A NOTE ON INFINITELY MANY ODD NONUNITARY ABUNDANT NUMBERS

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Abstract. In response to a recent article by K. R. S. Sastry, we exhibit infinitely many odd nonunitary abundant numbers.

In a recent article in *Mathematics and Computer Education*, K. R. S. Sastry [4] discussed a variety of results and problems involving nonunitary numbers. (A number of articles dealing with unitary and nonunitary numbers have appeared in the last several years. See, for example, [1], [2], and [3].) In particular, Sastry asked for an example of an odd nonunitary abundant number. The goal of this short note is to exhibit an infinite family of such integers.

A brief review of some key terms is in order. An integer N is said to be **nonunitary abundant** if the sum of its nonunitary divisors is bigger than N. A **nonunitary divisor** d of N is a divisor which satisfies $(d, N/d) > 1$, while a divisor d of N is called a **unitary divisor** of N if $(d, N/d) = 1$. Using Sastry's notation, we will denote the sum of the divisors of N by $\sigma(N)$, while the sum of the unitary divisors of N will be denoted by $\sigma^*(N)$.

Sastry [4] notes that if $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, then

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1} \quad \text{and}$$
$$\sigma^*(n) = \prod_{i=1}^{r} (p_i^{a_i} + 1).$$

We will use these facts below.

Thanks to a brief Maple search, the following was discovered.

Theorem 1. *The number* $N = 3^3 \cdot 5^2 \cdot 7^2 = 33075$ *is an odd nonunitary abundant number.*

Indeed, it is the case that 33075 is the smallest odd nonunitary abundant number.

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Proof. The proof involves a simple set of calculations.

$$\begin{aligned}\sigma(N)-\sigma^*(N) &= \frac{3^4-1}{2}\cdot\frac{5^3-1}{4}\cdot\frac{7^3-1}{6}-(3^3+1)(5^2+1)(7^2+1)\\ &= 70680-36400\\ &= 34280\\ &> 33075\\ &= N \qquad \blacksquare\end{aligned}$$

Hence, we have found one odd nonunitary abundant number. It is the case that a fairly large infinite family of such numbers can be exhibited.

Theorem 2. *Let $N = 3^m\cdot 5^l\cdot 7^k$ with $m\geq 3$, $l\geq 2$, and $k\geq 2$. Then N is an odd nonunitary abundant number.*

Proof. We begin by noting that for any prime q,

$$\frac{q^{s+1}-1}{q-1}\geq\frac{q^{s-a}(q^{a+1}-1)}{q-1} \tag{1}$$

and

$$q^s+1\leq q^{s-a}(q^a+1) \tag{2}$$

provided that s and a are natural numbers and $s-a\geq 0$. Thus,

$$\begin{aligned}&\sigma(3^m5^l7^k)-\sigma^*(3^m5^l7^k)\\ &\quad= \frac{3^{m+1}-1}{2}\cdot\frac{5^{l+1}-1}{4}\cdot\frac{7^{k+1}-1}{6}-(3^m+1)(5^l+1)(7^k+1)\\ &\quad\geq \frac{3^{m-3}(3^4-1)}{2}\cdot\frac{5^{l-2}(5^3-1)}{4}\cdot\frac{7^{k-2}(7^3-1)}{6}\\ &\qquad\qquad -3^{m-3}(3^3+1)\cdot 5^{l-2}(5^2+1)\cdot 7^{k-2}(7^2+1)\\ &\qquad\qquad\qquad\text{using (1) and (2) repeatedly}\\ &\quad= 3^{m-3}5^{l-2}7^{k-2}\left[\frac{3^4-1}{2}\cdot\frac{5^3-1}{4}\cdot\frac{7^3-1}{6}-(3^3+1)(5^2+1)(7^2+1)\right]\\ &\quad= 3^{m-3}5^{l-2}7^{k-2}\left[\sigma(3^35^27^2)-\sigma^*(3^35^27^2)\right]\\ &\quad> 3^{m-3}5^{l-2}7^{k-2}\left[3^35^27^2\right]\quad\text{via Theorem 1}\\ &\quad= 3^m5^l7^k. \qquad \blacksquare\end{aligned} \tag{*}$$

One final generalization is worth noting.

Theorem 3. *Let $N = 3^m\cdot 5^l\cdot 7^k\cdot p_1^{a_1}\cdot p_2^{a_2}\dots p_r^{a_r}$ where the p_i's are distinct, odd primes greater than 7, $m\geq 3$, $l\geq 2$, $k\geq 2$, and $a_i\geq 1$ for $i=1,2,\dots,r$ with $r\in\mathbb{N}$. Then N is an odd nonunitary abundant number.*

Proof. We need to show $\sigma(N)-\sigma^*(N)>N$.

We see that

$$\sigma(N) = \frac{3^{m+1}-1}{2} \cdot \frac{5^{l+1}-1}{4} \cdot \frac{7^{k+1}-1}{6} \cdot \prod_{i=1}^{r} \frac{p_i^{a_i+1}-1}{p_i-1}$$

and

$$\sigma^*(N) = (3^m+1)(5^l+1)(7^k+1)\prod_{i=1}^{r}(p_i^{a_i}+1).$$

Next we have

$$\begin{aligned}\prod_{i=1}^{r} \frac{p_i^{a_i+1}-1}{p_i-1} &= \prod_{i=1}^{r}(p_i^{a_i}+p_i^{a_i-1}+p_i^{a_i-2}+\cdots+p_i+1)\\ &\geq \prod_{i=1}^{r}(p_i^{a_i}+1) \quad \text{for each prime } p_i.\end{aligned}$$

Hence,

$$\begin{aligned}\sigma(N)-\sigma^*(N) &\geq \left[\prod_{i=1}^{r}(p_i^{a_i}+1)\right]\left[\sigma(3^m5^l7^k)-\sigma^*(3^m5^l7^k)\right]\\ &> \left[\prod_{i=1}^{r}p_i^{a_i}\right]\left[\sigma(3^m5^l7^k)-\sigma^*(3^m5^l7^k)\right]. \qquad (**)\end{aligned}$$

Therefore,

$$\begin{aligned}\sigma(N)-\sigma^*(N) &> \left[\prod_{i=1}^{r}p_i^{a_i}\right]\left[\sigma(3^m5^l7^k)-\sigma^*(3^m5^l7^k)\right] \quad \text{by } (**)\\ &> \left[\prod_{i=1}^{r}p_i^{a_i}\right]\left[3^m5^l7^k\right] \quad \text{by } (*)\\ &= N.\end{aligned}$$

This completes the proof of Theorem 3. ■

These three theorems clearly satisfy the request of Sastry concerning the existence of odd nonunitary abundant numbers. Certainly, a classification of all odd nonunitary abundant numbers is desirable. Theorem 3 may be a beginning to such a task.

Acknowledgements

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References

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