

# Rational Numbers with Non-Terminating, Non-Periodic Modified Engel-Type Expansions

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*Abstract.*

Recently, Kalpazidou, Knopfmacher, and Knopfmacher asked if there exist rational numbers whose “modified Engel-type” expansion is neither finite nor ultimately periodic. In this note we answer their question by explicitly providing an infinite sequence of such numbers.

In a recent paper [3], Kalpazidou, Knopfmacher, and Knopfmacher discussed expansions for real numbers of the form

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_1 + 1} \cdot \frac{1}{a_2} + \frac{1}{(a_1 + 1)(a_2 + 1)} \cdot \frac{1}{a_3} - \dots \quad (1)$$

which they called a “modified Engel-type” alternating expansion. Here  $a_0$  is an integer and  $a_i$  is a positive integer for  $i \geq 1$ . If  $a_{i+1} \geq a_i$ , this expansion is essentially unique. To save space we will abbreviate Eq. (1) by

$$A = \{a_0, a_1, a_2, \dots\}.$$

They say, “The question of whether or not all rationals have a finite or recurring expansion has not been settled.” (By “recurring” we understand “ultimately periodic”.)

In this note, we prove that the rational numbers  $\frac{2}{2r+1}$  ( $r$  an integer  $\geq 2$ ) have modified Engel-type expansions that are neither finite nor ultimately periodic.

**Theorem.**

Let  $r$  be an integer  $\geq 1$ . Then

$$\frac{2}{2r+1} = \{a_0, a_1, a_2, \dots\}$$

where  $a_0 = 0$ , and  $a_{2i-1} = b_i$ ,  $a_{2i} = 2b_i - 1$  for  $i \geq 1$ , and  $b_1 = r$ ,  $b_{n+1} = 2b_n^2 - 1$  for  $n \geq 1$ .

**Proof.**

As in [3], we have  $a_0 = \lfloor A \rfloor$ ,  $A_1 = A - a_0$ ,  $a_n = \lfloor 1/A_n \rfloor$  for  $n \geq 1$  and  $A_{n+1} = (1/a_n - A_n)(a_n + 1)$  for  $n \geq 1$ .

From this we see that  $a_0 = \lfloor \frac{2}{2r+1} \rfloor = 0$ .

We now prove the following four assertions by induction on  $n$ : (i)  $A_{2n-1} = \frac{2}{2b_n+1}$ ; (ii)  $a_{2n-1} = b_n$ ; (iii)  $A_{2n} = \frac{b_n+1}{b_n(2b_n+1)}$ ; and (iv)  $a_{2n} = 2b_n - 1$ .

It is easy to verify these assertions for  $n = 1$ , as we find

$$(i) \quad A_1 = \frac{2}{2r+1} = \frac{2}{2b_1+1};$$

$$(ii) \quad a_1 = \left\lfloor \frac{1}{A_1} \right\rfloor = r = b_1;$$

$$(iii) \quad A_2 = \left(\frac{1}{r} - \frac{2}{2r+1}\right)(r+1) = \frac{r+1}{r(2r+1)} = \frac{b_1+1}{b_1(2b_1+1)};$$

$$(iv) \quad a_2 = \left\lfloor \frac{1}{A_2} \right\rfloor = \left\lfloor \frac{r(2r+1)}{r+1} \right\rfloor = \left\lfloor 2r - 1 + \frac{1}{r+1} \right\rfloor = 2r - 1 = 2b_1 - 1.$$

Now assume the result is true for all  $i \leq n$ . We prove it for  $n + 1$ :

(i)

$$\begin{aligned} A_{2n+1} &= \left( \frac{1}{a_{2n}} - A_{2n} \right) (a_{2n} + 1) \\ &= \left( \frac{1}{2b_n - 1} - \frac{b_n + 1}{b_n(2b_n + 1)} \right) (2b_n) \\ &= \frac{2}{4b_n^2 - 1} \\ &= \frac{2}{2b_{n+1} + 1}. \end{aligned}$$

(ii)

$$a_{2n+1} = \left\lfloor \frac{1}{A_{2n+1}} \right\rfloor = \left\lfloor \frac{2b_{n+1} + 1}{2} \right\rfloor = b_{n+1}.$$

(iii)

$$\begin{aligned} A_{2n+2} &= \left( \frac{1}{a_{2n+1}} - A_{2n+1} \right) (a_{2n+1} + 1) \\ &= \left( \frac{1}{b_{n+1}} - \frac{2}{2b_{n+1} + 1} \right) (b_{n+1} + 1) \\ &= \frac{b_{n+1} + 1}{b_{n+1}(2b_{n+1} + 1)}. \end{aligned}$$

(iv)

$$\begin{aligned} a_{2n+2} &= \left\lfloor \frac{1}{A_{2n+2}} \right\rfloor \\ &= \left\lfloor \frac{b_{n+1}(2b_{n+1} + 1)}{b_{n+1} + 1} \right\rfloor \\ &= \left\lfloor 2b_{n+1} - 1 + \frac{1}{b_{n+1} + 1} \right\rfloor \\ &= 2b_{n+1} - 1. \end{aligned}$$

This completes the proof. ■

### Corollary.

For  $r \geq 2$ , the rational numbers  $\frac{2}{2r+1}$  have non-terminating, non-ultimately-periodic modified Engel-type expansions.

### Additional Remarks.

- For  $r = 1$ , the theorem gives the ultimately periodic expansion

$$2/3 = \{0, 1, 1, 1, 1, \dots\}.$$

- For  $r \geq 2$ , the expansion is not ultimately periodic; e.g.

$$2/5 = \{0, 2, 3, 7, 13, 97, 193, 18817, \dots\}.$$

In this case, we have the following brief table:

$n$	$a_n$	$b_n$	$A_n$
1	2	2	2/5
2	3	7	3/10
3	7	97	2/15
4	13	18817	8/105
5	97	708158977	2/195
6	193	1002978273411373057	98/18915

• The sequence  $b_1, b_2, \dots = 2, 7, 97, 18817, 708158977, \dots$ , corresponding to  $r = 2$ , appears to have been discussed first by G. Cantor in 1869 [1], who gave the infinite product

$$\sqrt{3} = \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{7}\right) \left(1 + \frac{1}{97}\right) \cdots.$$

For more on this product of Cantor, see Spiess [9], Sierpiński [7], Engel [2], Stratemeyer [10,11], Ostrowski [6], and Mendès France and van der Poorten [5]. The sequence 2, 7, 97, 18817, ... was also discussed by Lucas [4]. It is sequence #720 in Sloane [8].

• The sequence  $b_1, b_2, \dots = 3, 17, 577, 665857, \dots$ , corresponding to  $r = 3$ , was also discussed by Cantor [1], who gave the infinite product

$$\sqrt{2} = \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{17}\right) \left(1 + \frac{1}{577}\right) \cdots.$$

Also see the papers mentioned above. The sequence was also discussed by Wilf [12]. It is sequence #1234 in Sloane [8].

• It is easy to prove that  $b_{n+1} = B_{2^n}$  where  $B_0 = 1$ ,  $B_1 = r$ , and  $B_n = 2rB_{n-1} - B_{n-2}$  for  $n \geq 2$ . This gives a closed form for the sequence  $(b_n)$ :

$$b_{n+1} = \frac{(r + \sqrt{r^2 - 1})^{2^n} + (r - \sqrt{r^2 - 1})^{2^n}}{2}.$$

• 3/7 is the “simplest” rational for which no simple description of the terms in its modified Engel-type expansion is known. The first forty terms are as follows:

$$\begin{aligned} 3/7 = \{ & 0, 2, 4, 5, 7, 8, 10, 25, 53, 62, 134, 574, 2431, 13147, 27167, 229073, 315416, \\ & 435474, 771789, 1522716, 3853889, 7878986, 7922488, 8844776, 9182596, 9388467, \\ & 14781524, 135097360, 1374449987, 1561240840, 4408239956, 11166053604, 12014224315, \end{aligned}$$

23110106464, 553192836372, 900447772231, 1189661630241, 2058097840143484,  
6730348855426376, 12928512475357529,  $\dots$  }.

More generally, it would be of interest to know whether it is possible to characterize the modified Engel expansion of every rational number.

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