)$ is rational, then by essentially constructing the Padé approximant one can confidently *conjecture* the solution. Typically, one may generate 50-100 terms in the expansion — which are therefore known exactly — and use perhaps 10 or 20 of them to find a rational function with numerator and denominator of degree 5 or 10. Thus the first 10 or 20 terms of the series are used to identify this conjectured rational function, which can then be expanded, and the coefficients compared with the exact coefficients that were not used in its discovery. Hence while this is not a rigorous derivation, the chance of it being incorrect is extraordinarily small.

It should be said explicitly that this technique is computationally demanding. That is to say, the generation of sufficient terms in the generating function is usually quite difficult. Only with improved algorithms — most notably the combination of the finite lattice method [21,22] with a transfer matrix formulation — and computers with large physical memory which are needed for the efficient implementation of such algorithms, has it been possible to obtain expansions of the required length in a reasonable time. The technique is still far from routine, with each problem requiring a significant calculational effort. Other important aspects of the method, such as the connection of these ideas with concepts of integrability, and with the existence of a Yang-Baxter equation are not explored here.

An additional, and exceptionally valuable feature of the method comes when the numerical work, described above, is combined with certain functional relations that the anisotropised generating functions must satisfy. These key functional relations or *inversion relations* imply a connection between the generating function and its analytic continuation, usually involving the reciprocal of one or more of the expansion variable(s). As first shown by Baxter [14], the existence of these inversion relations, coupled with any obvious symmetries (usually a symmetry with respect to the interchange of x and y), coupled with the *observed* behaviour of the functions H_n — described above — can yield an *implicit* solution to the underlying problem with no further calculation. An example of this is the solution [14] of the zero-field free energy of the two-dimensional Ising model.

This same approach has been used to show [11,20] that the susceptibility of the two-dimensional Ising model, and the generating function for square lattice self-avoiding walks, is *D-unsolvable.*

In the next sections we apply the method to the problem of square lattice and hexagonal lattice site animals, and hence show that the former is solvable while the latter appears to be *D-unsolvable.*

3 Directed square site animals

As mentioned above, the generating function for square lattice directed site animals is exactly known. It has recently been generalised to include the number of sites supported in exactly one way, as well as the total number of sites [16]. The result is

$$F_{sq}(x,s) = \frac{1}{2}\left((1 - \frac{4x}{(1+x)(1+x-sx)})^{-\frac{1}{2}} - 1\right). \tag{9}$$

Writing this as

$$F_{sq}(x,s) = \sum_n H_n(x)s^n, \tag{10}$$

expansion readily yields

$$\begin{aligned}
H_0(x) &= x/(1-x), \\
H_1(x) &= x^2/(1-x)^3, \\
H_2(x) &= x^3(1+x+x^2)/(1-x)^5(1+x), \\
H_3(x) &= x^4(1+2x+4x^2x+2x^3+x^4)/(1-x)^7(1+x)^2, \\
H_4(x) &= x^5(1+3x+9x^2+9x^3+9x^4+3x^5+x^6)/(1-x)^9(1+x)^3, \\
H_5(x) &= x^6(1+4x+16x^2+24x^3+36x^4+24x^5+16x^6+4x^7+x^8) \\
&\quad /(1-x)^{11}(1+x)^4.
\end{aligned}$$

Here it can be seen that the functions $H_n(x)$ have just two denominator zeros, at $x = 1$ and $x = -1$. As discussed above, this is the hallmark of a solvable model.

Inspection also shows that $F_{sq}(x,s)$ is invariant under the combined substitutions $x \to 1/x$ and $s \to sx$ in that order. However, if one applies this transformation to the functions H_n above, it appears not to hold. The reason for this relates to the choice of branches in the square-root function. $H_0(x)$ is in fact $\frac{1}{2}(\sqrt{\frac{(1+x)^2}{(x-1)^2}} - 1)$, where the negative branch of the square root is taken to obtain the result $H_0(x) = x/(1-x)$ above. For $n > 0$, each of the functions $H_n(x)$ contain a multiplicative factor $\sqrt{\frac{(1+x)^2}{(x-1)^2}}$, where the negative branch has again been taken in obtaining the results given above. When the transformation $x \to 1/x$ is applied to these results, the other branch must be taken. This then leaves the functions H_n invariant. (Without taking this into account, the transformation $x \to 1/x$ introduces a spurious factor $-1/x$ to each H function.)

The existence of this invariance is, together with certain other observations, sufficient to implicitly solve the problem, by obtaining the functions H_n order by order. To be precise, we first observe that the structure of the functions H_n is $H_n(x) = x^{n+1}P_{2n-2}(x)/(1-x)^{2n+1}(1+x)^{n-1}$ for $n > 0$, where $P_{2n-2}(x)$ is a symmetric unimodal polynomial with all coefficients positive. (The symmetry is a reflection of the invariance under the transformation $x \to 1/x$.) Thus we have $2n-1$ unknown polynomial coefficients, of which n follow from symmetry. To obtain the $n-1$ unknowns, one studies the coefficients of the series expansion of the functions H_n systematically, and observes that the lowest order non-zero coefficients are all 1. The next lowest order coefficients are given by a linear recurrence (the coefficients are $1, 3, 5, 7, 9, \cdots$) the next lowest by a quadratic recurrence, and so on. This observation is sufficient to determine the unknown coefficients order by order (and to identify the recurrences order by order). In this way we could have experimentally "solved" the problem.

Turning now to hexagonal animals, we see quite different behaviour.

4 Directed hexagonal site animals

We have enumerated site animals on the hexagonal lattice, in the manner described above, anisotropising the area by distinguishing between those sites of the animal supported only one particular way, and the total number of sites. Let

$$F_h(x,t) = \sum_{m,n} a_{m,n} x^m t^n = \sum_n H_n(x) t^n, \tag{11}$$

be the site generating function, where $a_{m,n}$ gives the number of hexagonal lattice site animals, with n sites supported one particular way, and m sites in total.

The generating function for those animals with precisely n sites supported one particular way is $H_n(x)$.

The enumeration is carried out by a dynamic programming algorithm, as follows: Consider a diagonal line perpendicular to the preferred direction, and passing through sites. Any development of the animal in the preferred direction of this line cannot use information on the other side of the line, and thus the animal can be divided this way.

One can define a function which takes a set of sites along this line, together with a flag signifying whether it is an even line or an odd line (with respect to the brick work), and the number of each of the two types of sites left to consider, and which produces the number of animals. This function can be defined recursively by looking at all ways of going to the next diagonal line.

This recursive function was then evaluated using dynamic programming (technically known as memoisation). This is a very efficient method of counting in terms of computational time, but it does require a large amount of memory to be used.

In this way, the first few functions H_n were found to be

$$\begin{aligned}
H_0(x) &= x/(1-x), \\
H_1(x) &= x/(1-x)^3(1+x), \\
H_2(x) &= x^2(1+x+x^3)/(1-x)^5(1+x)^2(1+x^2), \\
H_3(x) &= x^3(1+x)(1+x+3x^3-x^4+x^5)/(1-x)^7(1+x)^3(1+x^2)^2, \\
H_4(x) &= x^4[1,3,4,10,12,14,16,13,14,7,6,4,0,1]/ \\
&\quad (1-x)^9(1+x)^4(1+x^2)^3(1-x-x^2)(1+x+x^2).
\end{aligned}$$

In $H_4(x)$ we have used the obvious notation $[a_0, a_1, \cdots, a_n]$ for the polynomial with those coefficients. Our enumerations are complete up to $H_9(x)$. The first 5 numerators are shown above, while the remaining 5 are shown in the table below.

The structure of the denominators is clearly that characteristic of a *D-unsolvable* model. All denominators have a zero at $x = 1$. Then $H_1(x)$ sees the first occurence of the factor $(1 - x^2)$ in the denominator. Then new factors appear at every second order. H_2 and H_3 see the first appearance of $(1 - x^4)$ and H_4 and H_5 see the first appearance of the factor $(1 - x^6)$. This pattern persists as far as our enumerations have gone. Indeed, the denominator can be written, as also mentioned in the tables, as:

$$\begin{aligned}
&(1-x)^{2n+1}(1+x)^n(1+x^2)^{n-1}(1+x+x^2+x^3+x^4)^{n-7}(1-x+x^2)^{n-3} \\
&\qquad (1+x+x^2)^{n-3}(1+x^4)^{n-5}(1-x+x^2-x^3+x^4)^{n-7}\cdots.
\end{aligned}$$

for $n < 10$.

Thus it appears that, for the problem we wish to solve, the structure is such that the numerator polynomials appear to increase in degree non-linearly (paralleling the growth of the denominator polynomials). The structure of the denominator is such that steadily higher degree roots of unity occur. It follows that, as n increases, the denominators of $H_n(x)$ contain zeros given by the $2k^{th}$ roots of unity, where $2k = n + 1$ or $n + 2$ according as n is odd or even. Hence as n increases, so does k. Hence the structure of the functions $H_n(x)$ is that of a rational function whose poles all lie on the unit circle in the complex x-plane, such that the poles become dense on the unit circle as n gets large. This behaviour of the functions $H_n(x)$ implies that $F_h(x, t)$ as a formal power

m	$t_{5,m}$	$t_{6,m}$	$t_{7,m}$	$t_{8,m}$	$t_{9,m}$	m	$t_{8,m}$	$t_{9,m}$
0	1	1	1	1	1	43	554727	301855120
1	4	5	6	7	8	44	274989	316203750
2	8	13	19	27	36	45	279041	253527720
3	22	40	65	105	156	46	123574	261829201
4	37	85	164	310	524	47	122507	198731706
5	56	160	358	806	1552	48	49192	202955241
6	88	297	763	1970	4259	49	45825	145078927
7	98	453	1377	4234	10394	50	17342	147020846
8	137	711	2474	8755	24183	51	14045	98362175
9	118	956	3936	16652	51875	52	5400	99296288
10	145	1276	6181	30224	106494	53	3354	61720320
11	113	1614	8979	52466	206745	54	1428	62329753
12	104	1798	12546	85594	384905	55	594	35690869
13	79	2192	16753	137723	687199	56	280	36212572
14	50	2064	21422	206033	1183788	57	87	18925621
15	40	2446	26258	309999	1963561	58	28	19365970
16	17	1949	31321	430890	3173730	59	13	9150529
17	10	2220	34969	610219	4927653	60		9463816
18	6	1546	39772	795737	7548433	61	1	4010458
19		1642	40007	1066153	11034284	62		4184507
20	1	1011	44030	1311798	16135051	63		1584852
21		959	39493	1671253	22312164	64		1651470
22		558	42663	1946718	31309061	65		563029
23		427	33716	2369655	41115663	66		570676
24		249	36011	2615436	55583921	67		179940
25		135	24891	3057147	69539866	68		168001
26		85	26363	3194751	90846746	69		51908
27		32	15853	3603847	108550608	70		40488
28		15	16467	3556844	137370735	71		13386
29		8	8717	3892590	157068764	72		7606
30			8644	3614528	192938081	73		2910
31		1	4115	3858021	211378885	74		1082
32			3667	3352815	252481864	75		456
33			1683	3509709	265248214	76		132
34			1201	2836166	308590485	77		36
35			581	2927694	310927383	78		16
36			281	2182896	352907485	79		
37			154	2234387	340888419	80		1
38			51	1523809	378112259			
39			21	1554040	349778052			
40			10	960476	379840056			
41				979475	335930553			
42			1	543813	357880527			

Table 1: Coefficients of numerator polynomials. $H_n(x)$ has numerator $x^n \sum_m x^m t_{n,m}$ and denominator $(1-x)^{2n+1}(1+x)^n(1+x^2)^{n-1}(1+x+x^2+x^3+x^4)^{n-7}(1-x+x^2)^{n-3}(1+x+x^2)^{n-3}(1+x^4)^{n-5}(1-x+x^2-x^3+x^4)^{n-7}\cdots$

series in t with coefficients in $\mathbb{C}(x)$ (a) has a natural boundary (b) is neither algebraic nor D-finite, nor CDA.

This conclusion is then consistent with earlier observations that suggested that the hexagonal lattice problem was qualitatively different to the same problem on the square or triangular lattice, and provides quantitative support for that observation. For some D-unsolvable problems inversion relations exist, and these are often discoverable from the functions H_n. In this case however we have been unable to find one. So the question as to the exact form of the generating function remains unanswered, though we suggest a likely candidate is the large class of q-generalisations of the standard functions of mathematical physics or possibly modular functions.

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