18.S66 PROBLEMS #3

Spring 2003

Beginning with this assignment we will (subjectively) indicate the difficulty level of each problem as follows:

[1] easy
[2] moderately difficult
[3] difficult.

In general, these difficulty ratings are based on the assumption that the solutions to the previous problems are known.

A *partition* λ of $n \geq 0$ (denoted $\lambda \vdash n$ or $|\lambda| = n$) is an integer sequence $(\lambda_1, \lambda_2, \ldots)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum \lambda_i = n$. Trailing 0's are often ignored, e.g., $(4,3,3,1,1)$ represents the same partition of 12 as $(4,3,3,1,1,0,0)$ or $(4,3,3,1,1,0,0,\ldots)$. The terms $\lambda_i > 0$ are called the *parts* of λ. The *conjugate partition* to λ, denoted λ', has $\lambda_i - \lambda_{i+1}$ parts equal to i for all $i \geq 1$. The (Young) *diagram* of λ is a left-justified array of squares with λ_i squares in the ith row. Notation such as $u = (2,3) \in \lambda$ means that u is the square of the diagram of λ in the second row and third column.

69. [1] Let λ be a partition. Then

$$\sum_i (i-1)\lambda_i = \sum_i \binom{\lambda'_i}{2}.$$

70. [1] Let λ be a partition. Then

$$\begin{aligned}
\sum_i \left\lceil \frac{\lambda_{2i-1}}{2} \right\rceil &= \sum_i \left\lceil \frac{\lambda'_{2i-1}}{2} \right\rceil \\
\sum_i \left\lfloor \frac{\lambda_{2i-1}}{2} \right\rfloor &= \sum_i \left\lceil \frac{\lambda'_{2i}}{2} \right\rceil \\
\sum_i \left\lfloor \frac{\lambda_{2i}}{2} \right\rfloor &= \sum_i \left\lfloor \frac{\lambda'_{2i}}{2} \right\rfloor.
\end{aligned}$$

71. [1] The number of partitions of n with largest part k equals the number of partitions of n with exactly k parts.

72. [2] Fix $k \geq 1$. Let λ be a partition. Define $f_k(\lambda)$ to be the number of parts of λ equal to k, e.g., $f_3(8,5,5,3,3,3,3,3,2,1,1) = 4$. Define $g_k(\lambda)$ to be the number of integers i for which λ has at least k parts equal to i, e.g., $g_3(8,8,8,8,6,6,3,2,2,2,1) = 2$. Then

$$\sum_{\lambda \vdash n} f_k(\lambda) = \sum_{\lambda \vdash n} g_k(\lambda).$$

73. [2] The number of partitions of n with odd parts equals the number of partitions of n with distinct parts.

74. [2] Let $\sigma(n)$ denote the sum of all (positive) divisors of $n \in \mathbb{P}$; e.g., $\sigma(12) = 1+2+3+4+6+12 = 28$. Let $p(n)$ denote the number of partitions of n (with $p(0) = 1$). Then

$$n \cdot p(n) = \sum_{i=1}^{n} \sigma(i) p(n-i).$$

75. [2] The number of self-conjugate partitions of n equals the number of partition of n into distinct odd parts.

76. [3] Let $f(n)$ be the number of partitions of n into an even number of parts, all distinct. Let $g(n)$ be the number of partitions of n into an odd number of parts, all distinct. For instance, $f(7) = 3$, corresponding to $6+1 = 5+2 = 4+3$, and $g(7) = 2$, corresponding to $7 = 4+2+1$. Then

$$f(n) - g(n) = \begin{cases} (-1)^k, & \text{if } n = k(3k \pm 1)/2 \text{ for some } k \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

NOTE. This result is usually stated in generating function form, viz.,

$$\prod_{n \geq 1} (1 - x^n) = 1 + \sum_{k \geq 1} (-1)^k \left(x^{k(3k-1)/2} + x^{k(3k+1)/2} \right),$$

and is known as *Euler's pentagonal number formula.*

77. [2] Let $f(n)$ (respectively, $g(n)$) be the number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n into distinct parts, such that the largest part λ_1 is even (respectively, odd). Then

$$f(n) - g(n) = \begin{cases} 1, & \text{if } n = k(3k+1)/2 \text{ for some } k \geq 0 \\ -1, & \text{if } n = k(3k-1)/2 \text{ for some } k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

78. [3] For $n \in \mathbb{N}$ let $f(n)$ (respectively, $g(n)$) denote the number of partitions of n into distinct parts such that the smallest part is odd and with an even number (respectively, odd number) of even parts. Then

$$f(n) - g(n) = \begin{cases} 1, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise.} \end{cases}$$

79. (a) (∗) The number of partitions of n into parts $\equiv \pm 1 \pmod 5$ is equal to the number of partitions of n whose parts differ by at least 2.

    (b) (∗) The number of partitions of n into parts $\equiv \pm 2 \pmod 5$ is equal to the number of partitions of n whose parts differ by at least 2 and for which 1 is not a part.

    NOTE. This is the combinatorial formulation of the famous *Rogers-Ramanujan identities*. One of the known proofs of this result has been converted into a complicated recursive bijection. What is wanted is a "direct" bijection whose inverse is easy to describe.

80. [3] The number of partitions of n into parts $\equiv 1,\ 5, \text{ or } 6 \pmod 8$ is equal to the number of partitions into parts that differ by at least 2, and such that odd parts differ by at least 4.

81. [3] A *lecture hall partition* of length k is a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ (some of whose parts may be 0) satisfying

$$0 \leq \frac{\lambda_k}{1} \leq \frac{\lambda_{k-1}}{2} \leq \cdots \leq \frac{\lambda_1}{k}.$$

The number of lecture hall partitions of n of length k is equal to the number of partitions of n whose parts come from the set $\{1, 3, 5, \ldots, 2k-1\}$ (with repetitions allowed).

82. (*) The *Lucas numbers* L_n are defined by $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$ for $n \geq 2$. Let $f(n)$ be the number of partitions of n all of whose parts are Lucas numbers L_{2n+1} of odd index. For instance, $f(12) = 5$, corresponding to

$$\begin{array}{l} 1+1+1+1+1+1+1+1+1+1+1+1 \\ 4+1+1+1+1+1+1+1+1 \\ 4+4+1+1+1+1 \\ 4+4+4 \\ 11+1 \end{array}$$

Let $g(n)$ be the number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_i/\lambda_{i+1} > \frac{1}{2}(3+\sqrt{5})$ whenever $\lambda_{i+1} > 0$. For instance, $g(12) = 5$, corresponding to

$$12, \quad 11+1, \quad 10+2, \quad 9+3, \quad 8+3+1.$$

Then $f(n) = g(n)$ for all $n \geq 1$.

83. [2.5] Let $A(n)$ denote the number of partitions $(\lambda_1, \ldots, \lambda_k) \vdash n$ such that $\lambda_k > 0$ and

$$\lambda_i > \lambda_{i+1} + \lambda_{i+2}, \quad 1 \leq i \leq k-1$$

(with $\lambda_{k+1} = 0$). Let $B(n)$ denote the number of partitions $(\mu_1, \ldots, \mu_j) \vdash n$ such that

- Each μ_i is in the sequence $1, 2, 4, \ldots, g_m, \ldots$ defined by

$$g_1 = 1, \ g_2 = 2, \ g_m = g_{m-1} + g_{m-2} + 1 \text{ for } m \geq 3.$$

- If $\mu_1 = g_m$, then every element in $\{1, 2, 4, \ldots, g_m\}$ appears at least once as a μ_i.

Then $A(n) = B(n)$ for all $n \geq 1$.

Example. $A(7) = 5$ because the relevant partitions are (7), $(6,1)$, $(5,2)$, $(4,3)$, $(4,2,1)$, and $B(7) = 5$ because the relevant partitions are $(4,2,1)$, $(2,2,2,1)$, $(2,2,1,1,1)$, $(2,1,1,1,1,1)$, $(1,1,1,1,1,1,1)$.

84. (*) Let $S \subseteq \mathbb{P}$ and let $p(S,n)$ denote the number of partitions of n whose parts belong to S. Let

$$\begin{aligned} S &= \pm\{1,4,5,6,7,9,10,11,13,15,16,19 \ (\mathrm{mod}\ 40)\} \\ T &= \pm\{1,3,4,5,9,10,11,14,15,16,17,19 \ (\mathrm{mod}\ 40)\}, \end{aligned}$$

where

$$\pm\{a, b, \ldots \pmod{m}\} = \{n \in \mathbb{P} : n \equiv \pm a, \pm b, \ldots \pmod{m}\}.$$

Then $p(S, n) = p(T, n-1)$ for all $n \geq 1$.

NOTE. In principle the known proof of this result and of Problem 85 below can be converted into a complicated recursive bijection, as was done for Problem 79. Just as for Probem 79, what is wanted is a "direct" bijection whose inverse is easy to describe. To my knowledge no one has tried to give a bijective solution to this problem and the next, so perhaps they are not so difficult.

85. (*) Let

$$\begin{aligned} S &= \pm\{1, 4, 5, 6, 7, 9, 11, 13, 16, 21, 23, 28 \pmod{66}\} \\ T &= \pm\{1, 4, 5, 6, 7, 9, 11, 14, 16, 17, 27, 29 \pmod{66}\}. \end{aligned}$$

Then $p(S, n) = p(T, n)$ for all $n \geq 1$ *except* $n = 13$ (!).

86. [1.5] Prove the following identities by interpreting the coefficients in terms of partitions.

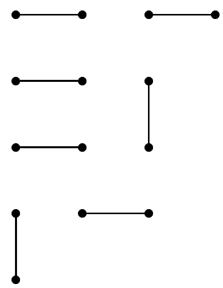
$$\begin{aligned} \prod_{i \geq 1} \frac{1}{1 - qx^i} &= \sum_{k \geq 0} \frac{x^k q^k}{(1-x)(1-x^2)\cdots(1-x^k)} \\ \prod_{i \geq 1} \frac{1}{1 - qx^i} &= \sum_{k \geq 0} \frac{x^{k^2} q^k}{(1-x)\cdots(1-x^k)(1-qx)\cdots(1-qx^k)} \\ \prod_{i \geq 1} (1 + qx^i) &= \sum_{k \geq 0} \frac{x^{\binom{k+1}{2}} q^k}{(1-x)(1-x^2)\cdots(1-x^k)} \\ \prod_{i \geq 1} (1 + qx^{2i-1}) &= \sum_{k \geq 0} \frac{x^{k^2} q^k}{(1-x^2)(1-x^4)\cdots(1-x^{2k})}. \end{aligned}$$

87. [3] Show that

$$\sum_{n=-\infty}^{\infty} x^n q^{n^2} = \prod_{k \geq 1} (1 - q^{2k})(1 + xq^{2k-1})(1 + x^{-1} q^{2k-1}).$$

This famous result is *Jacobi's triple product identity.*

88. [3] Let $f(n)$ be the number of partitions of $2n$ whose Ferrers diagram can be covered by n edges, each connecting two adjacent dots. For instance, $(4,3,3,3,1)$ can be covered as follows:



Then $f(n)$ is equal to the number of ordered pairs (λ, μ) of partitions satisfying $|\lambda| + |\mu| = n$.

89. (*) Given a partition λ and $u \in \lambda$, let $a(u)$ (called the *arm length* of u) denote the number of squares directly to the right of u (in the diagram of λ), counting λ itself exactly once. Similarly let $l(u)$ (called the *leg length* of u) denote the number of squares directly below u, counting u itelf once. Thus if $u = (i,j)$ then $a(u) = \lambda_i - j + 1$ and $l(u) = \lambda'_j - i + 1$. Define

$$\gamma(\lambda) = \#\{u \in \lambda \, : \, a(u) - l(u) = 0 \text{ or } 1\}.$$

Then

$$\sum_{\lambda \vdash n} q^{\gamma(\lambda)} = \sum_{\lambda \vdash n} q^{\ell(\lambda)},$$

where $\ell(\lambda)$ denotes the length (number of parts) of λ.

90. [2.5] If $0 \leq k < \lfloor n/2 \rfloor$, then $\binom{n}{k} \leq \binom{n}{k+1}$.

NOTE. To prove an inequality $a \leq b$ combinatorially, find sets A, B with $\#A = a$, $\#B = b$, and either an injection (one-to-one map) $f : A \to B$ or a surjection (onto map) $g : B \to A$.

91. [2.5] Let $1 \leq k \leq n-1$. Then $\binom{n}{k}^2 \geq \binom{n}{k-1}\binom{n}{k+1}$. Note that this result is even stronger than Problem 90 above (assuming $\binom{n}{k} = \binom{n}{n-k}$) [why?].

92. [1] Let $p(j,k,n)$ denote the number of partitions of n with at most j parts and with largest part at most k. Then $p(j,k,n) = p(j,k,jk-n)$.

NOTE. A standard result in enumerative combinatorics states that

$$\sum_{n=0}^{jk} p(j,k,n)q^n = \begin{bmatrix} j+k \\ j \end{bmatrix},$$

where $\left[\begin{smallmatrix} m \\ i \end{smallmatrix}\right]$ denotes the q-*binomial coefficient*:

$$\begin{bmatrix} m \\ i \end{bmatrix} = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-i+1})}{(1-q^i)(1-q^{i-1})\cdots(1-q)}.$$

93. [3] Continuing the previous problem, if $n < jk/2$ then $p(j,k,n) \le p(j,k,n+1)$.

    NOTE. A (difficult) combinatorial proof is known. What is really wanted, however, is an injection $f : A_n \to A_{n+1}$, where A_m is the set of partitions counted by $p(j,k,m)$, such that for all $\lambda \in A_n$, $f(\lambda)$ is obtained from λ by adding 1 to a single part of λ. It is known that such an injection f exists, but no explicit description of f is known.

94. [1] Let $\bar{p}(k,n)$ denote the number of partitions of n into *distinct* parts, with largest part at most k. Then

$$\bar{p}(k,n) = \bar{p}(k, \binom{k+1}{2} - n).$$

    NOTE. It is easy to see that

$$\sum_{n=0}^{\binom{k+1}{2}} \bar{p}(k,n)q^n = (1+q)(1+q^2)\cdots(1+q^k).$$

95. (*) Continuing the previous problem, if $n < \frac{1}{2}\binom{k+1}{2}$ then $\bar{p}(k,n) \le \bar{p}(k,n+1)$.

    NOTE. As in Problem 93 it would be best to give an injection $g : B_n \to B_{n+1}$, where B_m is the set of partitions counted by $\bar{p}(k,m)$, such that for all $\lambda \in B_n$, $f(\lambda)$ is obtained from λ by adding 1 to a single part of λ. It is known that such an injection g exists, but no explicit description of g is known. However, unlike Problem 93, *no* explicit injection $g : B_n \to B_{n+1}$ is known.

96. [2] A *partition* π of a set S is a collection of nonempty pairwise disjoint subsets (called the *blocks* of π) of S whose union is S. Let $B(n)$ denote the number of partitions of an n-element set. $B(n)$ is called a *Bell number*. For instance, $B(3) = 5$, corresponding to the partitions (written in an obvious shorthand notation) 1-2-3, 12-3, 13-2, 1-23, 123. The number of partitions of $[n]$ for which no block contains two consecutive integers is $B(n-1)$.