Permutations and Patterns

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Abstract

A permutation $\pi \in \mathcal{S}_n$, where $\mathcal{S}_n$ is the set of permutations of length n, is said to have an occurrence of the *pattern* 132, if it contains a subword $\pi_i\pi_j\pi_k$, with $1 \leq i < j < k \leq n$, such that $\pi_i < \pi_k < \pi_j$. If there is no such subword, π is said to be 132-*avoiding*. In the same way we define pattern avoiding permutations for every $\tau \in \mathcal{S}_k$.

In the first part of this paper we will first present some results in pattern counting problems. When dealing with pattern avoidance one will often encounter the Catalan numbers, and therefore we give in the second chapter an introduction to these. A useful technique when working with patterns is *generating trees*. We will show how one can use this concept in proving pattern avoidance theorems. We conclude the first part by first giving a brief account of *generalised patterns*, where the constraint, that two entries adjacent in the pattern must also be adjacent in the permutation, is admitted. We then give a new proof of a result given by Claesson and Mansour on the enumeration of permutations with exactly one occurrence of the generalised pattern 2-13.

In the second part we consider the following problem. With n and k fixed, we count for every permutations of length n the number of patterns of length k that have occurrences; How will this distribution look like? Here we concentrate on cases where k is just smaller than n and in particular where $k = n - 2$. Computations show that we get marked peaks in the distribution. We provide an explanation for this and in doing so we introduce the concept of *links*. By a link we mean a pair of consecutive numbers m and $(m+1)$ that are adjacent in a permutation. We derive a pair of recurrence formulae which produce these numbers, which we denote by $A_{n,k}$. We will also study the diagonals, $A_{n+m+1,n}$, with m fixed and $n \geq 1$. We will show that these diagonal elements are given by polynomials, $P_m(n)$, of degree m. We give a recurrence relation for the leading term of these polynomials. Finally we present conjectures on the values of $P_m(0)$ and $P_m(-1)$.

Acknowledgements

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Part I

Pattern Avoidance

Chapter 1

A Brief Survey on Patterns

1.1 Introduction

The pattern problems dealt with in this paper is a new but rapidly growing field of combinatorics. The birth of pattern avoidance can be ascribed to D. Knuth's ground-breaking work [13] from 1969, where it originated from certain sorting problems. After that this subject lay more or less dormant(with a few exceptions) until 1985 when R. Simion and F. Schmidt published their much celebrated paper [20]. Since then this field has virtually exploded with a couple of hundred papers being written in it. We give here some of the results obtained, mainly without proofs.

1.2 Preliminaries

Let $V = \{v_1, v_2, \ldots, v_n\}$ with $v_1 < v_2 < \cdots < v_n$ be a finite subset of $\mathbb{N}$. A *permutation* π of V is a bijection from V onto itself. We shall usually regard π as a word over V, that is if $\pi(v_i) = a_i$ we write

$$\pi = a_1 a_2 \ldots a_n.$$

The set of all permutations of V is denoted by $\mathcal{S}_V$. An often used set is $\{1, 2, \ldots, n\}$, which we will denote by $[n]$. The set of all permutations on $[n]$ is called the *symmetric group of order* n and is denoted by $\mathcal{S}_n$. We also put $\mathcal{S} = \cup_{n \geq 0} \mathcal{S}_n$.

Definition 1. By the *reduction* of a permutation π on the set V, we mean the permutation π' obtained from π when replacing v_i with i for all $i \in \{1, 2, \ldots, n\}$. We shall write $\pi' = \text{red}(\pi)$. $\diamondsuit$

Example 1. $\text{red}(62814) = 42513$ $\heartsuit$

Definition 2. An element of a permutation is called a *left-to-right minimum* of the permutation, if all entries preceding it, are bigger than that element. $\diamondsuit$

In the same manner we define left-to-right maximum, right-to-left minimum and right-to-left maximum.

Definition 3. For given $\sigma \in \mathcal{S}_n$ and $\tau \in \mathcal{S}_k$, where $k \leq n$, we say that a subword

$$\pi = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_n}$$

of σ such that $\mathrm{red}(\pi) = \tau$, is an *occurrence* of the *pattern* τ in σ. If there exists no such subword we say that σ is τ*-avoiding.* $\diamondsuit$

Example 2. The permutation $\sigma = 416235$ has two occurrences of the pattern 231, namely 462 and 463, but no occurrences of 321. Thus σ is 321-avoiding. $\heartsuit$

We denote by $\mathcal{S}_n(\tau)$ the set $\{\sigma \in [n] : \sigma \text{ avoids } \tau\}$ and more generally, with $V \subset \mathcal{S}$, we have that $\mathcal{S}_n(V) = \cap_{\tau \in V}\mathcal{S}_n(\tau)$. For the cardinality we define

$$s_n(V) = |\mathcal{S}_n(V)|.$$

Definition 4. Two patterns, τ_1 and τ_2, are said to be *Wilf equivalent* or belong to the same *Wilf class* if $s_n(\tau_1) = s_n(\tau_2)$ for all n. $\diamondsuit$

We now introduce some symmetry operations on permutations.

Definition 5. Let π be a permutation of $[n]$. We define the *reverse* of π, denoted π^r, by

$$\pi^r(i) = \pi(n+1-i), \qquad i \in [n].$$

$\diamondsuit$

Definition 6. Let π be a permutation of $[n]$. We define the *complement* of π, denoted π^c, by

$$\pi^c(i) = n+1-\pi(i), \qquad i \in [n].$$

$\diamondsuit$

We also define π^i as the usual inverse operation on $\mathcal{S}_n$. We denote the group generated by these three operations on $\mathcal{S}_n$ by $\mathcal{G}_p$. It's not hard to show that $\mathcal{G}_p$ is isomorphic to the dihedral group D_8, i.e. the symmetric operations on a square. The orbits under the action of $\mathcal{G}_p$ are called *symmetry classes.* It's plain to see that if $g \in \mathcal{G}_p$, then there are as many occurrences of τ in π as there are of $g(\tau)$ in $g(\pi)$. In particular, $s_n(\tau_1)$=$s_n(\tau_2)$ whenever τ_1 and τ_2 are in the same symmetry class. Thus all the members of a symmetry class are Wilf equivalent. The converse is not true, however. Two Wilf eqivalent patterns do not necessarily have to belong to the same symmetry class. As an illustration of this we present the following theorem.

Theorem 1.1. *For any n we have that the patterns*

$$12\cdots n-1\, n \quad and \quad 12\cdots n\, n-1$$

are Wilf eqivalent.

The most famous problem concerning pattern avoidance, namely the Stanley-Wilf conjecture from around 1990, which states that

Conjecture 1.2. *For any pattern $\tau \in \mathcal{S}_k$, the limit*

$$\lim_{n\to\infty} s_n(\tau)^{1/n}$$

exists, and is finite

was recently settled by A. Marcus and G. Tardos [15].

1.3 Patterns of Length Three

We have that $\mathcal{S}_n$ is divided into two symmetry classes, namely

$$\{123, 321\} \text{ and } \{132, 213, 231, 312\}.$$

The first ever result in the field of pattern avoidance was Knuth's enumeration of $\mathcal{S}_n(213)$:

Theorem 1.3. *For all $n \in \mathbb{N}$ we have that*

$$s_n(213) = C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{1.1}$$

where C_n is the nth Catalan number.

We will provide a proof for this result in the next chapter, where we will instead use 132 to represent this symmetry class. Hammersley [11] found the enumeration of the other symmetry class:

Theorem 1.4. *For all $n \in \mathbb{N}$ we have that*

$$s_n(321) = C_n. \tag{1.2}$$

Thus for patterns of length 3 there is only one Wilf class. This is not surprising in view of Theorem 1.1. We will give another proof for this in the chapter on generating trees.

When giving an account of pattern avoidance it's impossible to get pass the paper of Simion and Schmidt mentioned above. Here, among many things, they give the first first bijection between $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ and they also enumerate $\mathcal{S}_n(V)$ for all $V \subset \mathcal{S}_3$. Here we give their result for $|V| = 2$. It's easy to check, by using the symmetry operations of $\mathcal{G}_p$, that in this case there are six symmetry classes. It turns out that these six symmetry classes are divided into three Wilf classes as Table 1.1 shows.

V	$s_n(V)$
$\{123,132\},\{123,213\},\{231,321\},\{312,321\}$ $\{132,213\},\{231,312\}$ $\{132,231\},\{213,312\}$ $\{132,312\},\{213,231\}$	2^{n-1}
$\{132,321\},\{123,231\},\{123,312\},\{213,321\}$	$1+\binom{n}{2}$
$\{123,321\}$	0, when $n \geq 5$

Table 1.1: The cardinality of $\mathcal{S}_n(V)$, where $|V| = 2$.

1.4 Patterns of Length Four

For $\mathcal{S}_4$ there are 7 symmetry classes, but by using Theorem 1.1 and other rather intricate tricks one can show that there are only 3 Wilf classes, which we represent by 1234, 1342 and 1324. Computations show that for $n \leq 8$ we get for the numbers $s_n(\tau)$:

- $s_n(1342)$: 1, 2, 6, 23, 103, 512, 2740, 15485,
- $s_n(1234)$: 1, 2, 6, 23, 103, 513, 2761, 15767,
- $s_n(1324)$: 1, 2, 6, 23, 103, 513, 2762, 15793.

These numbers serve as an illustration for the following theorem.

Theorem 1.5. *For all $n \geq 7$ the inequality*

$$s_n(1342) < s_n(1234) < s_n(1324)$$

holds.

The first enumeration for a pattern of length 4 was given by Gessel in [10], where he used the theory of symmetric functions to show that

Theorem 1.6. *For all $n \geq 0$ we have*

$$s_n(1234) = 2 \cdot \sum_{i=0}^{n} \binom{2i}{i} \cdot \binom{n}{i}^2 \cdot \frac{3k^2 + 2k + 1 - n - 2kn}{(k+1)^2(k+2)(n-k+1)}.$$

Bóna [2] followed by proving

Theorem 1.7. *For all $n \geq 0$ we have*

$$s_n(1342) = \frac{7n^2 - 3n - 2}{2} \cdot (-1)^{n-1} + 3 \sum_{i=2}^{n} 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \binom{n-i+2}{2} (-1)^{n-1}.$$

As a consequence of these two theorems we get this rather surprising corollary

Corollary 1.8.

$$\lim_{n\to\infty} \frac{s_n(1342)}{s_n(1234)} = 0.$$

Thus the series $s_n(1342)_{n=0}^{\infty}$ is not just smaller than $s_n(1234)_{n=0}^{\infty}$ but also asymptotically smaller. The enumeration of $\mathcal{S}_n(1324)$ is still open.

1.5 Counting Patterns

We conclude this survey by saying something about the other basic problem concerning patterns, namely counting permutations with a prescribed number of occurrences of a given pattern(there is also some work done on the hybrid of those two problems, i.e. counting permutations which avoid one pattern and has a prescribed number of occurrences of another, e.g. [18, 19]). We will denote by $\mathcal{S}_n^r(\tau)$ the permutations of length n that has exactly r occurrences of the pattern t and, in analogy with the denotation for pattern avoiding permutations, we put $s_n^r(\tau) = |\mathcal{S}_n^r(\tau)|$.

Most of the papers in this field consider only patterns of length three. It's obvious that we in this case get the same two symmetry classes, $\{123, 321\}$ and $\{132, 213, 231, 312\}$, as we get with pattern avoidance problems. The first result in this area was Noonan's enumeration of $\mathcal{S}_n^r(123)$ [16]:

$$s_n^1 123 = \frac{3}{n}\binom{2n}{n-3}. \tag{1.3}$$

A new approach to this problem was suggested by Noonan and Zeilberger [17]. They gave a new proof on Noonan's result above and they also conjectured that

$$s_n^2(123) = \frac{59n^2 + 117n + 100}{2n(2n-1)(n+5)}\binom{2n}{n-4} \tag{1.4}$$

and

$$s_n^1(132) = \binom{2n-3}{n-3}. \tag{1.5}$$

Fulmek [9] proved the first conjecture, while Bóna [3] proved the second. In [14] Mansour and Vainshtein give an finite algorithm for computing the generating function

$$\Psi_r(x) = \sum_{j=0}^{\infty} s_j^r(132)x^j$$

for every $r > 0$. In their paper they give explicit results for $\Psi_r(x)$ and the corresponding numbers $s_n^r(132)$ for r up to 5(they have also computed $\Psi_6(x)$

and $s_n^6(132)$ but they consider the expressions too long to be put in the paper). We display their results for $s_n^r(132)$ for $r = 2$ and 3:

$$s_n^2(132) = \frac{n^3 + 17n^2 - 80n + 8}{2n(n-1)} \binom{2n-6}{n-2} \tag{1.6}$$

$$s_n^3(132) = (n^6 + 51n^5 - 407n^4 - 99n^3 + 7750n^2 - 22416n + 20160) \frac{(2n-9)!}{6n!(n-5)!}. \tag{1.7}$$

It should be obvious to the reader why we restrict ourselves to show only these two cases.

Here we finish our survey on pattern problems. For a more extensive read see [12] or [4, chapter 14].

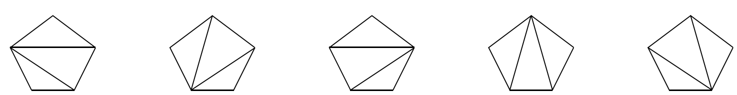
Chapter 2

The Catalan Numbers

2.1 Introduction

The Catalan numbers are omnipresent in the world of combinatorics. R. Stanley has collected 150(!) different statistics that are enumerated by them. Three examples are

I dividing convex $(n+2)$-gons into n triangles with non-crossing diagonals.



II ordering n pairs of parentheses.

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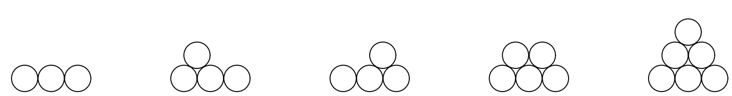
III stacking coins with a base of n consecutive coins.

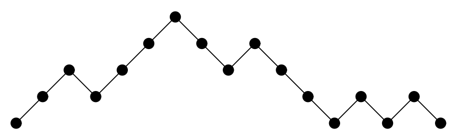


Here we will take a closer look at yet another example, the *Dyck paths.* The reason for choosing this representation of the Catalan numbers is that it is quite easy to establish a bijection between the Dyck paths and 132-avoiding permutations. We will do this and we will also derive a recursion formula obeyed by the Catalan numbers and their ordinary generating function. Finally we show that the nth Catalan number is given by $\frac{1}{n+1}\binom{2n}{n}$.

2.2 Dyck Paths

Definition 1. A Dyck path of length $2n$(or semi-length n) is a lattice path from $(0,0)$ to $(2n,0)$, with up-steps $(1,1)$ and down-steps $(1,-1)$, which never goes below the x-axis. $\diamondsuit$

We denote the set of all Dyck paths of length n with $\mathcal{D}_n$ and we let $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$. We also let $C_n = |\mathcal{D}_n|$, and as pointed out above C_n is the nth Catalan number. For convenience we represent the up-steps and down-steps by u and d respectively. For example the Dyck path



is coded by *uuduuuddudddudud*.

Every nonempty Dyck path γ can be uniquely written as $\gamma = u\gamma_1 d\gamma_2$, where γ_1 and γ_2 are Dyck paths. This decomposition is called the *first return decomposition* of γ. The following theorem shows that Dyck paths and 132-avoiding permutations have the same enumeration.

Theorem 2.1. *The Dyck paths of length* $2n$ *are in 1-to-1 correspondence with* 132*-avoiding permutations of length* n*.*

Proof. We will prove this by constructing a bijection, $\Phi : \mathcal{S}_n(132) \to \mathcal{D}_n$, recursively. First note that any $\pi \in \mathcal{S}_n(132)$ can be written as $\pi_1 n \pi_2$, where $\pi_1, \pi_2 \in \mathcal{S}_n(132)$, and the entries in π_1 are bigger than those in π_2. First we set

$$\Phi(\varepsilon) = \varepsilon,$$

where ε is the empty Dyck path/permutation, and next we set

$$\gamma = \Phi(\pi) = u\Phi(\pi_1)d\Phi(\pi_2).$$

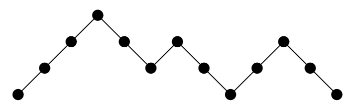
The function defined in this way clearly produces a well-defined Dyck path.

To produce the inverse, $\Phi^{-1}(\gamma)$, we label the n down-steps of γ as follows. First we write γ using the decomposition of first return, $\gamma = u\gamma_1 d\gamma_2$, where γ_1 and γ_2 are of lengths $2(i-1)$ and $2(n-i)$ respectively. The d in the decomposition above is the ith down-step and gets the label n. We proceed by recursively labelling the $(i-1)$ down-steps of γ_1 and the $(n-i)$ down-steps of γ_2 in the same manner, where γ_1 gets the labels $\{n-i+1, n-i+2, \ldots, n-1\}$ and γ_2 the labels $\{1, 2, \ldots, n-1\}$. We now read the labels from left to right to get our desired permutation. $\spadesuit$

Example 1. We use our function Φ defined above, on the 132-avoiding permutation 453612:

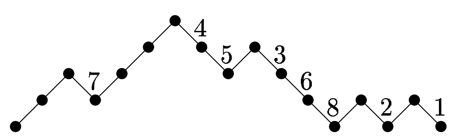
$$\Phi(453612) = u\Phi(453)d\Phi(12) = uu\Phi(4)d\Phi(3)du\Phi(1)d = uuudu dduudd.$$

Thus $\Phi(453612)$ produces the Dyck path



$\heartsuit$

Example 2. We illustrate the inverse, Φ^{-1}, by using the algorithm, described in the proof, to label the down-steps of the Dyck path displayed earlier:



Thus $\Phi^{-1}(uuduuudduddduddudud) = 74536821$. $\heartsuit$

Theorem 2.2. *The Catalan numbers satisfy the recurrence formula*

$$C_n = \sum_{i=0}^{n-1} C_i\, C_{n-1-i}. \tag{2.1}$$

Proof. Let $\mathcal{C}_{n,i}$ denote the set of Dyck paths of length $2n$ with first return at $2i$ and let $C_{n,i} = |\mathcal{C}_{n,i}|$. Any Dyck path $\sigma \in \mathcal{C}_{n,i}$ can be written as $\sigma = u\sigma_1 d\sigma_2$, where σ_1 and σ_2 are Dyck paths of length $2(i-1)$ and $2(n-i)$ respectively. Thus $C_{n,i} = C_{i-1}C_{n-i}$. Summing over i gives us

$$C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i} = \sum_{i=0}^{n-1} C_i\, C_{n-1-i}.$$

$\spadesuit$

Let $G(x)$ be the ordinary generating function for the Catalan numbers. From (2.1) we can derive the functional equation

$$G = 1 + xG^2 \tag{2.2}$$

with the solutions

$$G = \frac{1 \pm \sqrt{1-4x}}{2x}. \tag{2.3}$$

Since we know that $G(0) = 1$ we must choose the minus-sign above and thus we get

Theorem 2.3. *The ordinary generating function for the Catalan numbers is*

$$G = \frac{1 - \sqrt{1-4x}}{2x}. \tag{2.4}$$

It's interesting to notice that $G(x)$ has the simple continued fraction representation

$$G = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \ddots}}} \tag{2.5}$$

We conclude this chapter by giving the closed form for the Catalan numbers

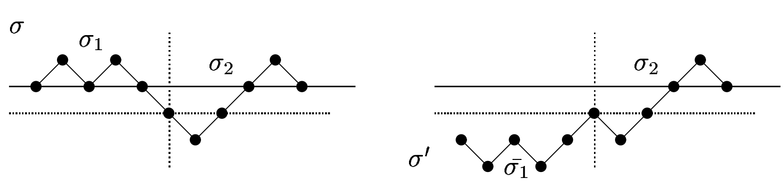
Theorem 2.4.

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{2.6}$$

Proof. An unrestricted path from $(0,0)$ to $(2n,0)$ has n up-steps and n down-steps. Thus there are $\binom{2n}{n}$ such paths. We shall call a path that dips below the x-axis, i.e. is not a Dyck-path, for a *bad path*. We denote the number of bad paths of length $2n$ by B_n. Then we clearly have the relation

$$C_n + B_n = \binom{2n}{n}. \tag{2.7}$$

Any given bad path, σ, can be written as $\sigma = \sigma_1\sigma_2$, where σ_1 ends at the point where σ goes below the x-axis for the first time. Now we create a new path $\sigma' = \bar{\sigma_1}\sigma_2$, where $\bar{\sigma_1}$ is the path we get from σ_1 when we change u to d and vice versa. Another way to put it, is to say that $\bar{\sigma_1}$ is the image of σ_1 under reflection in the line $x = -1$.



We see that σ' is a path from $(0,-2)$ to $(2n,0)$. This process can be reversed. Any path from $(0,-2)$ to $(2n,0)$ must touch the line $x = -1$ a first time. Interchanging the up-steps and down-steps of the path up to this point, produces a bad path. Thus the number of bad paths of length $2n$ is the same as the number of paths from $(0,-2)$ to $(2n,0)$. Such a path has $(n+1)$ up-steps and $(n-1)$ down-steps, so we get that

$$B_n = \binom{2n}{n+1}. \tag{2.8}$$

Now, using (2.7), it's straightforward to calculate C_n

$$\begin{aligned} C_n &= \binom{2n}{n} - \binom{2n}{n+1} \\ &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\ &= \frac{(n+1)(2n)!}{(n+1)!n!} - \frac{n(2n)!}{(n+1)!n!} \\ &= \frac{(2n)!}{(n+1)!n!} \\ &= \frac{1}{n+1}\frac{(2n)!}{n!n!} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

♠

Chapter 3

Generating Trees

3.1 Introduction

In this chapter we will give a brief presentation of *generating trees*, which is a powerful tool when dealing with pattern avoidance problems. This technique was first used in this field by J. West in his papers [23, 24]. Among his achievements are enumerations of permutations avoiding a pair of patterns, e.g. the enumerations of $\mathcal{S}_n(3142, 2413)$, $\mathcal{S}_n(123, 3241)$ and $\mathcal{S}_n(123, 3241)$. He also proves that $\mathcal{S}_n(1234)$, $\mathcal{S}_n(1243)$ and $\mathcal{S}_n(2143)$ are Wilf equivalent. Most of the results presented here are taken from his papers. Others who have made use of generating trees are Z. Stankova [21, 22, joint with J. West] and M. Bousquet-Mélou [5].

First we need to define what we mean by a generating tree.

Definition 1. A *generating tree* is a rooted, labelled tree with the property that one can derive the children and their labels from the label of a node. ♢

This mean, what is required to specify a particular generating tree, is a recursive definition consisting of

(1) the label of the root,

(2) a set of *succession rules* describing the number of children and their labelling. There is exactly one rule for every label.

We are in general interested in the total number of nodes at one label(*the level number*) and sometimes the distribution of nodes at every level.

Example 1. The complete binary tree:

$$\begin{array}{ll} \text{Root:} & (2) \\ \text{Rule:} & (2) \longrightarrow (2)(2) \end{array}$$

It's trivial that the level numbers are 2^n. ♡

Example 2. The Fibonacci tree:

Root:	(1)
Rules:	$(1) \longrightarrow (2)$
	$(2) \longrightarrow (1)(2)$

We show that this set of rules really gives the nth Fibonacci number at the nth level. We denote by f_n^1 and f_n^2 the numbers of nodes at level n with label 1 and label 2 respectively, and by f_n the total number of nodes at level n. Then for $n \geq 2$ we have $f_n^2 = f_{n-1}^1 + f_{n-1}^2 = f_{n-1}$ and $f_n^1 = f_{n-1}^2 = f_{n-2}$ and thus $f_n = f_n^1 + f_n^2 = f_{n-1} + f_{n-2}$. With the initial values $f_0 = f_1 = 1$ this is exactly the recurrence formula which defines the Fibonacci numbers. ♡

3.2 Generating Trees and Pattern Avoidance

For a given permutation $\pi \in \mathcal{S}_n$ and a given $i \in [n+1]$, let

$$\pi^i = (\pi_1, \pi_2, \dots, \pi_{i-1}, n+1, \pi_i, \pi_{i+1}, \dots, \pi_n).$$

We will call this *inserting* $(n+1)$ *into site* i. With respect to a given pattern τ, we will call a site i of $\pi \in \mathcal{S}_n(\tau)$ an *active site* if $\pi^i \in \mathcal{S}_{n+1}(\tau)$.

Proposition 3.1. *The patterns* 123 *and* 213 *are Wilf equivalent.*

Proof. We shall prove this by showing that $T(123) \cong T(213)$. For a permutation $\pi_1 \in \mathcal{S}_n(123)$ we set t to be the position of the first ascent in π_1, or, if there are no ascents, we set $t = n+1$. It's clear that the t first sites of π_1 are all active, but none of the sites following this first ascent are. Inserting $(n+1)$ into the first position give us a permutation with first ascent at $(t+1)$, while inserting $(n+1)$ into site i, $2 \leq i \leq m$, produces a permutation with first ascent at t. Thus the succession rule for $T(123)$ is

$$(t) \longrightarrow (2)(3)\cdots(t)(t+1) \tag{3.1}$$

Now, let s be the position of the first *descent* of a permutation $\pi_2 \in \mathcal{S}_n(213)$, or, if π_2 is the ascending permutation, let $s = n+1$. Once again, the first s sites are active, but none others are. Inserting $(n+1)$ into an active site i of π_2 creates a permutation $\pi_2^i \in \mathcal{S}_{n+1}(213)$ which has its first descent at $(i+1)$, and this gives us the succession rule

$$(s) \longrightarrow (2)(3)\cdots(s)(s+1) \tag{3.2}$$

for $T(213)$. Since the roots for $T(123)$ and $T(213)$ both are labelled (1) and their succession rules are the same, it is immediate that $T(123) \cong T(213)$ and this ensures that $s_n(123) = s_n(213)$ for all n. ♠

As expected, things get more complicated when the pattern we want to avoid is of length 4. For each node of the tree $T(1234)$ we now associate an ordered pair (x, y) as follows. Let x represent the first ascent(or $(n+1)$ if there is none) of the node $\pi \in \mathcal{S}_n(1234)$ and y the number of active sites in π. In this case y is the last element of the first increasing subsequence of length 3. With this notation we get the following lemma

Lemma 3.2. *In* $T(1234)$

$$(x, y) \longrightarrow (2, y+1)(3, y+1)\cdots(x+1, y+1)(x, x+1)(x, x+2)\cdots(x, y). \quad (3.3)$$

Proof. Let π be a node of $T(1234)$ having label (x, y). The active sites of π are obviously the y first sites. We create new nodes by inserting $(n+1)$ into site i, where $1 \leq i \leq y$. It's plain to see that the nodes thus produced will be labelled

$$\begin{array}{rl} (x+1, y+1) & \text{if } i = 1, \\ (i, y+1) & \text{if } 2 \leq i \leq x, \\ (x, i) & \text{if } x+1 \leq i \leq y, \end{array}$$

and our lemma is thereby proved. ♠

Next we consider the tree $T(2143)$. Also for this tree we label the nodes (x, y), with x being the position of the first *descent* and y, as before, being the number of active sites. In this case the y active sites are in general not the y first sites.

Lemma 3.3. *In* $T(2143)$

$$(x, y) \longrightarrow (2, y+1)(3, y+1)\cdots(x+1, y+1)(x(x+1)(x, x+2)\cdots(x, y). \quad (3.4)$$

Proof. Let the y active sites be numbered from left-to-right as $a_1, a_2, \ldots, a_y$. Note that the first x sites are active, since there is no decreasing pair which together with $(n+1)$ can form a 2143-pattern.

Inserting $(n+1)$ into site a_i of π splits it into two sites, both potentially active. If a site was active in π, it's easy to check that it will remain active in π^{a_i}, unless it's to the right of x and to the left of a_i. Now it's straightforward to show that for the pair (x, y) associated with a^i we get

$$\begin{array}{rl} (i+1, y+1) & \text{if } 1 \leq i \leq x, \\ (x, x+y+1-i) & \text{if } x+1 \leq i \leq y. \end{array}$$

A little bit of rearranging produces (3.4). ♠

From Lemma 3.2 and Lemma 3.3 we get the following theorem and its immediate corollary

Theorem 3.4. $T(1234) \cong T(2143)$.

Proof. $T(1234)$ and $T(2143)$ both have the root labelled (2,2), and since they have the same succession rules they are isomorphic. ♠

Corollary 3.5. *For all* $n \geq 0$ *we have that*

$$s_n(1234) = s_n(2143).$$

Chapter 4

Generalised Patterns

4.1 Introduction and Some Results

There are several ways of extending the concept of pattern avoidance. In this chapter we shall deal with one of these, introduced by Babson and Steingrímsson in [1]. Here they considered *generalised patterns*, where the requirement that adjacent letters in a pattern must also be adjacent in the permutations, is admitted. We shall adopt the convention that a dash in a generalised pattern means that the elements joined by the dash are *not* required to be adjacent in the permutation. With this notation a classical pattern has dashes in all positions.

Example 1. The generalised pattern 1-23 is avoided by the permutation 342165 but there are two occurrences of the classical pattern 1-2-3, namely 346 and 345. ♡

The enumeration of permutations avoiding generalised patterns with one dash and of length three has been investigated in depth by A.Claesson [6] and by Claesson together with T.Mansour [7]. We shall present some of their results, mostly without proofs.

4.1.1 Avoiding One Pattern

We begin with a rather trivial observation, which can readily be proved by looking at the symmetries of the permutations of length n.

Proposition 4.1. *With respect to being equidistributed, the twelve patterns of length three with one dash fall into three classes.*

(i) 1-23, 12-3, 3-21, 32-1.

(ii) 1-32, 3-12, 21-3, 23-1.

(iii) 2-13, 2-31, 13-2, 31-2.

We give the enumerations of these classes and start with this beautiful proof from Claesson [6].

Proposition 4.2. *We have that* $s_n(1\text{-}23)$ *is the same as the number of partitions of* $[n]$*, which equals* B_n*, where* B_n *is the* n*th Bell number.*

Proof. It is well-known that the number of ways of partitioning $[n]$ is counted by the Bell numbers. Now to prove the first part we introduce a standard representation of a given partition π of $[n]$ by demanding that:

(1) Each block is written with its smallest element first and the rest of the elements in decreasing order.

(2) The blocks are ordered so that their least elements form a decreasing sequence and we put a dash between the blocks.

We define $\hat{\pi}$ as the permutation we get from π by writing it in standard form and removing the dashes. We claim that $\hat{\pi} = a_1 a_2 \dots a_n$ is (1-23)-avoiding. Indeed, if $a_i < a_{i+1}$ then they are the first two elements of a block. By our construction a_i is a left-to-right minimum, which means that there exist no $j < i$ such that $a_j < a_i$.

It remains to show that we from any (1-23)-avoiding permutation, $\hat{\pi}$, can uniquely determine a partition in standard form. This is easily done by inserting a dash before every left-to-right minimum of $\hat{\pi}$, except the first one. ♠

We have the same enumeration for $\mathcal{S}_n(1\text{-}32)$:

Proposition 4.3. $s_n(1\text{-}32) = s_n(1\text{-}23) = B_n$.

The third class, however, is, like the classical patterns of length three, counted by the Catalan numbers. We will make use of this result in the next section.

Proposition 4.4. $s_n(2\text{-}13) = s_n(2\text{-}1\text{-}3) = C_n$.

Proof. The second part we have already proved and hence we concentrate on the first one. It's obvious that $\mathcal{S}_n(2\text{-}1\text{-}3) \subseteq \mathcal{S}_n(2\text{-}13)$. Hence it's enough to show that every permutation π having a (2-1-3)-occurrence also has a (2-13)-occurrence.

Let z be the leftmost entry of π such that there exist x and y so that xyz form a (2-1-3)-pattern. Let y' be the entry just preceding z. Then we must have $y' < x$, otherwise xyy' would form a (2-1-3)-pattern, which is a contradiction. Now $xy'z$ form a (2-13)-pattern and we're done. ♠

Thus there are two Wilf-classes in the case of generalised patterns of length three with one dash.

4.1.2 Counting Permutations With Pattern Occurrences

In [8] Claesson and Mansour study the enumeration of permutations with exactly r occurrences of a generalised pattern of length 3 with one dash. We give some of their main results. We start with a couple of recurrence relations for the numbers $s_n^1(1\text{-}23)$ and $s_n^1(1\text{-}32)$.

Theorem 4.5. *Let $u_1(n) = s_n^1(1\text{-}23)$ and let B_n be the nth Bell number. The numbers $u_1(n)$ satisfy the recurrence*

$$u_1(n+2) = 2u_1(n+1) + \sum_{k=0}^{n-1} \binom{n}{k} \left[u_1(k+1) + B_{k+1}\right],$$

when $n \geq -1$, with the initial condition $u_1(0) = 0$.

Theorem 4.6. *Let $v_1(n) = s_n^1(1\text{-}32)$. The numbers $v_1(n)$ satisfy the recurrence*

$$v_1(n+1) = v_1(n) + \sum_{k=1}^{n-1} \left[\binom{n}{k} v_1(k) + \binom{n-1}{k-1} B_k\right],$$

when $n \geq 0$, with the initial condition $v_1(0) = 0$.

With the aid of a certain continued fraction they manage to give closed formulae for $s_n^r(2\text{-}13)$ when $r = 1, 2, 3$.

Proposition 4.7. *The number of permutations of length n with exactly one occurrence of the pattern 2-13 is*

$$s_n^1(2\text{-}13) = \binom{2n}{n-3}.$$

In the next section we will provide a new combinatorial proof of this proposition.

Proposition 4.8. *The number of permutations of length n with exactly two occurrences of the pattern 2-13 is*

$$s_n^2(2\text{-}13) = \frac{n(n-3)}{2(n+4)} \binom{2n}{n-3}.$$

Proposition 4.9. *The number of permutations of length n with exactly three occurrences of the pattern 2-13 is*

$$s_n^3(2\text{-}13) = \frac{1}{3} \binom{n+2}{2} \binom{2n}{n-5}.$$

4.2 Enumerating $\mathcal{S}_n^1(2-13)$

As mentioned above we will give a new proof of the following

Theorem 4.10. *For the numbers of n-permutations with exactly one occurrence of the generalised pattern 2-13 we have the simple formula*

$$s_n^1(2\text{-}13) = \binom{2n}{n-3}. \tag{4.1}$$

In proving this we will very much follow the same path as M.Bóna [4, chapter 14:exer. 8-10] when he proved that $s_n^1(1\text{-}3\text{-}2)$ equals the number of ways of partitioning a convex $(n+1)$-gon into triangles and one quadrilateral which is $\binom{2n-3}{n-3}$. The difference is that here we will partition a convex $(n+4)$-gon into triangles and one *heptagon*. We start by enumerating our partitions.

Theorem 4.11. *The number of ways of dividing a convex $(n+4)$-gon into triangles and one heptagon with non-crossing diagonals is* $\binom{2n}{n-3}$.

Proof. We denote the number of partitions by d_n and label the vertices of our $(n+4)$-gon $1, 2, \ldots, (n+4)$.

First we compute the partitions where 1 is a vertex of the heptagon. We denote these numbers by d_n^1. Let $\{a, b, c, d, e, f\}$ be the six other vertices of the heptagon such that $1 < a < b < \cdots < f \leq (n+4)$. With these vertices fixed the rest of the $(n+4)$-gon is divided into 7 convex polygons all of which are to be triangulated. The number of sides for these polygons are: a, $(b-a+1)$, $(c-b+1)$,..., $(n+4-f)$(here we allow for two-sided polygons). It's well-known that there are C_{n-2} triangulations of a convex n-gon(See the first example on page 12), and thus we are lead to the following formula for d_n^1:

$$d_n^1 = \sum_{a=2}^{n-1} \sum_{b=a+1}^{n} \sum_{c=b+1}^{n+1} \sum_{d=c+1}^{n+2} \sum_{e=d+1}^{n+3} \sum_{f=e+1}^{n+4} C(a, b, c, d, e, f),$$

where

$$C(a, b, c, d, e, f) = C_{a-2} \cdot C_{b-a-1} \cdot C_{c-b-1} \cdot C_{d-c-1} \cdot C_{e-d-1} \cdot C_{f-e-1} \cdot C_{n+4-f}.$$

This may look like a formidable sum to handle, but it's actually not very hard to evaluate it. We deal with one sum at a time using the recurrence relation (2.1) on page 14.

$$\begin{aligned}
\sum_{f=e+1}^{n+4} C_{f-e-1} \cdot C_{n+4-f} &= C_0 \cdot C_{n+3-e} + \cdots + C_{n+3-e} \cdot C_0 \\
&= C_{n+4-e} \\
\sum_{e=d+1}^{n+3} C_{e-d-1} \cdot C_{n+4-e} &= C_0 \cdot C_{n+3-d} + \cdots + C_{n+2-d} \cdot C_1 \\
&= C_{n+4-d} - C_0 \cdot C_{n+3-d} \\
&= C_{n+4-d} - C_{n+3-d}
\end{aligned}$$

The rest of the sums are evaluated in the same way. We save the reader from some tedious calculations and display the final result

$$d_n^1 = C_{n+3} - 5C_{n+2} + 6C_{n+1} - C_n. \tag{4.2}$$

We rewrite this using the closed form of the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$.

$$\begin{aligned} d_n^1 &= \frac{1}{n+4}\binom{2n+6}{n+3} - \frac{5}{n+3}\binom{2n+4}{2n+2} + \frac{6}{n+2}\binom{2n+2}{n+1} - \frac{1}{n+1}\binom{2n}{n} \\ &= \frac{(2n)!}{n!(n+4)!}(7n^3 - 21n^2 + 14n) \\ &= \frac{7}{n+4}\binom{2n}{n-3}. \end{aligned} \tag{4.3}$$

Since the vertices of the $(n+4)$-gon are indistinguishable we would get the same number of partitions if we fixed any other vertex than 1. Consider the sum $\sum_{i=1}^{n+4} d_n^i = (n+4)d_n^1$. Here every partition will be counted once for every vertex of the heptagon, and hence we get that

$$\begin{aligned} d_n &= \frac{n+4}{7} \cdot d_n^1 \\ &= \frac{n+4}{7} \cdot \frac{7}{n+4}\binom{2n}{n-3} \\ &= \binom{2n}{n-3}, \end{aligned}$$

and this completes our proof. ♠

We continue by deriving a recurrence relation satisfied by d_n.

Proposition 4.12.

$$d_n = 2\left(\sum_{i=1}^{n} d_{i-1}C_{n-i}\right) + C_{n+2} - 4C_{n+1} + 3C_n. \tag{4.4}$$

Proof. Let P be a convex $(n+4)$-gon. When partitioning P into triangles and one heptagon we regard two separate cases

I First we consider the case when there is no diagonal going into vertex 1. Now there are two possibilities. If $(n+4, 2)$ is a diagonal then our problem is reduced by one in size and we get d_{n-1} partitions.

If $(n+4, 2)$ is not a diagonal then $\{n+4, 1, 2\}$ are vertices of the heptagon. Let $\{a, b, c, d\}$ denote the four remaining vertices. Then a can be positioned from 3 to n, b from $(a+1)$ to $(n+1)$, c from $(b+1)$ to $(n+2)$ and d from $(c+1)$ to $(n+3)$. The rest of P is divided by the heptagon into five convex polygons. The number of sides for these polygons are $(a-1)$, $(b-a+1)$, $(c-b+1)$, $(d-c+1)$ and $(n+5-d)$. To complete the partitioning all of these polygons must be triangulated and this can be done in

$$C_{a-1}C_{b-a+1}C_{c-b+1}C_{d-c+1}C_{n+3-d}$$

different ways. Summing over a, b, c and d gives us

$$\sum_{a=3}^{n}\sum_{b=a+1}^{n+1}\sum_{c=b+1}^{n+2}\sum_{d=c+1}^{n+3} C_{a-1}C_{b-a+1}C_{c-b+1}C_{d-c+1}C_{n+3-d} = C_{n+1} - 3C_n + C_{n-1} \qquad (4.5)$$

partitions. Thus altogether in this first case we get

$$d_{n-1} + C_{n+1} - 3C_n + C_{n-1} \qquad (4.6)$$

different partitions.

II Now we look at the case where there is a diagonal going into 1. Let i be the lowest labelled vertex such that $(1, i)$ is a diagonal. Then the heptagon is either in the part $12\ldots i$ or in the part $i(i+1)\ldots(n+4)1$. We handle the latter case first. Here the part with the heptagon is a polygon with $(n+6-i)$ sides, and hence it can be partitioned in d_{n+2-i} ways. Next we have to triangulate $12\ldots i$ so that no diagonal goes into 1. Now $(2, i)$ must be a diagonal, thus this is reduced to the problem of triangulating $23\ldots i$. This can be done in C_{i-3} ways. Summing over i yields(to get a heptagon we must have $i \leq n-1$)

$$\sum_{i=3}^{n-1} d_{n+2-i}C_{i-3}. \qquad (4.7)$$

In the first case we must first partition $12\ldots i$ into triangles and one heptagon so that no diagonal goes to 1. But this is exactly what we computed in case **I** and the number of ways of doing this is

$$d_{i-5} + C_{i-3} - 3C_{i-4} + C_{i-5}.$$

Secondly we need to triangulate $i(i+1)\ldots(n+4)1$. This can be done in C_{n+4-i} different ways. Summing over i we get

$$\sum_{i=7}^{n+3}(d_{i-5} + C_{i-3} - 3C_{i-4} + C_{i-5})C_{n+4-i} = C_{n+2} - 5C_{n+1} + 6C_n - C_{n-1} + \sum_{i=7}^{n+3} d_{i-5}C_{n+4-i}. \qquad (4.8)$$

Notice that the sums in (4.7) and (4.8) almost have the same summands. We find that in this second case we get, after some rewriting, the total number of partitions to be

$$C_{n+2} - 5C_{n+1} + 6C_n - C_{n-1} + 2\left(\sum_{i=3}^{n} d_{i-3}C_{n-i}\right) - d_{n-1}. \qquad (4.9)$$

Finally, adding case **I** and **II** we get the recurrence formula

$$d_n = C_{n+2} - 4C_{n+1} + 3C_{n+2} + 2\sum_{i=3}^{n} d_{i-1}C_{n-i}.$$

This is clearly equivalent to (4.4) since $d_n = 0$ when $n < 3$. ♠

We now turn our attention to permutations in order to fulfil the final step of our proof of Theorem 4.10. In the next proposition we show that the numbers $s_n^1(2\text{-}13)$ satisfy the same recurrence relation as the partition numbers d_n. For convenience we denote $s_n^1(2\text{-}13)$ by b_n.

Proposition 4.13. *The permutations with exactly one occurrence of the pattern* 2-13 *satisfy the recurrence*

$$b_n = 2(\sum_{i=1}^{n} b_{i-1}C_{n-i}) + C_{n+2} - 4C_{n+1} + 3C_n \tag{4.10}$$

Proof. When determining b_n we consider three possibilities depending on the placement of the (2-13)-occurrence relative to the entry 1.

I With our (2-13) subsequence on the left-hand side of 1 we must have all elements on the left of 1 to be bigger than the entries on the right. For if we had elements x and y, $x < y$, to the left and right of 1 respectively, $x1y$ would form a (2-1-3)-pattern and from Lemma 4.4 we know that we would then get an extra (2-13)-occurrence. Thus, with 1 in position i, the $(i-1)$ entries $(n-i+2,\ n-i+3, \ldots,\ n)$ preceding 1 can be ordered in b_{i-1} different ways and the $(n-i)$ entries $(1, 2, \ldots, n-i+1)$ succeeding 1 must be (2-13)-avoiding and this can be done in C_{n-i} ways. Summing over i give a total from this first case of

$$\sum_{i=1}^{n} b_{i-1}C_{n-i} \tag{4.11}$$

different permutations.

II If instead the occurrence of 2-13 is to the right of 1 we must still, by the same argument as in **I**, have the left entries to be bigger than the right entries. With 1 in position i we now must order the $(i-1)$ entries preceding 1 to be (2-13)-avoiding, which can be done in C_{i-1} ways, and the $(n-i)$ entries succeeding 1 in b_{n-i} ways. Summing yields

$$\sum_{i=1}^{n} b_{n-i}C_{i-1} \tag{4.12}$$

permutations from this case. We notice that the summands in (4.11) and (4.12) are the same, thus adding the contributions from **I** and **II** give a total of

$$2\sum_{i=1}^{n} b_{i-1}C_{n-i} \tag{4.13}$$

permutations.

III Now we focus on the most intricate and interesting case, namely when the entry 1 is placed between 2 and 3 in the (2-13)-occurrence. With yxz being the (2-13)-subword we choose to handle this possibility in two separate cases:

(i) First we consider the case where $x = 1$. The entries $\notin \{1, y, z\}$ are divided into three intervals

(1). $2, 3, \ldots, y-1$,
(2). $y+1, y+2, \ldots, z-1$,
(3). $z+1, z+2, \ldots, n$.

These entries must be ordered so we don't get any more occurrences of the pattern 2-13. We see that the elements in (1) and (2) must be placed after z. Otherwise, with $w \in (1) \cup (2)$, $w1z$ would form a (2-13)-pattern. Moreover we must have (2) preceding (1).

The entries in (3) have a larger degree of freedom. They can be placed in three blocks: before y, between y and 1 and between z and (2). We denote these blocks (3a), (3b) and (3c). Here we must have the elements in (3a) bigger than those in (3b), which must be bigger than those in (3c). Hence with the sizes of these blocks fixed the elements belonging to each of them are determined. If (3a) contains i elements, (3b) contains j elements, then there are $(n-z-i-j)$ elements left to be placed in (3c).

To summarise we get the following schematic structure of a permutation from this case

$$(3a)\ y\ (3b)\ 1z\ (3c)\ (2)\ (1)$$

Thus we get five blocks, all of which must be (2-13)-avoiding. Since y ranges from 2 to $(n-1)$ and z from $(y+1)$ to n we get

$$\sum_{y=2}^{n-1}\sum_{z=y+1}^{n}\sum_{i=0}^{n-z}\sum_{j=0}^{n-z-i} C_{y-2}C_{z-y-1}C_iC_jC_{n-z-i-j} \tag{4.14}$$

different permutations in this case.

(ii) When $x \neq 1$ things get a little bit more complicated. Now we get four different intervals for the entries $\notin \{x, y, z\}$:

(1) $2, 3, \ldots, x-1$,
(2) $x+1, x+2, \ldots, y-1$,
(3) $y+1, y+2, \ldots, z-1$,
(4) $z+1, z+2, \ldots, n$.

The intervals (2)-(4) correspond to (1)-(3) of case (i) and are handled in the same way.

For the elements of (1) however, we need to make a deeper investigation. To avoid occurrences of 2-13 these elements can be placed either between 1 and x or at the end of the permutation. Moreover, the elements between 1 and x must be increasing. In addition the $(x-2)$ entries of (1) must of course be (2-13)-avoiding. Thus for any permutation $\pi \in \mathcal{S}_{x-2}$, we can make a cut before the first *descent* of π and place the entries preceding the cut between 1 and x, and the remaining entries at the back of our permutation.

Let t be the position of this first descent(or let $t = x - 1$ if there isn't any). Then there are t ways of making the cut. Thus all in all, with $s_k(2\text{-}1\text{-}3, t)$ denoting the number of permutations of length k with first descent t, we can arrange the elements of (1) in

$$\sum_{t=1}^{x-1} t \cdot s_{x-2}(2\text{-}1\text{-}3, t) \tag{4.15}$$

different ways. Now, recall the generating tree for $\mathcal{S}_n(2\text{-}1\text{-}3)$ on page 18. Here a node labelled (t), where (t) denotes the first descent of the corresponding permutation, has t children. Thus summing the labels at one level i yields the next level number, i.e. C_{i+1}. But this sum is precisely (4.15) with $i = x - 2$, and we therefore get that (4.15) equals C_{x-1}.

From the discussion above we can conclude that a permutation satisfying the conditions of this case has the following structure

$$(4a)\ y\ (4b)\ 1\ (1a)\ xz\ (4c)\ (3)\ (2)\ (1b)$$

where x ranges from 2 to $(n-2)$, y from $(x+1)$ to $(n-1)$ and z from $(y+1)$ to n. Summing up we get

$$\sum_{x=2}^{n-2} \sum_{y=x+1}^{n-1} \sum_{z=y+1}^{n} \sum_{i=0}^{n-z} \sum_{j=0}^{n-z-i} C_{x-1}C_{y-2}C_{z-y-1}C_iC_jC_{n-z-i-j} \tag{4.16}$$

permutations from this case.

If we let x start from 1 in (4.16) we see that the extra term we get is exactly (4.14) and thus the total contribution from this third case is

$$\sum_{x=1}^{n-2} \sum_{y=x+1}^{n-1} \sum_{z=y+1}^{n} \sum_{i=0}^{n-z} \sum_{j=0}^{n-z-i} C_{x-1}C_{y-2}C_{z-y-1}C_iC_jC_{n-z-i-j} = C_{n+2} - 4C_{n+1} + 3C_n \tag{4.17}$$

Adding these three cases gives us (4.10) ♠

Since we obviously have $b_n = d_n = 0$ when $n < 3$ and from Proposition 4.12 and Proposition 4.13 we know that these numbers satisfy the same recurrence equation, we must have $b_n = d_n$ for all n. Now Theorem 4.11 immediately give us Theorem 4.10.

Part II

Pattern Count Distribution

Chapter 5

A Distributional Surprise

5.1 Introduction and Preliminaries

The starting point of this second part is the following problem. For fixed n and k we count, for each of the permutations of length n, the number of different patterns of length k it contains. Now we ask, what will this distribution look like? This question is not very interesting when n is much larger than k, since in this case almost all permutations will contain all or almost all patterns of length k. Thus we will concentrate on the situation where k is roughly of the same size as n. In particular we will target the case when $k = n - 2$. For this study we'll need some machinery and hence the following definitions. We define

$$\tau_k(\pi) = |\{\tau \in \mathcal{S}_k : \pi \text{ contains } \tau\}|,$$

that is, $\tau_k(\pi)$ is the number of patterns of length k that occurs in π. We say that π contains $\tau_k(\pi)$ patterns(of length k) or has a *pattern count* of $\tau_k(\pi)$. Usually k is implied and we will simply write $\tau(\pi)$. Further we define the numbers

$$R_i^{(n,k)} = |\{\pi \in \mathcal{S}_n : \tau_k(\pi) = i\}|.$$

When k is slightly smaller than n it's quite conceivable that only a few permutations contain few patterns and only a few permutations contain many patterns, and one might assume that the distribution in between is, or is close to, unimodular. But this guess turns out to be far from the truth as seen in Table 5.1 and the corresponding diagrams in Figures 5.1-5.3.

The obvious question is: why do we get these peaks in our distributions? We will answer this question in part, but we do not expect to get an exact formula for $R_i^{(n,k)}$, since this a much to complicated problem.

5.2 Factors that reduce the number of patterns

From now on in this chapter we assume, if nothing else is stated, that $k = n - 2$, so we use 'patterns' as short for 'patterns of length $n - 2$'. The number of

$R_i^{(8,6)}$		$R_i^{(9,7)}$				$R_i^{(9,6)}$					
i		i		i		i		i		i	
1	2	1	2	29	8936	1	2	29	10176	57	4636
2	8	2	8	30	4580	2	8	30	13076	58	2470
3	46	3	52	31	11364	3	32	31	12396	59	1804
4	62	4	72	32	14984	4	56	32	9668	60	2546
5	212	5	266	33	9896	5	120	33	10068	61	2648
6	296	6	430	34	3932	6	160	34	13804	62	2078
7	580	7	780	35	984	7	298	35	12978	63	1268
8	970	8	1588	36	158	8	398	36	9570	64	998
9	1002	9	2168			9	690	37	8418	65	1048
10	1916	10	2796			10	732	38	8418	66	848
11	2512	11	5716			11	956	39	11764	67	484
12	2196	12	8098			12	1476	40	12468	68	272
13	2090	13	6396			13	2098	41	10232	69	252
14	3942	14	8324			14	1924	42	6898	70	104
15	4296	15	16672			15	3182	43	8548	71	118
16	2438	16	20900			16	3020	44	10780	72	50
17	1678	17	15432			17	3592	45	11106	73	24
18	3216	18	9832			18	5126	46	6772	74	2
19	4564	19	18824			19	5086	47	6220	75	4
20	2370	20	34500			20	5132	48	7536	76	4
21	696	21	32230			21	7286	49	6416		
22	624	22	17230			22	7330	50	4592		
23	1414	23	7108			23	7240	51	4452		
24	1784	24	12972			24	8236	52	5356		
25	1064	25	30388			25	9794	53	4296		
26	304	26	35260			26	9124	54	2472		
27	48	27	20000			27	10176	55	2907		
28	20	28	6520			28	11192	56	4852		

Table 5.1: $R_i^{(n,k)}$ for some values on n and k.

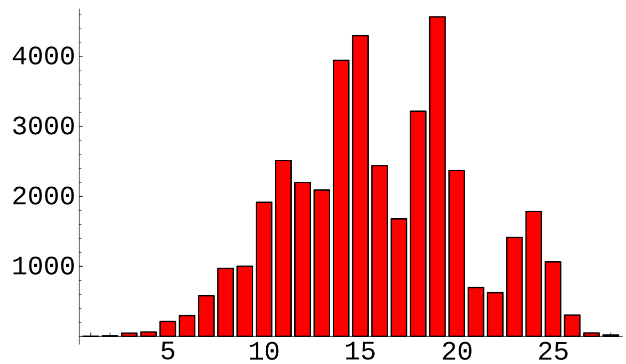


Figure 5.1: $R_i^{(8,6)}$

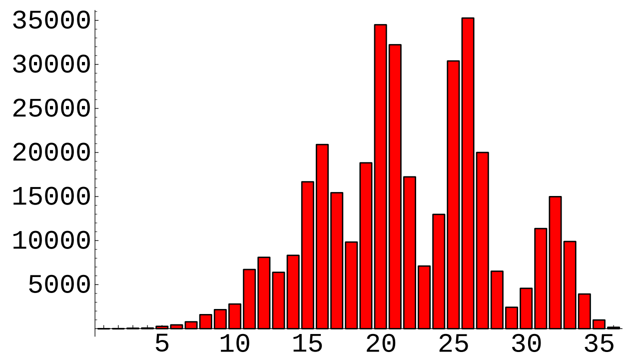


Figure 5.2: $R_i^{(9,7)}$

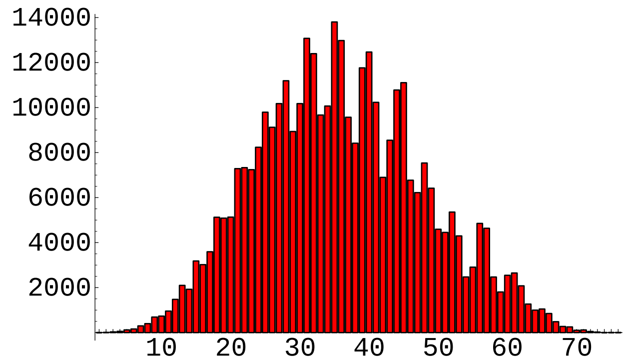


Figure 5.3: $R_i^{(9,6)}$

subsequences of length $(n-2)$ for a permutation $\pi \in S_k$ is $\binom{n}{n-2} = \binom{n}{2}$. The number of patterns is of course $(n-2)!$, and thus the theoretical maximum number of patterns π can contain is $\min(\binom{n}{2}, (n-2)!)$. Here we will assume that $n \geq 6$ so that this maximum number is $\binom{n}{2} = \frac{n(n-1)}{2}$. A permutation contains this maximum number of patterns if and only if no pattern is represented more than once. If a permutation has k pattern doublets we say that *the max is reduced by k*.

5.2.1 The Main Factor

Definition 1. Given a permutation $\pi \in \mathcal{S}_n$ and a set $s \subset [n]$ we define the function $\pi_\chi(s)$ to be the pattern we get when we from π exclude the elements in s. With $s = \{a_1, a_2, ..., a_i\}$ we will write $\pi_\chi(a_1, a_2, ..., a_i)$ instead of $\pi_\chi(\{a_1, a_2, ..., a_i\})$. $\diamondsuit$

Example 1. With $\pi = 2561374$ and $s = \{1, 3, 6\}$ we have that $\pi_\chi(s) = \pi_\chi(1, 3, 6) = \mathrm{red}(2574) = 1342$. $\heartsuit$

Assume that a permutation π can be written as $\pi = (\pi_1, i, i-1, \pi_2)$ and that $m \notin \{i, i-1\}$ is an element of π. Then $\pi_\chi(i, m) = \pi_\chi(i-1, m)$. The reason for this is that i will play the same role in $\pi_\chi(i, m)$ as $(i-1)$ does in $\pi_\chi(i-1, m)$. Since m can be chosen in $(n-2)$ ways, the max is reduced by $(n-2)$. From this we can conclude that a crucial factor for how many patterns a permutation π contains is the numbers of consecutive numbers that are adjacent in π and hence the following definition.

Definition 2. If the numbers i and $(i+1)$ are adjacent in a permutation π, we say that they *form a link in π*. $\diamondsuit$

We will use the denotation $(i) = (i+1)$ (or $(i+1) = (i)$), for saying that i and $(i+1)$ are linked. We define the function $\alpha(\pi)$ to be the number of links in π and we also define

$$A_{n,k} = |\{\pi \in S_n : \alpha(\pi) = k\}|.$$

We will take a closer look at the numbers $A_{n,k}$ in the next chapter.

What if π has two links? Those links can be two pairs, $\{i, i+1\}$ and $\{j, j+1\}$, with $i+1 < j$, or one chain of consecutive numbers, $\{i, i+1, i+2\}$. In the first case, the inclusion-exclusion principle tells us that we get $2(n-2)-1 = 2n-5$ doublets of patterns. In the latter case $\pi_\chi(m, i) = \pi_\chi(m, i+1) = \pi_\chi(m, i+2)$ for any $m \notin \{i, i+1, i+2\}$ and $\pi_\chi(i, i+1) = \pi_\chi(i, i+2) = \pi_\chi(i+1, i+2)$. Since m can be chosen in $(n-3)$ ways we get $n-3+1 = n-2$ triplets of patterns and hence the max is reduced by $(2n-4)$. We can reason in the same way if we have more than two links.

5.2.2 Other Factors

There are other factors apart from the number of links that reduce the number of patterns in a permutation. Suppose the element i is adjacent to $(i+2)$ in π. Than $\pi_\chi(i, i+1) = \pi_\chi(i+2, i+1)$. Now assume the elements j and $(j+1)$ are separated by only one element, say k, in π. In this case $\pi_\chi(j,k) = \pi_\chi(j+1,k)$. We shall call these two types of subsequences, which reduce the max by one, for *bonds*. For the first kind of bond we write $(i) - (i+2)$ and for the second kind we write $(i) ^\frown (i+1)$.

Example 2. The permutation $\pi = 513462$ has one link$(3 = 4)$ and two bonds $(1-3, 4-6)$ and thus we may think that $\tau(\pi) = 15 - 4 - 2 = 9$. Counting the different number of patterns π contains shows that this is really the case. ♡

Unfortunately things are not so easy that we, by just counting the links and bonds of a permutation, can determine how many patterns a given permutation contains, as the following examples show.

Example 3. The permutation $\pi = 143652$ has two links$(4 = 3, 6 = 5)$ and no bonds, and from the discussion above we might expect that $\tau(\pi) = 15 - 7 = 8$, but the true number is 7. The reason for this 'disappearing' pattern is that $\pi_\chi(4,3) = \pi_\chi(6,5) = 1432$, which is easily understood. ♡

There are more subtle ways in which we get pattern doublets that can't be explained from bonds. For instance in the permutation 416235 we have that $\pi_\chi(4,1) = \pi_\chi(6,5) = 4123$, which maybe is a little bit unexpected.

Example 4. Now let $\pi = 524361$. Here we have that π has one link$(4 = 3)$ and three bonds$(5 ^\frown 4, 2-4, 2 ^\frown 3)$ and hence the expected value for $\tau(\pi)$ would be $15 - 4 - 3 = 8$. This time, however, the true value is 9 and thus bigger than the expected. The cause for this lies in the segment 243, which contains one link and two bonds. When checking which doublets we get, we see that one pair ($\pi_\chi(2,4)$ and $\pi_\chi(2,3)$) is counted twice. This double counting explains why $\tau(\pi)$ is 9 and not 8. ♡

Thus we see that there are tertiary factors that might adjust the expected numbers of patterns up and down. It's probable, but not a certainty, that with n large these tertiary effects would diminish.

5.3 Explaining the Distributions

With n large we take an arbitrary permutation $\pi \in \mathcal{S}_n$. The probability for an entry i of π to be adjacent with $(i+1)$ is roughly $2/n$, and thus we would get an average of about $n \cdot 2/n = 2$ links for π. In any case, we would expect that the bulk of the permutations has less than, say 5 links. We will see in the next chapter that this is the case. Likewise we can argue that the average number of bonds for π is about $4(2+2)$. Also, if a permutation has many links it's less likely it has many bonds and vice versa.

With this in mind, we can see why our distributions for $R_i^{(8,6)}$ and $R_i^{(9,7)}$ look like they do. Going from right to left we simply explain the peaks, as the mean of the number of patterns, for permutations with $0, 1, 2$ etc. number of links. We take $R_i^{(9,7)}$ as an illustration. Here the maximum number of patterns is 36. One link would reduce the patterns by 7 to 29. Thus the permutations containing between 30 and 36 patterns are all void of links. We would guess that the average number of bonds for these permutations is slightly more than 4, and this is the case as readily seen in Figure 5.2. The same reasoning applies to the other peaks.

We end this chapter by saying something about the case where $k = n - 3$. Now one link reduces the number of patterns by $\binom{n-2}{2} = \frac{(n-2)(n-3)}{2}$, and a bond reduces the numbers by $(n-3)$. There are several tertiary effects that reduce the numbers by one, e.g. segments like

$$\{\ldots, i, i-3, \ldots\} \qquad \text{and} \qquad \{\ldots, i, m, i+2, \ldots\}.$$

Thus here we expect to get peaks for permutations with the same number of links but different number of bonds and hence more, but less marked, peaks. Figure 5.3 confirms this for $R_i^{(9,6)}$.

Chapter 6

Getting Linked

6.1 Introduction

In this chapter we will take a look at the numbers $A_{n,k}$ defined in the previous chapter. Computations give us the following table for $A_{n,k}$.

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	2	14	90	646	5242	47622	479306
1		2	4	10	40	239	1580	12434	110329	1090270
2			2	10	48	256	1670	12846	112820	1108612
3				2	16	120	888	7198	64968	650644
4					2	22	226	2198	22120	236968
5						2	28	366	4448	54304
6							2	34	540	7900
7								2	40	748
8									2	46
9										2

Table 6.1: $A_{n,k}$

We will in the first section derive a system of recurrences which produces $A_{n,k}$. By using these recurrences we get numerical evidence to conjecture that $A_{n,k_1}/A_{n,k_2}$ has a rational limit as n tends to infinity.

6.2 Deriving our Recurrences

For our recurrences we will need the following auxiliary numbers

Definition 1. We define $b_{n,k}$ as the numbers of permutations of length n with k links where n is linked. $\diamondsuit$

Obviously $b_{n,k}$=0 if $k < 1$ or $k \geq n$.

Theorem 6.1. *The numbers $A_{n,k}$ and $b_{n,k}$ satisfy the following recurrence equation system*

$$\begin{aligned} A_{n,k} &= 2A_{n-1,k-1} - b_{n-1,k-1} + (n-k-2)A_{n-1,k} + 2b_{n-1,k} + \\ &\quad (k+1)A_{n-1,k+1} - b_{n-1,k+1} \qquad (6.1) \\ b_{n,k} &= 2A_{n-1,k-1} - b_{n-1,k-1} + b_{n-1,k}, \qquad (6.2) \end{aligned}$$

with the additional conditions

$$\begin{cases} A_{n,k} = 0 & \textit{when } k < 0 \textit{ or } k \geq n \\ b_{n,k} = 0 & \textit{when } k < 1 \textit{ or } k \geq n \\ A_{1,0} = 1 & . \end{cases} \qquad (6.3)$$

Proof. The conditions (6.3) are pretty obvious. One could argue that $A_{0,0}$ should be equal to 1, but for convenience we choose to define $A_{0,0}$ to be 0.

We first concentrate on (6.1). Let π' represent an arbitrary permutation in S_{n-1}. By inserting n into π' we distinguish between three different cases.

I If π' has $(k-1)$ links we can increase the number of links by one, by linking n to $(n-1)$. If $(n-1)$ is not linked in π' there are two possible positions for n while if $(n-1)$ is linked there is one. Thus here we get $2A_{n-1,k-1} - b_{n-1,k-1}$.

II If $\alpha(\pi') = k$ we consider two cases. First, if $(n-1)$ isn't linked we must place n so that none of the k links is broken and we must also avoid to place n adjacent to $(n-1)$. There are $(n-k-2)$ positions that achieves this, so this case contributes with $(n-k-2)(A_{n-1,k} - b_{n-1,k})$ to $A_{n,k}$. Secondly, when $(n-i)$ is linked it is possible to place n between $(n-1)$ and $(n-2)$ since this breaks up a link while creating a new one. This give us $k-1+1 = k$ forbidden positions for n so we get the contribution $(n-k)b_{n-1,k}$ to $A_{n,k}$. Adding these two cases we get $(n-k-2)A_{n-1,k} + 2b_{n-1,k}$.

III When $\alpha(\pi') = k+1$ we can reduce the links by one by breaking up one of the links. There are k possible positions for n if $n-1$ is linked and $(k+1)$ positions if $(n-1)$ isn't. Thus we get the contribution $(k+1)A_{n-1,k+1} - b_{n-1,k+1}$.

Adding these three cases give us (6.1). To get (6.2) we just need to count the $A_{n,k}$ in which n is linked. In the first case above n is always linked. In the second case n is only linked when n is placed between $(n-2)$ and $(n-1)$, which give us a contribution of $b_{n-1,k}$. And finally, in case **III** n is never linked. ♠

It would of course be desirable to have a recursion formula for $A_{n,k}$ without involving $b_{n,k}$, but this looks pretty difficult to accomplish, so we will have to settle for this system of equations for now.

6.3 Some Computations and a Conjecture

From Table 6.1 it seems likely that, with a fixed $n \geq 5$, $A_{n,2}$ is the largest of the numbers $A_{n,k}$, closely followed by $A_{n,1}$. In fact it seems like the quotient $A_{n,2}/A_{n,1}$ is approaching 1 when n is growing. If this really is the case, is it also true that $A_{n,2}/A_{n,k}$ have nice limits for some other values on k? To investigate this we use Theorem 6.1 to compute $A_{n,k}$ for $k \leq 4$ up to $n = 500$.

$n \setminus k$	0	1	2	3	4	$\times$
100	1.2680	2.5516	2.5518	1.6840	0.8248	$\times 10^{157}$
200	1.0673	2.1454	2.1454	1.4231	0.7044	$\times 10^{373}$
300	4.1419	8.3118	8.3119	5.5227	2.7428	$\times 10^{613}$
400	0.8666	1.7376	1.7376	1.1555	0.5748	$\times 10^{867}$
500	1.6513	3.3092	3.3092	2.2017	1.9064	$\times 10^{1133}$

Table 6.2: $A_{n,k}$ for $k \leq 4$

From this table we get for the quotients $A_{n,2}/A_{n,k}$ the following table

$n \setminus k$	0	1	3	4
100	2.0208	1.0001	1.5154	3.0941
200	2.0102	1.0000	1.5076	3.0460
300	2.0068	1.0000	1.5050	3.0304
400	2.0051	1.0000	1.5038	3.0227
500	2.0040	1.0000	1.5030	3.0182

Table 6.3: $A_{n,2}/A_{n,k}$

It seems very likely that the limits for these quotients, as n tends to infinity, are 2, 3, $\frac{3}{2}$ and 3. If this is the case then it also seems probable that we would get a rational limit for any quotient $A_{n,2}/A_{n,k}$ as n tends to infinity. But, if so, the following conjecture holds

Conjecture 6.2. *For all $k_1, k_2 \in \mathbb{N}$ the limit*

$$r(k_1, k_2) = \lim_{n \to \infty} \frac{A_{n,k_1}}{A_{n,k_2}}$$

exists and is rational.

Chapter 7

A Diagonal View

7.1 Introduction

In this chapter we will take a look at the diagonals of Table 6.1 with $k \geq 1$. The first diagonal is just the boring sequence $2, 2, 2, \ldots$, which is easily explained from the fact that there are only the two monotonic permutations of length n which have $(n-1)$ links. The second diagonal is slightly more interesting. It starts $4, 10, 16, 20, \ldots$, and the obvious guess is that this sequence is given by $(6k-2)$. We will give a combinatorial proof that this is the case. We then continue, by using 6.1, to show that the third diagonal is given by $17k^2 - 13k + 6$. From this we might suspect that the mth diagonal is given by a polynomial of degree $(m-1)$. We will prove that this suspicion actually holds and we will further derive a recursion for the coefficient of the highest term of these polynomials.

7.2 The Proofs

For convenience we introduce the notations $D_k^m = A_{k+m+1,k}$ and $c_k^m = b_{k+m+1,k}$. In words D_k^m is the number of permutations of length $m+k+1$ with k linked pairs and m unlinked. Holding m constant we get the elements in the $(m+1)$th diagonal.

Proposition 7.1. *The numbers of the second diagonal, D_k^1, are given by* $6k-2$.

Proof. D_k^1 counts the number of permutations of length $(k+2)$ where all but one pair is linked. Let σ_i denote the sequence $1\,2 \ldots i$ and τ_i the sequence $i+1 \ldots k+2$. Then for every $i \in 2, 3, \ldots, k$ the 6 permutations $\sigma_i(\tau_i)^r$, $(\sigma_i)^r\tau_i$, $(\sigma_i)^r(\tau_i)^r$, $\tau_i\sigma_i$, $\tau_i(\sigma_i)^r$, and $(\tau_i)^r\sigma_i$ will all have k links. With $i = 1$ there are only two permutations with k links, namely $1\; k+2\; k+1 \ldots 2$ and $2\;3 \ldots\; k+2\;1$. Likewise, there are two such permutations when $i = k+1$: $k+1\; k \ldots 1\; k+2$ and $k+2\;1\;2 \ldots k+1$. Thus all in all there are $6(k-1) + 4 = 6k-2$ permutations of length $k+2$ with k links. ♠

We notice that of the permutations which contribute to D_k^1, $(k+2)$ is linked in all, except in the last two ones. This immediately give us the following corollary

Corollary 7.2. $c_k^1 = 6k - 4$.

Proposition 7.3. *The third diagonal numbers are given by* $17k^2 - 13k + 6$.

Proof. Setting $n = k + 3$ in 6.1 and 6.2, and changing notation give us

$$D_k^2 = 2D_{k-1}^2 - c_{k-1}^2 + D_k^1 + 2c_k^1 + (k+1)D_{k+1}^0 - c_{k+1}^0 \quad (7.1)$$
$$c_k^2 = 2D_{k-1}^2 - c_{k-1}^2 + c_k^1 \quad (7.2)$$

Using 7.1 and 7.2 plus the fact that $D_k^0 = c_k^0 = 2$ we get

$$D_k^2 = 2D_{k-1}^2 - b_{k-1}^2 + 20k - 10, \quad (7.3)$$
$$c_k^2 = 2D_{k-1}^2 - b_{k-1}^2 + 6k - 4. \quad (7.4)$$

By replacing k with $(k+1)$ in 7.3 and using that $D_k^2 - b_k = 14k - 6$ we get the recurrence equation

$$D_{k+1}^2 = D_k^2 + 34k + 4. \quad (7.5)$$

Combined with the starting value $D_1^2 = 10$ which we get from Table 6.1, this equation is easy to solve and the solution is the claimed $D_k^2 = 17k^2 - 13k + 6$. We also note *en passant* that

$$c_k^2 = 17k^2 - 27k + 12. \quad (7.6)$$

♠

The proof of the following lemma is not hard and not very interesting and therefore we choose to omit it.

Lemma 7.4. *For the sums of powers the formula*

$$\sum_{k=1}^{m} k^n = Q_{n+1}(m) \quad (7.7)$$

holds, where $Q_n(x)$ *is a polynomial of degree* n *with leading coefficient* $\frac{1}{n+1}$.

Theorem 7.5. *The numbers* D_k^m *and* c_k^m*, with* m *fixed, are given by polynomials of degree* m*. The leading coefficient,* g_m*, of these polynomials are the same and satisfy the recurrence relation* $mg_m = (2m+1)g_{m-1} + 2g_{m-2}$.

Proof. We shall prove this by induction. From Proposition 7.1, Corollary 7.2 and Proposition 7.3 we know that our statement holds for $m = 1$ and $m = 2$. Assuming that it's true for $(m-1)$ and $(m-2)$, i.e.

$$D_k^{m-1} = P_{m-1}(k) \quad (7.8)$$
$$D_k^{m-2} = P_{m-2}(k) \quad (7.9)$$
$$c_k^{m-1} = Q_{m-1}(k) \quad (7.10)$$
$$c_k^{m-2} = Q_{m-2}(k), \quad (7.11)$$

where $P_p(k)$ and $Q_p(k)$ are polynomials of degree p with leading term g_p, we need to show that it's also true for m. Following the lines of the proof of Proposition 7.3 with $n = (k + m + 1)$ in 6.1 we get

$$D_k^m = 2D_{k-1}^m - c_{k-1}^m + (m-2)D_k^{m-1} + 2c_k^{m-1} + (k+1)D_{k+1}^{m-2} - c_{k+1}^{m-2} \tag{7.12}$$

$$c_k^m = 2D_{k-1}^m - c_{k-1}^m + c_k^{m-1} \tag{7.13}$$

Taking the difference of these equations and setting $j = k - 1$ gives us

$$D_{j-1}^m - c_{j-1}^m = (m-2)D_{j-1}^{m-1} + c_{j-1}^{m-1} + jD_j^{m-2} - c_j^{m-2} \tag{7.14}$$

this inserted into 7.12, with k replaced by j, yields

$$\begin{aligned} D_j^m - D_{j-1}^m &= (m-1)D_{j-1}^{m-1} + c_{j-1}^{m-1} + jD_j^{m-2} - c_j^{m-2} + (m-1)D_k^{m-1} \\ &\quad +2c_j^{m-1} + (j+1)D_{j+1}^{m-2} - c_{j+1}^{m-2} \end{aligned} \tag{7.15}$$

Now using our assumptions 7.8-7.11 we get that

$$D_j^m - D_{j-1}^m = ((2m+1)g_{m-1} + 2g_{m-2})j^{m-1} + \text{TLD}, \tag{7.16}$$

where TLD is short for 'terms of lower degree'. Finally, summing over j and using Lemma 7.7 brings us home

$$\begin{aligned} D_k^m &= D_1^m + \sum_{j=2}^{k} \left[(2m+1)g_{m-1} + 2g_{m-2})j^{m-1} + \text{TLD}\right] \\ &= ((2m+1)g_{m-1} + 2g_{m-2})k^m/m + \text{TLD}. \end{aligned} \tag{7.17}$$

♠

We shall call $P_m(k)$ *diagonal polynomials.*

7.3 Diagonal Polynomials Conjectures

With the aid of *Mathematica* it's quite easy to compute the polynomials $P_m(k)$, and we list the first 6 below.

$$\begin{aligned} P_0(k) &= 2 \\ P_1(k) &= -2 + 6k \\ P_2(k) &= 6 - 13k + 17k^2 \\ P_3(k) &= -22 + \frac{217}{3}k - 54k^2 + \frac{131}{3}k^3 \\ P_4(k) &= 134 - \frac{1811}{6}k + \frac{1897}{4}k^2 - \frac{1099}{6}k^3 + \frac{427}{4}k^4 \\ P_5(k) &= -630 + \frac{25321}{10}k - \frac{28133}{12}k^2 + \frac{27907}{12}k^3 - \frac{6667}{12}k^4 + \frac{15139}{60}k^5 \end{aligned}$$

Our first guess concerning these polynomial is

Conjecture 7.6. *The coefficients of the diagonal polynomial have alternating signs, i.e.*

$$P_m(k) = \sum_{j=0}^{m} (-1)^{m+j} a_{m,j} k^j,$$

where $a_{m,j} > 0$, for all m and j.

These polynomials are obviously not valid for $k = 0$, that is $P_m(k) = A_{m+k+1,k}$ if $k \geq 1$ but not when $k = 0$. We might still suspect that there could be a connection between $P_m(0)$ and $A_{m+1,0}$ though. A listing of these values together with their differences further nurtures our suspicions:

m	$P_m(0)$	$A_{m+1,0}$	$P_m(0) - A_{m+1,0}$
0	2	1	1
1	−2	0	−2
2	6	0	6
3	−22	2	−24
4	134	14	120
5	−630	90	−720

Thus we arrive at the following conjecture

Conjecture 7.7. *For the values of $P_m(0)$ and $A_{m+1,0}$ we have the relation*

$$P_m(0) - A_{m+1,0} = (-1)^m (m+1)! \tag{7.18}$$

There might(should?) exist a simple combinatorial explanation for this relation but the author of this paper isn't clever enough to see it.

It may also be interesting to look at the values of the diagonal polynomials at $k = -1$, since if Conjecture 7.6 holds we have that

$$P_m(-1) = (-1)^m \sum_{j=0}^{m} a_m.$$

Computations give rise to our third and final conjecture concerning these polynomials.

$$\begin{aligned}
P_0(-1) &= 2 = 2 \cdot 1 \cdot 1! \\
-P_1(-1) &= 8 = 2 \cdot 2 \cdot 2! \\
P_2(-1) &= 36 = 2 \cdot 3 \cdot 3! \\
-P_3(-1) &= 192 = 2 \cdot 4 \cdot 4! \\
P_4(-1) &= 1200 = 2 \cdot 5 \cdot 5! \\
-P_5(-1) &= 8640 = 2 \cdot 6 \cdot 6!
\end{aligned}$$

Conjecture 7.8. *For the values of the diagonal polynomials taken at -1 we have that*

$$(-1)^m P_m(-1) = 2 \cdot (m+1) \cdot (m+1)!.$$

Once again, there may be an combinatorial explanation for this simple expression, but the proof is left for the reader as an exercise.

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