Partition Identities

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Introduction

A *partition* of a positive integer n (or a partition of weight n) is a non-decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of non-negative integers λ_i such that $\sum_{i=1}^{k} \lambda_i = n$. The λ_i's are the *parts* of the partition λ. Integer partitions are of particular interest in combinatorics, partly because many profound questions concerning integer partitions, solved and unsolved, are easily stated, but not easily proved. Even the most basic question "How many partitions are there of weight n?" has no simple solution. Remarkably, however, there are a variety of partition identities of the form "The number of partitions of n satisfying condition A is equal to the number of partitions of n satisfying condition B," even though no simple formulas are known for the number of partitions of n satisfying A or B. The motivating example of such a partition identity is due to Euler: The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts. We will start here and our ultimate goal will be to examine a very recent development in the theory of partition identities: the so-called "Lecture Hall Partitions" of Bousquet-Mélou and Eriksson. Along the way, we will briefly visit some earlier results such as the Rogers-Ramanujan identities which have also extended the study of partition identities.

1 Preliminaries

Definition 1.1. *A* partition *of a positive integer n (or a partition of weight n) is a non-decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of non-negative integers λ_i such that $\sum_{i=1}^{k} \lambda_i = n$. The weight, $|\lambda|$, of the partition λ is defined to be $\sum_{i=1}^{k} \lambda_i$.*

For example, there are 5 partitions of 4: $(1,1,1,1), (1,1,2), (2,2), (1,3), (4)$.

The study of partitions identities begins with the following result due to Euler. The proof of this result will demonstrate the power of using generating functions to prove partitions identities. The use of generating function permeates the study of partition identities and we will again encounter similar uses of generating functions when we examine more recent development in partition identities.

Theorem 1.2 (Euler). *The number of partitions of n into distinct parts (i.e. no part is repeated more than once) is equal to the number of partitions of n into odd parts.*

Proof. Let $p_{\mathcal{D}}(n)$ be the number of partitions of n into distinct parts. For instance, $p_{\mathcal{D}}(5) = 3$, since $(1,4), (2,3)$ and (5) are the only partitions of 5 into distinct parts.

Similarly, let $p_{\mathcal{O}}(n)$ be the number of partitions of n into odd parts. For instance, $p_{\mathcal{O}}(5) = 3$, since $(1,1,1,1,1), (1,1,3)$ and (5) are the only partitions of 5 into odd parts.

We will show that $p_{\mathcal{D}}(n) = p_{\mathcal{O}}(n)$ by showing that their generating functions, $P_{\mathcal{D}}(x)$ and $P_{\mathcal{O}}(x)$, are equal:

$$P_{\mathcal{D}}(x) = p_{\mathcal{D}}(0) + p_{\mathcal{D}}(1)x + p_{\mathcal{D}}(2)x^2 + \cdots = p_{\mathcal{O}}(0) + p_{\mathcal{O}}(1)x + p_{\mathcal{O}}(2)x^2 + \cdots = P_{\mathcal{O}}(x)$$

Consider the product $F(x) = \prod_{i=1}^{\infty}(1 + x^i)$. When this product is expanded, every term has the form $x^{i_1+i_2+\cdots+i_\ell}$ where the i_j's are distinct. Furthermore, for any set of distinct i_j's, the term $x^{i_1+i_2+\cdots+i_\ell}$ occurs only once. Therefore, the number of occurrences of $x^n = x^{i_1+i_2+\cdots+i_\ell}$ is exactly the number of ways to partition n into distinct parts. So the coefficient of x^n in $F(x)$ is $p_{\mathcal{D}}(n)$. Therefore $P_{\mathcal{D}}(x) = \prod_{i=1}^{\infty}(1 + x^i)$.

Similarly, we consider the product $G(x) = \prod_{i \text{ odd}}^{\infty}(1 + x^i + x^{2i} + x^{3i} + \cdots)$. If we expand this product, each term has the form $x^n = x^{k_1 i_1 + k_2 i_2 + \cdots + k_\ell i_\ell}$. This term represents a natural partition of n into odd parts; in particular, if we construct a partition with k_1 parts equal to i_1, and k_2 parts equal to i_2 and so forth, we have a partition of n into odd parts (since all the i_j's are odd). Conversely, if we take any partition of n into odd parts, we can write that partition (uniquely) as $k_1 i_1 + k_2 i_2 + \cdots + k_\ell i_\ell$, where the i_j's are the (odd) parts that occur and the k_j's are the number of times each i_j occurs. Therefore, the coefficient of x^n in $G(x)$ is equal to the number of partitions of n into odd parts, or $p_{\mathcal{O}}(n)$. It follows that $G(x) = P_{\mathcal{O}}(x)$.

To prove that $P_{\mathcal{D}}(x) = P_{\mathcal{O}}(x)$ we must show that

$$\prod_{i=1}^{\infty}(1 + x^i) = \prod_{i \text{ odd}}^{\infty}(1 + x^i + x^{2i} + x^{3i} + \cdots)$$

This can be achieved by manipulating $P_{\mathcal{O}}(x)$ as follows:

$$P_{\mathcal{O}}(x) = \prod_{i \text{ odd}}^{\infty}(1 + x^i + x^{2i} + x^{3i} + \cdots) = \prod_{i \text{ odd}}^{\infty}\frac{1}{1 - x^i} = \prod_{i=1}^{\infty}\frac{1}{1 - x^{2i-1}}$$

and then

$$\begin{aligned} P_{\mathcal{D}}(x) = \prod_{i=1}^{\infty}(1 + x^i) &= \prod_{i=1}^{\infty}\frac{1 - x^{2i}}{1 - x^i} = \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} \\ &= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = \prod_{i=1}^{\infty}\frac{1}{1 - x^{2i-1}} = P_{\mathcal{O}}(x) \end{aligned}$$

Thus $P_{\mathcal{D}}(x) = P_{\mathcal{O}}(x)$ and so $p_{\mathcal{D}}(n) = p_{\mathcal{O}}(n)$ for all n.

□

2 Some Generalizations of Euler's Theorem

Now it is natural to ask whether there exist other similar partition identities, and indeed there are many. The *Rogers-Ramanujan identites*, stated here without proof, are two other examples of this type of partition identity:

Proposition 2.1 (The First Rogers-Ramanujan Identity). *The number of partitions of n such that the difference between any two parts is at least 2 is equal to the number of partitions of n such that each part is congruent to 1 or 4 modulo 5.*

Example 2.2. *There are 4 partitions of 9 such that each part differs by at least 2: $(1,3,5),(1,8)$, $(2,7),(3,6)$. There are also exactly 4 partitions of 9 such that each part is congruent to 1 or 4 modulo 5: $(1,1,1,1,1,1,1,1,1),(1,1,1,1,1,4),(1,1,1,6),(9)$.*

Proposition 2.3 (The Second Rogers-Ramanujan Identity). *The number of partitions of n such that 1 is not a part and the difference between any two parts is at least 2 is equal to the number of partitions of n such that each part is congruent to 2 or 3 modulo 5.*

Example 2.4. *There are 2 partitions of 9 such that each part differs by at least 2 and where 1 is not a part: $(2,7),(3,6)$. There are also exactly 2 partitions of 9 such that each part is congruent to 2 or 3 modulo 5: $(2,2,2,3),(2,7)$.*

It turns out that the Rogers-Ramanujan identities can be generalized even further, as is demonstrated by the following result due to Gordon from which the Rogers-Ramanujan identities follow as corollaries. Again, the proof is omitted.

Theorem 2.5. *The number of partitions $(\lambda_1,\lambda_2,\ldots,\lambda_r)$ of n such that no more than $i-1$ of the parts are 1 and where $\lambda_j - \lambda_{j-k+1} \geq 2$ is equal to the number of partitions of n into parts not congruent to 0 or $\pm i$ modulo $2k+1$.*

Example 2.6. *Letting $i=2$ and $k=2$, we obtain the first Rogers-Ramanujan identity. Letting $i=1$ and $k=2$, we obtain the second Rogers-Ramanujan identity.*

Example 2.7. *Letting $i=1$ and $k=3$, we find that the number of partitions of n such that $\lambda_j - \lambda_{j-2} \geq 2$ and where 1 is not a part is equal to the number of partitions of n into parts not congruent to 0 or ± 1 modulo 7.*

The proof of Theorem 2.5 is similar to that of Theorem 1.2 in that two generating functions are shown to be equal in order to show that these partitions are equinumerous for all n. However, the generating functions for the proof of Theorem 2.5 are considerably more complicated than those of Theorem 1.2 and the manipulations are far from trivial. The proof and necessary background material are developed in [1].

3 Lecture Hall Partitions

Thus far, there has been a common theme among the generalizations of Theorem 1.2: all these identities have been concerned with the *difference* between parts in partitions. For example, the distinctness condition in Theorem 1.2 is equivalent to requiring that all parts differ by at least 1, whereas one of the conditions of the Rogers-Ramanujan identities is that all parts differ by at least 2. However, one could just as easily interpret the distinctness condition in Theorem 1.2 to require that the quotient of successive parts be greater than 1. This is exactly the approach that Bousquet-Mélou and Eriksson have taken, and they have developed the theory of "Lecture Hall Partitions" as a very different generalization of Theorem 1.2.

Definition 3.1. *A* lecture hall partition of length n *is a partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *satisfying*

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_n}{n}$$

Such partitions are called *lecture hall partition* because they correspond to all the possible ways in which an n-row lecture hall can be constructed such that every seat has a clear view of the lecturer (assuming that all rows are at integer heights). If a person in row j is to have an unobstructed view of the lecturer (at height 0), then $h(j)/j$ must be greater than or equal to $h(i)/i$ for all $i < j$ where $h(i)$ denotes the height of the i-th row. However, we will see that the applications of lecture hall partitions are far more interesting than simply enumerating possible lecture hall designs.

The first result is the so-called Lecture Hall Theorem, first presented by Bousquet-Mélou and Eriksson in [2].

Theorem 3.2 (Lecture Hall Theorem). *Let* $\mathcal{L}_n$ *be the set of lecture hall partitions of length* n*. Then*

$$\sum_{\lambda \in \mathcal{L}_n} q^{|\lambda|} = \prod_{i=0}^{n-1} \frac{1}{1 - q^{2i+1}}$$

The left-hand side of this expression is simply the weight generating function of lecture hall partitions. That is, the coefficient of q^N is the number of lecture hall partitions of length n and weight N. The right-hand side should look familiar from Theorem 1.2; for the same reason that $\prod_{i=0}^{\infty} \frac{1}{1-q^{2i+1}}$ was the weight generating function for partitions with odd parts, $\prod_{i=0}^{n-1} \frac{1}{1-q^{2i+1}}$ is the weight generating function for partitions with small odd parts: $1, 3, 5, \ldots, 2n-1$. Therefore, Theorem 3.2 can be interpreted as saying: "The number of lecture hall partitions of length n and weight N is equal to the number of partitions of N into small odd parts: $1, 3, 5, \ldots, 2n-1$."

Example 3.3. *There are* 4 *lecture hall partitions of length* 3 *and weight* 7*:*

$$(0, 0, 7), (0, 1, 6), (0, 2, 5), (1, 2, 4)$$

There are also exactly 4 *partitions of* 7 *into odd parts no greater than* $5 = 2 \cdot 3 - 1$*:*

$$(1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 3), (1, 1, 5), (1, 3, 3)$$

In fact, Bousquet-Mélou and Eriksson prove the following more general result.

Theorem 3.4 (Refined Lecture Hall Theorem). *Given a lecture hall partition* λ*, define its even weight,* $|\lambda|_e$ *and its odd weight,* $|\lambda|_o$*, by*

$$|\lambda|_e = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \lambda_{n-2k} \quad |\lambda|_o = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \lambda_{n-2k-1}$$

Then

$$\sum_{\lambda \in \mathcal{L}_n} x^{|\lambda|_e} y^{|\lambda|_o} = \prod_{i=0}^{n-1} \frac{1}{1 - x^{i+1} y^i}$$

The usefulness of this bivariate generating function identity will become clearer when we examine the theory of a-lecture hall partitions. For the moment, we will simply note that the substitution of $q = x = y$ yields Theorem 3.2.

Now, with this background in the basic ideas of lecture hall partitions, we will study the most general and most powerful aspect of Bousquet-Mélou and Eriksson's work in [3]: a-lecture hall partitions.

4 a-Lecture Hall Partitions

Thus far, we have looked at lecture hall partitions as defined in Definition 3.1. However, since we are not very interested in using lecture hall partitions to enumerate lecture hall designs, the choice of the values $1, 2, 3, \ldots$ as the denominators in the definition seems somewhat arbitrary. To generalize this definition of lecture hall partitions, we introduce the a-lecture hall partitions:

Definition 4.1. *Let a be a non-decreasing sequence of positive integers, $a = (a_1, a_2, \ldots, a_n)$. An* a-lecture hall partition of length n *is a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying*

$$0 \leq \frac{\lambda_1}{a_1} \leq \frac{\lambda_2}{a_2} \leq \cdots \leq \frac{\lambda_n}{a_n}$$

So the lecture hall partitions of Definition 3.1 are simply the $(1, 2, 3, \ldots, n)$-lecture hall partitions according to this new definition. We will denote the weight generating function of a-lecture hall partitions by $S_a(q)$, so for example, $S_{(1,2,3,\ldots,n)}(q) = \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}$, by Theorem 3.2.

The main objective of this generalization will be to find other sequences $a = (a_1, a_2, \ldots, a_n)$, such that $S_a(q)$ has the form

$$S_a(q) = \frac{1}{(1-q^{e_1})(1-q^{e_2})\cdots(1-q^{e_n})} \qquad e_i \neq e_j \tag{1}$$

for if $S_a(q)$ is of this form, then we may conclude that the number of a-lecture hall partitions of length n and weight N equals the number of partitions of N into elements of $\{e_1, e_2, \ldots, e_n\}$.

We will now study the major aspects of Bousquet-Mélou and Eriksson's results concerning a-lecture hall partitions. Since our main objective is to understand the novel aspects of Bousquet-Mélou and Eriksson's approach, the proofs of several results will be omitted in order to avoid obfuscating the overall structure of their work. Any omitted details are available in [3].

Ultimately we wish to understand the weight generating function, $S_a(q)$, of a-lecture hall partitions. For this pursuit to be fruitful, however, it is better to consider the bivariate case, as in Theorem 3.4. In particular, we will study

$$S_a(x, y) = \sum_{\lambda \in \mathcal{L}} x^{|\lambda|_e} y^{|\lambda|_o}$$

Definition 4.2. *Given a sequence $a = (a_1, a_2, \ldots, a_n)$, the standard a-lecture hall partitions, $\lambda^{(i)}$, are defined by:*

$$\lambda^{(i)} = (0, \ldots, 0, a_i, a_{i+1}, \ldots, a_n)$$

Definition 4.3. *An a-lecture hall partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is said to be* reduced *if, for all i, $\lambda - \lambda^{(i)}$ (as elements of $\mathbb{Z}^n$) is not an a-lecture-hall partition. We will denote the (finite) set of reduced a-lecture hall partitions by $\mathcal{R}$.*

It can be shown that any a-lecture hall partition of length n can be uniquely decomposed as $\lambda = \mu + \sum_{i=1}^{n} k_i \lambda^{(i)}$, where the μ are *reduced* lecture hall partitions, where the $\lambda^{(i)}$ are *standard* lecture hall partitions and where the k_i are positive integers. The proof of this fact is routine and available in either [2] or [3].

Furthermore, each $\mu + \sum_{i=1}^{n} k_i \lambda^{(i)}$ will constitute an a-lecture hall partition, since a sum of a-lecture hall partitions is clearly also an a-lecture hall partition. Therefore, there is a natural bijection between a-lecture hall partitions of length n and $(n+1)$-tuples, $(\mu, k_1, k_2, \ldots, k_n)$ where μ is a reduced partition and where the k_i's are non-negative integers.

Since we will be working in the bivariate case, let us first note that $|\lambda|_o = |\mu|_o + \sum_{i=1}^n k_i|\lambda^{(i)}|_o$ and similarly $|\lambda|_e = |\mu|_e + \sum_{i=1}^n k_i|\lambda^{(i)}|_e$, whenever $\lambda = \mu + \sum_{i=1}^n k_i\lambda^{(i)}$. Now we are ready to consider the following sum and product (where $\mathcal{R}$ denotes the set of reduced a-lecture hall partitions):

$$\sum_{\mu\in\mathcal{R}} x^{|\mu|_e} y^{|\mu|_o} \tag{2}$$

$$\prod_{i=1}^n \frac{1}{1-x^{|\lambda^{(i)}|_e} y^{|\lambda^{(i)}|_o}} = \prod_{i=1}^n (1 + x^{|\lambda^{(i)}|_e} y^{|\lambda^{(i)}|_o} + x^{2|\lambda^{(i)}|_e} y^{2|\lambda^{(i)}|_o} + x^{3|\lambda^{(i)}|_e} y^{3|\lambda^{(i)}|_o} + \cdots) \tag{3}$$

When the product in (3) is expanded, for every linear combination $\sum_{i=1}^n k_i\lambda(i)$ there will be a term of the form $x^{\sum_{i=1}^n k_i|\lambda^{(i)}|_o} \cdot y^{\sum_{i=1}^n k_i|\lambda^{(i)}|_e}$, and furthermore, each such term will arise exactly once. So if we expand the product of equations (2) and (3), every term will be constructed by choosing a term $x^{|\mu|_e} y^{|\mu|_o}$ from (2) and by choosing a term $x^{\sum_{i=1}^n k_i|\lambda^{(i)}|_o} \cdot y^{\sum_{i=1}^n k_i|\lambda^{(i)}|_e}$ from (3). Therefore, by the uniqueness of the decomposition $\lambda = \mu + \sum_{i=1}^n k_i\lambda^{(i)}$, every a-lecture hall partition λ is accounted for exactly once in the product of equations (2) and (3). So we may write:

$$S_a(x,y) = \left(\sum_{\mu\in\mathcal{R}} x^{|\mu|_e} y^{|\mu|_o}\right) \cdot \left(\prod_{i=1}^n \frac{1}{1-x^{|\lambda^{(i)}|_e} y^{|\lambda^{(i)}|_o}}\right)$$

Now, since we would like to compute $S_a(x,y)$, we need to find a way to compute the two factors of the above expression. Unfortunately, it is not clear how to compute both these factors for arbitrary sequences $a = (a_1, a_2, \ldots, a_n)$. Therefore, from this point onwards, Bousquet-Mélou and Eriksson restrict their study to a specific type of sequence for which both these factors can be computed more easily.

Definition 4.4. *A sequence $a = (a_1, a_2, \ldots, a_n)$ is called a (k,ℓ)-sequence if it satisfies the following recursion relation:*

$$\begin{cases} a_{2n} &= \ell a_{2n-1} - a_{2n-2} \\ a_{2n+1} &= k a_{2n} - a_{2n-1} \end{cases} \tag{4}$$

Example 4.5. *An interesting example of a (k,ℓ)-sequence is the sequence of every other Fibonacci number. If we choose $a_1 = 1$, $k = 3$ and $\ell = 3$ then $(a_1, a_2, a_3, \ldots) = (1, 3, 8, \ldots)$ and in general $a_i = F_{2i-1}$ where F_i denotes the i-th Fibonacci number. We will see this sequence again shortly when we examine a limit theorem concerning a-lecture hall partitions.*

Under the assumption that a is a (k,ℓ)-sequence, the following theorem is proved in [3].

Theorem 4.6. *Fix $k, \ell \geq 2$. Let a be a finite (k,ℓ)-sequence of length n and let a^* be the (k,ℓ)-sequence defined by $a_1^* = 0$, $a_2^* = 1$. If n is even, then*

$$S_a(x,y) = \prod_{i=1}^n \frac{1}{1-x^{a_i} y^{a_i^*}}$$

and if n is odd then

$$S_a(x,y) = \prod_{i=1}^{n} \frac{1}{1 - x^{a^*_{i+1}} y^{a_{i-1}}}$$

Now, we have an equation of the same form as (1), and there is an interpretation of this equation in terms of a-lecture hall partitions. If we set $q = x = y$, then we have

$$S_a(q) = \prod_{i=1}^{n} \frac{1}{1 - q^{a_i + a^*_i}} \quad \text{if } n \text{ is even}$$

and

$$S_a(q) = \prod_{i=1}^{n} \frac{1}{1 - q^{a^*_{i+1} + a_{i-1}}} \quad \text{if } n \text{ is odd}$$

which can be interpreted as saying: "If n is even, then the number of a-lecture hall partitions of weight N is equal to the number of partitions of N into parts belonging to the set $\{s | s = a_i + a^*_i\}$. And if n is odd, then the number of a-lecture hall partitions of weight N is equal to the number of partitions of N into parts belonging to the set $\{s | s = a^*_{i+1} + a_{i-1}\}$."

This identity is still a little cumbersome, however. For the next result we will set $\ell = k$ to obtain a clearer, albeit less general, result.

The last major result we will consider arises from taking a limit ($n \to \infty$) in Theorem 4.6. This will allow the set of parts to be infinite. That is, we will obtain a result which says "The number of partitions of N satisfying condition A is equal to the number of partitions of N into parts belonging to the set S," where the set S is infinite.

Set $k = \ell$ and let $a_1 = 1$. Now, the sequences a and a^* from Theorem 4.6 are related by: $a^*_{i+1} = a_i$. Substituting for all occurrences of a^*_i in Theorem 4.6, we obtain:

$$S_a(x,y) = \prod_{i=1}^{n-1} \frac{1}{1 - x^{a_{i+1}} y^{a_i}}$$

To glean a combinatorial interpretation from this formula, we can again substitute $q = x = y$ to obtain:

$$S_a(q) = \prod_{i=1}^{n-1} \frac{1}{1 - q^{a_i + a_{i+1}}} \tag{5}$$

This can be interpreted as saying that the number of a-lecture hall partitions of weight N is equal to the number of partitions of n into elements of $S = \{s | s = a_i + a_{i+1}\}$. Now, we will use (5) to show the following generalization of Theorem 1.2:

Proposition 4.7. *The number of partitions of n where the quotient of successive parts is greater than $\frac{k+\sqrt{k^2-4}}{2}$ is the same as the number of partitions of n into elements of $S = \{s | s = a_i + a_{i+1}\}$, where a is an infinite (k,k)-sequence (i.e. $a_1 = 1$ and $a_{i+1} = ka_i - a_{i-1}$).*

Proof. Let us consider the ratio $\frac{a_{i+1}}{a_i}$ as i goes to infinity. Denote this limit by θ:

$$\theta = \lim_{i\to\infty} \frac{a_{i+1}}{a_i} = \lim_{i\to\infty} \frac{ka_i - a_{i-1}}{a_i} = k - \frac{1}{\theta}$$

$$\theta = k - \frac{1}{\theta} \Rightarrow \theta^2 - \theta k + 1 = 0$$

and so

$$\theta = \frac{k \pm \sqrt{k^2-4}}{2}$$

But $\frac{a_{i+1}}{a_i} \geq 1$ for all i, since (k,ℓ)-sequences are non-decreasing. Therefore,

$$\theta = \frac{k + \sqrt{k^2-4}}{2}$$

The reason we are interested in this limit is because of the following observation: Given any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ where the quotient of successive terms, $\frac{\lambda_{i+1}}{\lambda_i}$, is greater than (or equal to[1]) θ, there exists a sufficiently large n such that $\lambda' = (0, 0, \ldots, 0, \lambda_1, \lambda_2, \ldots, \lambda_s)$ is an a-lecture hall partitions of length n. Conversely, if λ has two parts, a_i and a_{i+1}, whose quotient, a_{i+1}/a_i is not greater than θ, then there exists a sufficiently large n such that $\lambda' = (0, 0, \ldots, 0, \lambda_1, \lambda_2, \ldots, \lambda_s)$ is *not* an a-lecture hall partitions of length n. Therefore, we may take the limit of equation (5) as $n \to \infty$ and we will obtain exactly the weight generating function for partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ where the quotient of successive terms, $\frac{\lambda_{i+1}}{\lambda_i}$ is greater than θ. This may be interpreted as saying that the number of partitions of n where the quotient of successive parts is greater than θ is the same as the number of partitions of n into elements of $S = \{s | s = a_i + a_{i+1}\}$, where a is an infinite (k, k)-sequence. This is exactly the generalization of Theorem 1.2 that we wanted. □

Example 4.8. *Let a be the $(4,4)$-sequence defined by $a_1 = 1$ and $a_{i+1} = 4a_i - a_{i-1}$; so $a = (1, 4, 15, 56, 209, 780, 2911, \ldots)$. Now let $S = \{s | s = a_{2i-1} + a_{2i+1}\} = \{1, 16, 224, 3120, \ldots\}$. By Proposition 4.7, the number of partitions of an integer n into parts belonging to S is equal to the number of partitions of n such that the ratio of consecutive terms is greater than $\frac{4+\sqrt{4^2-4}}{2} = 2 + \sqrt{3}$.*

Example 4.9. *Let F_i denote the i-th Fibonacci number (i.e. $F_0 = 1, F_1, = 1, F_2 = 2, \ldots$), and let F denote the set $\{f | f = F_{2i-1} + F_{2i+1}, i \geq 0\} = \{1, 4, 11, 29, 76, \ldots\}$. The odd-indexed Fibonacci numbers, F_{2i-1}, are exactly the elements of the $(3,3)$-sequence $a = (1, 3, 8, 21, 55, \ldots)$. Therefore, the number of partitions of n into elements of F is equal to the number of partitions of n such that the ratio of consecutive terms is greater than $\frac{3+\sqrt{3^2-4}}{2} = \frac{3+\sqrt{5}}{2}$.*

5 Conclusions

By this point, we have seen a wealth of material motivated by Theorem 1.2: The Rogers-Ramanujan identities, Gordon's Generalization of the Rogers-Ramanujan identities, and several of Bousquet-Mélou and Eriksson's results concerning lecture hall partitions and the more general a-lecture hall partitions. In some ways, these results are all very similar. Not only are all these results motivated by Theorem 1.2, but they all employ the proof technique introduced by Euler in Theorem 1.2, namely the use of generating functions as a powerful tool for working with partition identities. On the other hand, these results have taken different directions in generalizing Euler's result. The

[1]Note: Equality is only possible when $k = 2$, since θ is irrational for $k > 2$.

most striking such difference is Bousquet-Mélou and Eriksson's reinterpretation of the "distinctness" condition of Theorem 1.2 as a multiplicative property, rather than an additive property. What is more, these results span a period of nearly 250 years, demonstrating that the study of partition identities is a fruitful subject with a wide variety of known results, and undoubtedly many more to be discovered.

References

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