Infinitely Many Composite NSW Numbers: An Inductive Proof

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1 Motivation

The NSW numbers were introduced approximately 20 years ago [3] in connection with the order of certain simple groups. These are the numbers f_n which satisfy the recurrence

$$f_{n+1} = 6f_n - f_{n-1} \tag{1}$$

with initial conditions $f_1 = 1$ and $f_2 = 7$.

In recent years, these numbers have been studied from a variety of perspectives [1], [2]. Moreover, the author, in collaboration with Hugh Williams, has proven that there are infinitely many composite NSW numbers [4] as requested in [1]. The goal of this note is to provide a purely inductive proof of the main theorem in [4]. We restate it here.

Theorem 1.1. *For all $m \geq 1$ and all $n \geq 0$, $f_m \mid f_{(2m-1)n+m}$.*

2 The Necessary Tools

To prove Theorem 1.1, we need to develop a few key tools.

Proposition 2.1. *For all integers $a, b \geq 0$, and for all $1 \leq j \leq a+b-2$, we have*

$$f_{a+b} = s_{j+1} f_{a+b-j} - s_j f_{a+b-j-1} \tag{2}$$

where $s_j = \sum_{i=1}^{j} (-1)^{i+j} f_i$.

Proof. We prove this proposition using induction on j. First, when $j = 1$, the right hand side of (2) is $(f_2 - f_1) f_{a+b-1} - f_1 f_{a+b-2}$ or $6 f_{a+b-1} - f_{a+b-2}$, which equals f_{a+b} thanks to (1).

Next, we assume

$$f_{a+b} = s_{j+1} f_{a+b-j} - s_j f_{a+b-j-1}$$

for $j < a + b - 2$. Thus, since $f_{a+b-j} = 6 f_{a+b-j-1} - f_{a+b-j-2}$, we have

$$\begin{aligned} f_{a+b} &= s_{j+1}(6 f_{a+b-j-1} - f_{a+b-j-2}) - s_j f_{a+b-j-1} \\ &= (6 s_{j+1} - s_j) f_{a+b-j-1} - s_{j+1} f_{a+b-j-2}. \end{aligned}$$

Then we note that

$$\begin{aligned} 6 s_{j+1} - s_j &= 6 \sum_{i=1}^{j+1} (-1)^{i+j+1} f_i - \sum_{i=1}^{j} (-1)^{i+j} f_i \\ &= 6(-1)^{j+2} f_1 + 6 \sum_{i=2}^{j+1} (-1)^{i+j+1} f_i - \sum_{i=1}^{j} (-1)^{i+j} f_i \end{aligned}$$

$$
\begin{aligned}
&= 6(-1)^{j+2}f_1 + 6\sum_{i=1}^{j}(-1)^{i+j+2}f_{i+1} - \sum_{i=1}^{j}(-1)^{i+j}f_i \\
&= (f_2 - f_1)(-1)^{j+2} + \sum_{i=1}^{j}(-1)^{i+j+2}(6f_{i+1} - f_i) \\
&= (f_2 - f_1)(-1)^{j+2} + \sum_{i=1}^{j}(-1)^{i+j+2}f_{i+2} \quad \text{by (1)} \\
&= (f_2 - f_1)(-1)^{j+2} + \sum_{i=3}^{j+2}(-1)^{i+j}f_i \\
&= \sum_{i=1}^{j+2}(-1)^{i+j}f_i \\
&= s_{j+2}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
f_{a+b} &= (6s_{j+1} - s_j)f_{a+b-j-1} - s_{j+1}f_{a+b-j-2} \\
&= s_{j+2}f_{a+b-j-1} - s_{j+1}f_{a+b-j-2},
\end{aligned}
$$

which completes the proof of Proposition 2.1. □

Proposition 2.2. *For all* $m \geq 1$ *and for all* $1 \leq c \leq m-1$, $f_m \mid f_{m+c} + f_{m-c}$.

Proof. For $c = 1$, we know from (1) that $6f_m = f_{m+1} + f_{m-1}$, so that $f_m \mid f_{m+1} + f_{m-1}$. Next, we assume $f_m \mid f_{m+c} + f_{m-c}$ for $1 \leq c \leq d$ for some value $d < m - 1$. Since

$$
f_{m+d+1} = 6f_{m+d} - f_{m+d-1} \quad \text{and} \quad f_{m-d-1} = 6f_{m-d} - f_{m-d+1},
$$

we know

$$
f_{m+d+1} + f_{m-(d+1)} = 6f_{m+d} - f_{m+d-1} + 6f_{m-d} - f_{m-d+1}
$$

$$= \quad 6(f_{m+d} + f_{m-d}) - (f_{m+d-1} + f_{m-(d-1)}).$$

By the induction hypothesis, the result follows. □

Proposition 2.3. *For all* $m \geq 1$, $f_m \mid s_{2m-1}$.

Proof. We see that

$$s_{2m-1} = \sum_{i=1}^{2m-1} (-1)^{i-1} f_i = f_1 - f_2 + f_3 - \ldots + (-1)^{m-1} f_m + \ldots + f_{2m-1}.$$

Notice that this sum is centered about f_m, which divides itself, and that the rest of the terms can be paired in such a way that Proposition 2.2 can be applied easily. □

We are now ready to prove Theorem 1.1.

Proof. When $n = 0$, the result is clear. Next, assume $f_m \mid f_{(2m-1)n+m}$ or $f_m \mid f_{2mn-n+m}$. We want to prove

$$f_m \mid f_{(2m-1)(n+1)+m} \quad \text{or} \quad f_m \mid f_{2mn-n+m+(2m-1)}.$$

Using Proposition 2.1 with $a = 2mn-n+m$, $b = 2m-1$, and $j = 2m-2$, we have

$$f_{2mn-n+m+(2m-1)} = s_{2m-1} f_{2mn-n+m+1} - s_{2m-2} f_{2mn-n+m}.$$

From Proposition 2.3, we know $f_m \mid s_{2m-1}$, and from the induction hypothesis, $f_m \mid f_{2mn-n+m}$. The result follows. □

References

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