Young Students Approach Integer Triangles

James Tanton
and David Batuner, Alexander Belyi, Owen Callen,
John Coyne, Adam Donovan, Rostic Gorbatov, Avi Levitan,
Sam Lichtenstein, Alan McAvinney, Benjamin Moody,
Peter Sergey Panov, David Plotkin, Ben Plotkin-Swing, Yuri Podpaly,
Noah Rosenblum, Teresa Shirkova, Sonya Shteingold, Matthew Tai.

April 11, 2002

How many different triangles of perimeter n can be made with a supply of n toothpicks? Let us assume all triangles created have an integral number of toothpicks per side and have positive area. We wish to count the number of incongruent triangles that can be so constructed. This is the problem students aged 12 - 17 grappled with over ten one-hour sessions of *The Math Circle* in the fall of 2001.

Organized as a school for the enjoyment of mathematics, *The Math Circle* offers programs of courses for young students, aged 5 through 18, who enjoy math and want more than the typical school curriculum offers. We hold classes at Harvard University and Northeastern University, and at community education centers. The school was founded by Robert Kaplan, Ellen Kaplan and Tomás Guillermo in 1994, and I have been fortunate to be involved with this program for the past two years.

We typically begin a semester by posing a mathematical problem to each of our classes and allowing the small groups of students to explore the mystery for the ten weeks that follow. The role of an instructor in *The Math Circle* is not so much to instruct as to guide and offer occasional hand-holds (or sometimes to throw spanners into the works). We want our students to experience for themselves the creative aspects of mathematics, to make mistakes, to experience the frustrations of doing math, as well as the deep joy when inspiration finally arrives. In short, we want our students to *own* the mathematics they encounter.

With all boundaries removed it is amazing just how far young students can, and do, go. This was shown by the accomplishments of my students in designing a novel solution to the toothpick triangle problem. It was a collaborative effort. The point of *The Math Circle* is not to foster competition but to allow, and encourage, students to take intellectual risks. The setting is collegial and relaxed. With free discussion of ideas and play of invention, innovation can blossom. The toothpick triangle problem, like all of our problems, has the added advantage of being immediately accessible and mathematically rich.

Trial and error shows that a supply of 13 toothpicks can produce five incongruent triangles. We denote them as triples: (6, 6,1), (6, 5, 2), (6, 4, 3), (5, 5, 3) and (5, 4, 4). There is a surprise if you add one toothpick: the count decreases. Only *four* integer triangles can be made with 14 toothpicks.

Let $T(n)$ denote the number of possible triangles for a given perimeter n (for n in $\mathbb{N}$). $T(13) = 5$ and $T(14) = 4$ shows that the function $T(n)$ is certainly not monotonic. Is there a general formula for it?

My students didn't know that this problem is well known and complete specification of the function $T(n)$ has already been established (see [1], [3] and also [2, ch. 3], [5] and [6, ch. 6]). The approaches taken in the literature make use of algebraic machinery that can only be described as hefty: generating functions, multiple-term partial fractions involving cubics, and the binomial theorem to extract coefficients (though see [3] for an alternative direct approach). I was astonished to see my students solve this problem completely using nothing more than formulas for quadratics.

This paper describes the evolution of their approach. They began classically by first discovering a connection to partition functions (and they made some new observations along the way). Thereafter, their method is surprisingly simple, elegant, and new!

1. WHAT MAKES A TRIANGLE? Physically manipulating toothpicks soon became tedious, so the students immediately looked to algebraic means to calculate values $T(n)$. They noted that three positive integers a, b and c, summing to n, form the sides of a triangle if, and only if, the three triangular inequalities hold:

$$
\begin{aligned}
a+b &> c \\
b+c &> a \\
c+a &> b.
\end{aligned}
$$

Adding c, b and a respectively to each of these inequalities yields:

$$n = a + b + c > 2a, 2b, 2c.$$

Thus a triple (a, b, c) forms a triangle of perimeter n if, and only if, each term in the triple is strictly less than $n/2$. We will assume for convenience that we always have $a \geq b \geq c$. It is now easy to compute values of $T(n)$: simply list the "3-partitions" of n (that is, list the ways n can be written as a sum of three positive integers, order immaterial) in a systematic way, and count all those 3-partitions that have terms all less than half n. For example, the 3-partitions of 12 are:

$$
\begin{gathered}
10, 1, 1 \\
9, 2, 1 \\
8, 3, 1 \\
8, 2, 2 \\
7, 4, 1 \\
7, 3, 2 \\
6, 5, 1 \\
6, 4, 2 \\
6, 3, 3 \\
\mathbf{5, 5, 2} \\
\mathbf{5, 4, 3} \\
\mathbf{4, 4, 4}
\end{gathered}
$$

of which only the last three correspond to a triangle. Thus $T(12) = 3$. The students readily computed the first 15 values of $T(n)$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$T(n)$	0	0	1	0	1	1	2	1	3	2	4	3	5	4	7

Table 1

2. THE SEARCH FOR PATTERNS. The desire to find patterns, it seems, is innate in the human psyche. Our pleasure at *The Math Circle* lies in nudging our younger students on to the next step - *why* are these patterns so? My older students, however, are eager to tackle such challenges without any urging on my part.

From inspection it seems that the value of $T(n)$ decreases at each even value of n larger than 6. The students offered the following observation by way of an explanation:

Lemma 1: *No integer triangle with even perimeter has a side of length 1.*

Proof: Suppose the triple $(a, b, 1)$ represents a triangle of perimeter $2n$. Then $a < n$ and $b < n$. But this is impossible as $a + b = 2n - 1$. $\square$

"With the option of 'one' as a side-length removed," commented my 12-14 year olds, "the number of possibilities for triangles is likely to decrease." They, and the class for juniors and seniors, later came up with the following proof based on the idea that any triangle of perimeter $2n$ can be transformed into a triangle of perimeter $2n - 1$ by removing a toothpick from a shortest side:

Lemma 2: $T(2n) < T(2n - 1)$ *for* $n > 3$.

Proof: Let (a, b, c) be the triple for a triangle of perimeter $2n$. Since $c \geq 2$, $(a, b, c - 1)$ is a triple representing a valid triangle of perimeter $2n - 1$. As no two triples of the first type map to the same triple of the second type we have $T(2n) \leq T(2n - 1)$. Moreover, any triangle of the form (a, b, b) of perimeter $2n - 1$ is not in the range of this mapping. It is easy to show that if $n > 3$, triangles of this form do exist and so $T(2n) < T(2n - 1)$.$\square$

The table suggests that $T(2n)$ is always one less than $T(2n-1)$, but this is not the case. $T(16)$, for example, equals five and is two less than $T(15)$. This provides a good argument for collecting a lot of data before formulating a theory.

One of the most striking patterns appearing in Table 1 is the placement of pairs of values. For example, the value "2" appears twice in the table, separated three places ($T(7) = 2 = T(10)$). The values "3" and "4" also appear twice, again separated three places. The same is true for the two pairs of "1's" that occur. This led students to conjecture:

$$T(2n) = T(2n - 3).$$

Moreover, some students recognized the sequence of values appearing in the even (and odd) positions of Table 1. Let $C(n)$ be the number of 3-partitions of n. Recall that 3-partitions were used for counting triangles. Fortunately, a few students, naturally inclined to collecting data, had had the good sense to keep track of the number of these along the way too:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$C(n)$	0	0	1	1	2	3	4	5	7	8	10	12	14	16	19

Table 2

It seems two copies of the terms $C(n)$ appear in Table 1 intertwined in its even and odd positions with one copy delayed three places behind the other.

Theorem 3: *For* $n \in \mathbb{N}$, $T(2n) = C(n) = T(2n-3)$

A specific example revealed the key to proving this. Consider the case $n = 8$:

$T(13)$	$C(8)$	$T(16)$
6 6 1	6 1 1	7 7 2
6 5 2	5 2 1	7 6 3
6 4 3	4 3 1	7 5 4
5 5 3	4 2 2	6 6 4
5 4 4	3 3 2	6 5 5

Reverse the entries of the triples of the first column and sum them termwise to the entries of the middle column. This produces a constant sum of 7. Similarly, reversing the entries of the third column and summing them termwise to the middle column produces a constant sum of 8. This suggests correspondences of the form:

$$(n-1-c, n-1-b, n-1-a) \leftrightarrow (a,b,c) \leftrightarrow (n-c, n-b, n-a).$$

Now it is a matter of checking that each triple constructed this way is of a valid form.

Proof of Theorem: Let $n = a + b + c$ be a 3-partition of n with $1 \leq c \leq b \leq a \leq n$. In fact we must have $a \leq n-2$.Then each quantity $n-a, n-b$ and $n-c$ is positive and less than n, and so $(n-c, n-b, n-a)$ is an integer triangle of perimeter $2n$. Since different 3-partitions of n produce different integer triangles we have $C(n) \leq T(2n)$.

Also each quantity $n-1-a, n-1-b, n-1-c$ is positive and less than $n-1$. Thus $(n-1-c, n-1-b, n-1-a)$ is an integer triangle of perimeter $2n-3$. We also have $C(n) \leq T(2n-3)$.

Conversely, suppose (a, b, c) is an integer triangle of perimeter $2n$. Then $a, b, c < n$ and $n = (n-c) + (n-b) + (n-a)$ is a partition of n with positive terms. Different integer triangles correspond to different 3-partitions of n and so $T(2n) \leq C(n)$.

If (a, b, c) is instead an integer triangle of perimeter $2n-3$, then $a, b, c < n-1$ and $n = (n-1-c) + (n-1-b) + (n-1-a)$ is a partition of n with positive terms. We have $T(2n-3) \leq C(n)$. $\square$

Thus to find a formula for $T(n)$ we should focus on the function $C(n)$.

3. PARTITION FUNCTIONS. Is there a general formula for $C(n)$? To examine this function the students decided to analyze difference terms

$$\Delta(n) = C(n+1) - C(n)$$

("How else is there to analyze a sequence?") but didn't find the outcome enlightening. "Do it again,"someone suggested noting that the double differences,

$$\Delta^2(n) = \Delta(n+1) - \Delta(n),$$

lead to something striking:

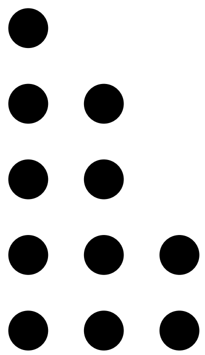
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$C(n)$	0	0	1	1	2	3	4	5	7	8	10	12	14	16	19
$\Delta(n)$	0	1	0	1	1	1	1	2	1	2	2	2	2	3	2
$\Delta^2(n)$	1	-1	1	0	0	0	1	-1	1	0	0	0	1	-1	1

Table 3

It seems the double difference values repeat in a cycle of length six, suggesting that there must be some relation between the terms $C(n)$ and $C(n+6)$ six places apart. Close examination of the table soon led to the conjecture:

$$(*) \qquad C(n+6) = C(n) + n + 3.$$

Attempts to prove this relation however proved fruitless, until one student in the junior and senior class had the inspiration to represent partitions pictorially. (I then later offered this hand-hold to students in the younger class). Represent each term in a 3-partition of a number as a column of equally spaced dots. For example, here is a picture depicting the partition $5+4+2$ of 11.



(Such diagrams are known as Ferrers graphs in the literature. See [2], for example.) Reading across the rows we obtain a multiple-term partition of the number 11 using only 1's, 2's and 3's:

$$11 = 1 + 2 + 2 + 3 + 3.$$

Every 3-partition of n yields a diagram with a row of three dots at the bottom (we insisted all terms be positive), and hence a multiple-term partition of n into 1's, 2's and 3's containing at least one 3, or, equivalently, a partition of $n-3$ into small numbers with no restrictions on the numbers of 3's present.

Let $P_{123}(n)$ count the number of ways to write n as a sum of these small digits (order irrelevant). We have

$$C(n) = P_{123}(n-3).$$

Relation (*) translates to:

$$(**) \qquad P_{123}(n+6) = P_{123}(n) + n + 6.$$

Can this form of the relation be proved?

When the going gets tough it is often helpful to take a break, and perhaps take a step backwards. Some students during this lull asked "What's special about '3' here?" and wondered about other partition functions that might be easier to handle. Almost jokingly, one student defined $P_1(n)$ to be the number of ways to write n as a sum of ones, and noted

$$P_1(n) = 1$$

for all n. The function $P_{12}(n)$, the number of ways to write n as a sum of just 1's and 2's (order irrelevant), is more interesting, and can be handled easily. Since any representation of n of this type can involve at most $\lfloor \frac{n}{2} \rfloor$ twos we have:

$$P_{12}(n) = \lfloor \frac{n}{2} \rfloor + 1.$$

(Here $\lfloor x \rfloor$ denotes the value of x rounded down to the nearest integer.)

Thinking along these lines, consider the number of threes that can be present in a representation of n as a sum of 1's, 2's and 3's. There are $P_{12}(n)$ such representations using no threes and $P_{12}(n-3)$ such representations using one three, $P_{12}(n-6)$ representations using two threes, and so on in a cascade down the number of threes used. This led students to the admittedly ungainly formula

$$\begin{aligned} P_{123}(n) &= P_{12}(n) + P_{12}(n-3) + \cdots + P_{12}(n - 3\lfloor \frac{n}{3} \rfloor) \\ &= \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} P_{12}(n - i\lfloor \frac{n}{3} \rfloor) \\ &= \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} \left(\lfloor \frac{n - i\lfloor \frac{n}{3} \rfloor}{2} \rfloor + 1 \right). \end{aligned}$$

The students in the upper class were able to use this formula to prove the relation (**) but felt their approach was far from elegant. The students in the younger class, with a little nudging from me, had the insight to stop the cascade after two steps. (My nudge was to suggest we start the cascade with the term $P_{123}(n+6)$.They noted, that after two steps, the term $P_{123}(n)$ appears.).

Lemma 4: $P_{123}(n+6) = P_{12}(n+6) + P_{12}(n+3) + P_{123}(n)$.

Proof: There are $P_{12}(n+6)$ representations of $n+6$ as a sum of 1's, 2's and 3's using no threes, $P_{12}(n+3)$ using precisely one three, and $P_{123}(n)$ using two or more threes. $\square$

This is helpful for relation (**) now translates to:

$$P_{12}(n+6) + P_{12}(n+3) = n+6$$

which the students were able to prove.

Lemma 5: *The relation above holds.*

Proof: We need to show

$$\lfloor\frac{n+6}{2}\rfloor + 1 + \lfloor\frac{n+3}{2}\rfloor + 1 = n+6$$

or equivalently

$$\lfloor\frac{n}{2}\rfloor + 3 + 1 + \lfloor\frac{n+1}{2}\rfloor + 1 + 1 = n+6.$$

That is,

$$\lfloor\frac{n}{2}\rfloor + \lfloor\frac{n+1}{2}\rfloor = n.$$

You can easily check this holds by verifying the cases n even and odd separately. $\square$

This establishes the relation (**) and consequently (*). The moment of play paid off.

4. SOLVING FOR $C(n)$. Making the cycles of six explicit, relation (*) can be written:

$$\begin{aligned} C(6n+a) &= C\left(6(n-1)+a\right) + 6(n-1) + a + 3 \\ &= C\left(6(n-1)+a\right) + 6n + a - 3 \end{aligned}$$

for $a \in \{0, 1, 2, 3, 4, 5\}$. This relation allows us to compute all values of the partition function recursively. By forming a telescoping sum, the students also obtained an explicit formula for $C(6n+a)$ - after all, every term in a sequence is the sum of the differences:

$$\begin{aligned} C(6n+a) &= \left(\sum_{i=1}^{n} C(6i+a) - C\left(6(i-1)+a\right)\right) + C(a) \\ &= \left(\sum_{i=1}^{n} 6n + a - 3\right) + C(a) \\ &= 6 \cdot \frac{1}{2} n(n+1) + an - 3n + C(a) \\ &= 3n^2 + an + C(a). \end{aligned}$$

This was the approach taken by the juniors and seniors without any hesitation. Students in the younger class however required a little coaching from me. I simply asked: *If we know the difference terms of a sequence, do we know the sequence itself?* Working with each value of a independently, this was enough to lead these students to the same six quadratic relations, which they later united via the same single formula obtained by the older class.

We would like to express this general quadratic relation in terms of its argument $6n+a$, that is, to write

$$C(6n+a) = x(6n+a)^2 + y(6n+a) + z$$

for some values x, y and z. Expanding the brackets and equating coefficients quickly yields $x = \frac{1}{12}$, $y = 0$ and $z = C(a) - \frac{1}{12}a^2$. So we have:

$$C(6n+a) = \frac{(6n+a)^2}{12} + C(a) - \frac{a^2}{12}$$

for $a \in \{0, 1, 2, 3, 4, 5\}$. Writing out these quadratics for each value of a explicitly gives:

$$\begin{aligned}
C(6n) &= \frac{(6n)^2}{12} \\
C(6n+1) &= \frac{(6n+1)^2}{12} - \frac{1}{12} \\
C(6n+2) &= \frac{(6n+2)^2}{12} - \frac{4}{12} \\
C(6n+3) &= \frac{(6n+3)^2}{12} + \frac{3}{12} \\
C(6n+4) &= \frac{(6n+4)^2}{12} - \frac{4}{12} \\
C(6n+5) &= \frac{(6n+5)^2}{12} - \frac{1}{12}.
\end{aligned}$$

The fractional quantities added or subtracted are precisely the values that round the quantity $\frac{(6n+a)^2}{12}$ to an integer! We thus have the general formula:

$$C(n) = \langle \frac{n^2}{12} \rangle$$

for $n \in \mathbb{N}$ where the angular brackets indicate rounding to the nearest integer.

5. SOLVING FOR $T(n)$. Now it is just a matter of putting everything together. Using theorem 3 we have:

For n even,

$$T(n) = C(\frac{n}{2}) = \langle \frac{n^2}{48} \rangle.$$

For n odd,

$$T(n) = T(n+3) = C(\frac{n+3}{2}) = \langle \frac{(n+3)^2}{48} \rangle.$$

These are the same formulae presented in [2] and [5], and not a generating function nor partial fraction in sight to get here.

We found it amusing to note that with 1000 toothpicks, one can produce precisely 20833 different integer triangles. The mind snaps up what the eye and hand can't begin to take in.

6. TWO INTERESTING RELATIONS. Since $P_{123}(n)$ equals $C(n+3)$ we have the identity:

$$1 + \lfloor \frac{n}{3} \rfloor + \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} \lfloor \frac{n - i \lfloor \frac{n}{3} \rfloor}{2} \rfloor = \langle \frac{(n+3)^2}{12} \rangle.$$

The junior and senior students first followed a slightly different approach in section 4, applying the methods described there directly to the function $P_{123}(n)$ and the relation (**). They obtained:

$$P_{123}(n) = \lfloor \frac{(n+3)^2 + 3}{12} \rfloor.$$

Since $C(n) = P_{123}(n-3)$ this establishes the identity:

$$\langle \frac{n^2}{12} \rangle = \lfloor \frac{n^2 + 3}{12} \rfloor.$$

For which other values a and b does $\langle \frac{n^2}{b} \rangle$ equal $\lfloor \frac{n^2+a}{b} \rfloor$ for all $n \in \mathbb{N}$?

ADDENDUM: To learn more about *The Math Circle* see our website `www.themathcircle.org`. Also see [4].

REFERENCES:

[1] G. Andrews, “A note on partitions and triangles with integer sides,” *American Mathematical Monthly*, **86** (1979), 477.

[2] R. Honsberger, *Mathematical Gems III*, Mathematical Association of America, Washington D.C., 1985.

[3] J. H. Jordan, R. Walch and R. J. Wisner, “Triangles with integer sides,” *American Mathematical Monthly*, **86** (1979), 686 - 689.

[4] R. Kaplan, “The Math Circle,” *AMS Notices*, **47** No 7, September 1995.

[5] N. Krier and B. Manvel, “Counting Integer Triangles,” *Mathematics Magazine*, **71** (1998) 291 - 295.

[6] D. Olivastro,*Ancient Puzzles*, Bantam Books, New York, 1993.