1 OEIS 70

Conjecture 1 *Consider the formal power series* $f(q) = \sum_{n\geq 0} a_n q^n$ *defined by* $a_0 = a_1 = 1$ *and*

$$a_n = \sum_{k=0}^{n-2} a_k \binom{n-2}{k}$$

for $n \geq 2$*. Then*

$$f(q) = 1 + q + \sum_{k=1}^{\infty} q^{2k} \frac{1-(k-1)q}{\prod_{m=1}^{k}(1-mq)^2}.$$

Proof.

$$\begin{aligned}
\sum_{n\geq 2} a_n q^n &= \sum_{n\geq 2}\sum_{k=0}^{n-2} a_k \binom{n-2}{k} q^n \\
f(q) - 1 - q &= \sum_{k=0}^{n-2} a_k \sum_{n\geq 2} \binom{n-2}{k} q^n \\
&= \sum_{k\geq 0} a_k \sum_{n\geq k+2} \binom{n-2}{k} q^n \\
&= \sum_{k\geq 0} a_k \sum_{n\geq k+2} \binom{n-2}{k} q^n \\
&= \frac{q^2}{1-q} \sum_{k\geq 0} a_k \left(\frac{q}{1-q}\right) \\
&= \frac{q^2}{1-q} \cdot f\left(\frac{q}{1-q}\right)
\end{aligned}$$

So we have

$$f(q) = 1 + q + \frac{q^2}{1-q} f\left(\frac{q}{1-q}\right).$$

which leads to

$$f\left(\frac{q}{1-mq}\right) = \frac{1-(m-1)q}{1-mq} + \frac{q^2}{(1-(m-1)q)(1-mq)} \cdot f\left(\frac{q}{1-(m+1)q}\right)$$

for all $m \geq 0$.

Iterating, we obtain

$$\begin{aligned}
f(q) &= 1 + q + \sum_{k=1}^{\infty} \frac{1-(k-1)q}{1-kq} \prod_{m=1}^{k} \frac{q^2}{(1-(m-1)q)(1-mq)} \\
&= 1 + q + \sum_{k=1}^{\infty} q^{2k} \frac{1-(k-1)q}{\prod_{m=1}^{k}(1-mq)^2}
\end{aligned}$$

■

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