ON MIKI'S IDENTITY FOR BERNOULLI NUMBERS

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Dedicated to the memory of Arnold E. Ross

Abstract. We give a short proof of Miki's identity for Bernoulli numbers,

$$\sum_{i=2}^{n-2} \beta_i\beta_{n-i} - \sum_{i=2}^{n-2} \binom{n}{i}\beta_i\beta_{n-i} = 2H_n\beta_n,$$

for $n \geq 4$ where, $\beta_i = B_i/i$, B_i is the ith Bernoulli number, and $H_n = 1 + 1/2 + \cdots + 1/n$.

1. Introduction. The Bernoulli numbers B_n are defined by

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}.$$

There are many identities involving binomial convolutions of Bernoulli numbers, the best known being Euler's identity

$$\sum_{i=2}^{n-2} \binom{n}{i} B_i B_{n-i} = -(n+1)B_n$$

for $n \geq 4$. Euler's identity is an easy consequence of the formula

$$b(x)^2 = (1-x)b(x) - xb'(x),$$

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where $b(x) = x/(e^x - 1)$. Many similar identities for Bernoulli numbers can be proved in the same way. (See, e.g., Dilcher [1] and Huang [4].)

A more mysterious identity was proved by Hiroo Miki in 1978: Let $\beta_n = B_n/n$ and let H_n be the harmonic number $1 + 1/2 + \cdots + 1/n$. Then

$$\sum_{i=2}^{n-2} \beta_i \beta_{n-i} - \sum_{i=2}^{n-2} \binom{n}{i} \beta_i \beta_{n-i} = 2H_n \beta_n. \tag{1}$$

This identity is unusual because it involves both a binomial convolution and an ordinary convolution.

Miki gave a complicated proof of his identity that was based on a formula for the Fermat quotient $(a^p - a)/p$ modulo p^2. He showed that both sides of (1) are congruent modulo p for every sufficiently large prime p, which implies that they are equal. Another proof of Miki's identity, using p-adic analysis, was given by Shiratani and Yokoyama [6].

We give here a simple proof of Miki's identity, based on two different expressions for Stirling numbers of the second kind $S(n,k)$, which may be defined by the ordinary generating function

$$\sum_{n=0}^{\infty} S(n,k) x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \tag{2}$$

or by the exponential generating function

$$\sum_{n=0}^{\infty} S(n,k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}. \tag{3}$$

The equivalence of (2) and (3) follows by comparing the partial fraction expansion

$$\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{1-jx}$$

with the binomial theorem expansion

$$\frac{(e^x - 1)^k}{k!} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} e^{jx}.$$

From each of (2) and (3) one can derive a formula that for fixed n expresses $S(m+n, m)$ as a polynomial in m. (For some combinatorial applications of these polynomials, see [3].) Equating coefficients of m^2 in these formulas gives Miki's identity. Similarly, equating coefficients of higher powers of m gives an infinite sequence of related, though more complicated, identities, of which the next one is (for $n \geq 4$)

$$\sum_{\substack{i+j+k=n\\ i,j,k\geq 2}} \binom{n}{i,j,k}\beta_i\beta_j\beta_k + 3H_n\sum_{i=2}^{n-2}\binom{n}{i}\beta_i\beta_{n-i} + 6H_{n,2}\beta_n = \sum_{\substack{i+j+k=n\\ i,j,k\geq 2}} \beta_i\beta_j\beta_k + \frac{n^2-3n+5}{4}\beta_{n-2}, \qquad (4)$$

where $H_{n,2} = \sum_{1\leq i<j\leq n}(ij)^{-1}$.

2. Lemmas. The *Nörlund polynomials* $B_n^{(z)}$ are defined by

$$\sum_{n=0}^{\infty} B_n^{(z)}\frac{x^n}{n!} = \left(\frac{x}{e^x-1}\right)^z.$$

(If z is a nonnegative integer, then $B_n^{(z)}$ is called a *Bernoulli number of order z.)* Note that $B_n^{(1)} = B_n$ and that for fixed n, $B_n^{(z)}$ is a polynomial in z of degree n. As is well known, the Stirling numbers of the second kind can be expressed in terms of these polynomials:

Lemma 1.

$$S(m+n,m) = \binom{m+n}{n}B_n^{(-m)}.$$

Proof. From (3) we have

$$\sum_{n=0}^{\infty} B_n^{(-m)}\frac{x^n}{n!} = \left(\frac{e^x-1}{x}\right)^m = \sum_{n=0}^{\infty} S(m+n,m)\frac{m!\,n!}{(m+n)!}\frac{x^n}{n!},$$

and the result follows. □

To find a formula for coefficients of powers of m in $S(m+n,m)$ we need to expand both $\binom{m+n}{n}$ and $B_n^{(-m)}$ in powers of m.

Let us define generalized harmonic numbers $H_{n,j}$ by

$$H_{n,j} = \sum_{1\leq k_1<k_2<\cdots<k_j\leq n}\frac{1}{k_1k_2\cdots k_j},$$

with $H_{n,0} = 1$. Then $H_n = H_{n,1}$. The numbers $H_{n,j}$ are closely related to the unsigned Stirling numbers of the first kind $c(n,j)$ which may be defined by

$$x(x+1)\cdots(x+n-1) = \sum_{j=0}^{n} c(n,j)x^j;$$

we have $H_{n,j} = c(n+1,j+1)/n!$.

Lemma 2.

$$\binom{m+n}{n} = \sum_{j=0}^{n} H_{n,j} m^j.$$

Proof. This follows easily from

$$\binom{m+n}{n} = \frac{m+n}{n} \cdot \frac{m+n-1}{n-1} \cdots \frac{m+1}{1} = \left(1+\frac{m}{1}\right)\left(1+\frac{m}{2}\right)\cdots\left(1+\frac{m}{n}\right). \quad \square$$

Now let us define β_n to be $(-1)^n B_n/n$ for $n \geq 1$. (Note $\beta_n = B_n/n$ except for $\beta_1 = -B_1 = 1/2$, since $B_n = 0$ when n is odd and greater than 1.) We define $\beta_n^{(j)}$ by

$$\beta_n^{(j)} = \frac{1}{j!} \sum_{i_1+i_2+\cdots+i_j=n} \binom{n}{i_1,\ldots,i_j} \beta_{i_1} \cdots \beta_{i_j}, \tag{5}$$

where the sum is over positive integers $i_1, \ldots, i_j$.

Lemma 3. *For $n > 0$,*

$$B_n^{(-m)} = \sum_{j=1}^{n} \beta_n^{(j)} m^j.$$

Proof. We have

$$\sum_{n=0}^{\infty} B_n^{(-m)} \frac{x^n}{n!} = \left(\frac{e^x-1}{x}\right)^m = \exp\left(m \log\left(\frac{e^x-1}{x}\right)\right) = \sum_{j=0}^{\infty} \frac{m^j}{j!} \left[\log\left(\frac{e^x-1}{x}\right)\right]^j.$$

Since

$$\frac{d}{dx} \log\left(\frac{e^x-1}{x}\right) = \frac{1}{x}\left(\frac{-x}{e^{-x}-1} - 1\right) = \sum_{n=1}^{\infty} (-1)^n B_n \frac{x^{n-1}}{n!},$$

we have

$$\log\left(\frac{e^x-1}{x}\right) = \sum_{n=1}^{\infty} (-1)^n \frac{B_n}{n} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \beta_n \frac{x^n}{n!},$$

and thus

$$\frac{1}{j!}\left[\log\left(\frac{e^x-1}{x}\right)\right]^j = \sum_{n=0}^{\infty} \beta_n^{(j)} \frac{x^n}{n!}. \quad \square$$

Now we apply formula (2), which we rewrite as

$$\frac{1}{(1-x)(1-2x)\cdots(1-mx)} = \sum_{n=0}^{\infty} S(m+n,n) x^n. \tag{6}$$

Lemma 4.

$$\sum_{n=0}^{\infty} S(m+n,n)x^n = \exp\Bigg[\sum_{k=1}^{\infty} \frac{x^k}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{m^{j+1}}{j+1} B_{k-j}\Bigg].$$

Proof. The left side of (6) may be written as

$$\exp\Bigg[\sum_{k=1}^{\infty}(1^k + \cdots + m^k)\frac{x^k}{k}\Bigg].$$

There is a well-known formula for expressing the power sum $1^k + \cdots + m^k$ in terms of Bernoulli numbers which may be derived as follows:

$$\begin{aligned}\sum_{k=0}^{\infty}(1^k + \cdots + m^k)\frac{x^k}{k!} &= e^x + \cdots + e^{mx} = \frac{e^{(m+1)x} - e^x}{e^x - 1} \\ &= \frac{e^{mx} - 1}{1 - e^{-x}} = \frac{e^{mx} - 1}{x} \frac{-x}{e^{-x} - 1}.\end{aligned} \tag{7}$$

Equating coefficients of x^k in (7) gives

$$1^k + \cdots + m^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{m^{j+1}}{j+1} B_{k-j}. \quad \square$$

3. The Proof. Combining Lemmas 1, 2, and 3, gives our first formula for expressing $S(m+n,m)$ as a polynomial in m: For $n > 0$,

$$S(m+n,m) = \beta_n m + \big(\beta_n^{(2)} + H_n\beta_n\big)m^2 + \big(\beta_n^{(3)} + H_n\beta_n^{(2)} + H_{n,2}\beta_n\big)m^3 + \cdots \tag{8}$$

On the other hand, expanding the right side of Lemma 4 gives the alternative formula (for $n \geq 2$)

$$S(m+n,n) = \beta_n m + \left(\frac{1}{2}(-1)^{n-1}B_{n-1} + \frac{1}{2}\sum_{i=1}^{n-1}\beta_i\beta_{n-i}\right)m^2 + Cm^3 + \cdots, \tag{9}$$

where

$$C = \frac{1}{6}\sum_{i+j+k=n}\beta_i\beta_j\beta_k + \frac{1}{2}\sum_{i=1}^{n-1}(-1)^{n-i-1}\beta_i B_{n-i-1} + \frac{1}{6}(-1)^n(n-1)B_{n-2}.$$

Equating coefficients of m^2 in (8) and (9) yields the identity

$$\beta_n^{(2)} + H_n\beta_n = \frac{1}{2}(-1)^{n-1}B_{n-1} + \frac{1}{2}\sum_{i=1}^{n-1}\beta_i\beta_{n-i},$$

which for n even and greater than 2 is Miki's identity. (For n odd, the coefficient of m^2 in both (8) and (9) is $\frac{n}{2}\beta_{n-1}$.)

Equating coefficient of m^3 in (8) and (9) yields the identity

$$\beta_n^{(3)} + H_n\beta_n^{(2)} + H_{n,2}\beta_n = \frac{1}{6}\sum_{i+j+k=n}\beta_i\beta_j\beta_k + \frac{1}{2}\sum_{i=1}^{n-1}(-1)^{n-i-1}\beta_i B_{n-i-1} + \frac{1}{6}(-1)^n(n-1)B_{n-2}. \tag{10}$$

For n odd and greater than 3, (10) reduces to Miki's identity. For n even, we may simplify (10) by separating all occurrences of β_1 or B_1, and multiplying by 6, obtaining (4) for $n \geq 4$.

4. A generalization.

An identity related to Miki's was found by Faber and Pandharipande, and proved by Zagier [2]. Their identity may be written

$$\frac{n}{2}\sum_{i=2}^{n-2}\frac{B_i(\frac{1}{2})}{i}\frac{B_{n-i}(\frac{1}{2})}{n-i} - \sum_{i=0}^{n-2}\binom{n}{i}B_i(\tfrac{1}{2})\beta_{n-i} = H_{n-1}B_n(\tfrac{1}{2}). \tag{11}$$

As before, $\beta_n = (-1)^n B_n/n$, and $B_n(\lambda)$ is the Bernoulli polynomial, defined by

$$\sum_{n=0}^{\infty} B_n(\lambda)\frac{x^n}{n!} = \frac{xe^{\lambda x}}{e^x - 1}.$$

Thus $B_n(0) = B_n$ and it is well known that $B_n(\frac{1}{2}) = (2^{1-n} - 1)B_n$.

By using the approach of this paper, one can prove a common generalization of the Faber-Pandharipande-Zagier identity and Miki's identity:

$$\frac{n}{2}\left(B_{n-1}(\lambda) + \sum_{i=1}^{n-1}\frac{B_i(\lambda)}{i}\frac{B_{n-i}(\lambda)}{n-i}\right) - \sum_{i=0}^{n-1}\binom{n}{i}B_i(\lambda)\beta_{n-i} = H_{n-1}B_n(\lambda), \tag{12}$$

for all $n \geq 1$. To derive (11) from (12), we observe that $B_i(\frac{1}{2})$ is 0 for i odd, so for $\lambda = \frac{1}{2}$ and n even, the terms in (12) that don't appear in (11) are all 0 (and for n odd all terms in (11) are 0).

We can derive Miki's identity (1) from the case $\lambda = 0$ of (12). We first observe that from

$$\frac{1}{i(n-i)} = \frac{1}{n}\left(\frac{1}{i} + \frac{1}{n-i}\right)$$

we get

$$\sum_{i=2}^{n-2}\binom{n}{i}\beta_i\beta_{n-i} = \frac{2}{n}\sum_{i=2}^{n-2}\binom{n}{i}B_i\beta_{n-i}.$$

Thus (1) multipled by $n/2$ is equivalent to

$$\frac{n}{2}\sum_{i=2}^{n-2}\beta_i\beta_{n-i} - \sum_{i=2}^{n-2}\binom{n}{i}B_i\beta_{n-i} = H_nB_n.$$

For n even and greater than 2, this can be obtained from the case $\lambda = 0$ of (12) by adding $\binom{n}{0}B_0\beta_n = B_n/n$ to both sides and deleting some terms that are 0.

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