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ON REPRESENTATIONS OF A NUMBER AS A SUM OF THREE TRIANGLES

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Abstract. Let $t(n)$ be the number of representations of n as a sum of three triangles. We prove infinitely many results concerning $t(n)$ of the sort

$$t(27n+12) = 3 \times t(3n+1), \quad t(27n+21) = 5 \times t(3n+2),$$

$$t(81n+3) = 4 \times t(9n), \quad t(81n+57) = 4 \times t(9n+6).$$

1. Introduction.

Let $t(n)$ be the number of representations of n as the sum of three triangular numbers. Then

$$\sum_{n\geq 0} t(n)q^n = \left(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\right)^3.$$

*This research was begun while the first author was enjoying leave.

Gauss[2] showed that $t(n) > 0$, in other words, every number is the sum of three triangles. Recently, George E. Andrews[1] provided a proof of this fact via q-series.

We shall show that $t(n)$ satisfies infinitely many arithmetic identities. Thus, for $\lambda \geq 1$,

$$t(3^{2\lambda+1}n + (11 \times 3^{2\lambda} - 3)/8) = 3^{\lambda} \times t(3n+1),$$

$$t(3^{2\lambda+1}n + (19 \times 3^{2\lambda} - 3)/8) = (2 \times 3^{\lambda} - 1) \times t(3n+2),$$

$$t(3^{2\lambda+2}n + (3^{2\lambda+1} - 3)/8) = ((3^{\lambda+1} - 1)/2) \times t(9n)$$

and

$$t(3^{2\lambda+2}n + (17 \times 3^{2\lambda+1} - 3)/8) = ((3^{\lambda+1} - 1)/2) \times t(9n+6).$$

It should be noted that

$$t(n) = r(8n+3)$$

where $r(n)$ is the number of representations of n in the form

$$n = k^2 + l^2 + m^2$$

with k, l, m odd and positive. Indeed,

$$n = (r^2 + r)/2 + (s^2 + s)/2 + (t^2 + t)/2$$

is equivalent to

$$8n + 3 = (2r+1)^2 + (2s+1)^2 + (2t+1)^2.$$

Our results can be written in terms of $r(n)$. Thus, we have

$$r(9^{\lambda}n) = \begin{cases} 3^{\lambda} \times r(n) & \text{if } n \equiv 11 \pmod{24} \\ (2 \times 3^{\lambda} - 1) \times r(n) & \text{if } n \equiv 19 \pmod{24} \\ ((3^{\lambda+1} - 1)/2) \times r(n) & \text{if } n \equiv 3 \text{ or } 51 \pmod{72}. \end{cases}$$

We also obtain the generating function formulae

$$\sum_{n\geq 0} t(3n+1)q^n = 3 \times \frac{(q^3;q^3)_\infty^3}{(q;q^2)_\infty^2},$$

$$\sum_{n\geq 0} t(3n+2)q^n = 3 \times \frac{(q^6;q^6)_\infty^3}{(q;q^2)_\infty},$$

$$\sum_{n\geq 0} t(9n)q^n = \frac{(q^3;q^3)_\infty}{(q;q^6)_\infty (q^5;q^6)_\infty} \left\{ 1 + 6 \sum_{n\geq 1} \left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}} \right) \right\},$$

and

$$\sum_{n\geq 0} t(9n+6)q^n = 6 \times \frac{(q^3;q^6)_\infty^2 (q^6;q^6)_\infty^4}{(q;q)_\infty}.$$

2. The generating functions for $t(3n+1)$ and $t(3n+2)$.

We have $t(3n+1) = r(24n+11)$, so we need to consider

$$24n + 11 = k^2 + l^2 + m^2$$

with k, l, m odd and positive. Modulo 6, this becomes

$$k^2 + l^2 + m^2 \equiv 5 \pmod 6.$$

This has the solutions, together with permutations,

$$(k, l, m) \equiv (\pm 1, \pm 1, 3) \pmod 6.$$

Conversely, if $(k, l, m) \equiv (\pm 1, \pm 1, 3) \pmod 6$ then $k^2 + l^2 + m^2 \equiv 5 \pmod{24}$.

It follows that

$$\sum_{n\geq 0} t(3n+1)q^{24n+11} = \sum_{n\geq 0} r(24n+11)q^{24n+11}$$
$$= 3\sum_{m\geq 0} q^{(6k+1)^2+(6l+1)^2+(6m+3)^2}.$$

Here we have used the facts that

$$\sum_{\substack{k>0 \\ k\equiv\pm 1 \pmod 6}} q^{k^2} = \sum q^{(6k+1)^2}$$

and

$$\sum_{\substack{k>0 \\ k\equiv 3 \pmod 6}} q^{k^2} = \sum_{k\geq 0} q^{(6k+3)^2}.$$

Thus we have

$$\sum_{n\geq 0} t(3n+1)q^{24n+11} = 3q^{11}\sum_{m\geq 0} q^{36k^2+12k+36l^2+12l+36m^2+36m}$$

so

$$\sum_{n\geq 0} t(3n+1)q^n = 3\sum_{m\geq 0} q^{(3k^2+k)/2+(3l^2+l)/2+3((m^2+m)/2)}$$

$$= 3 \times (-q;q^2)_\infty^2(-q^2;q^3)_\infty^2(q^3;q^3)_\infty^2\frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty}$$
$$= 3 \times \frac{(q^3;q^3)_\infty^3}{(q;q^2)_\infty^2}.$$

Similarly we find that

$$\begin{aligned}\sum_{n\geq 0} t(3n+2)q^{24n+19} &= \sum_{n\geq 0} r(24n+19)q^{24n+19}\\ &= 3\sum_{l,m\geq 0} q^{(6k+1)^2+(6l+3)^2+(6m+3)^2}\\ &= 3q^{19}\sum_{l,m\geq 0} q^{36k^2+12k+36l^2+36l+36m^2+36m}\end{aligned}$$

so

$$\begin{aligned}\sum_{n\geq 0} t(3n+2)q^n &= 3\sum_{l,m\geq 0} q^{(3k^2+k)/2+3((l^2+l)/2+3((m^2+m)/2)}\\ &= 3 \times (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty\left(\frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty}\right)^2\\ &= 3 \times \frac{(q^6;q^6)_\infty^3}{(q;q^2)_\infty}.\end{aligned}$$

3. The generating functions for $t(9n)$ and $t(9n+6)$.

We have $t(9n) = r(72n+3)$, so we need to consider

$$72n+3 = k^2+l^2+m^2.$$

with k, l, m odd and positive. Modulo 18, this becomes

$$k^2+l^2+m^2 \equiv 3 \pmod{18}.$$

The only solutions of this are, together with permutations,

$$(k,l,m) \equiv (\pm1,\pm1,\pm1), (\pm5,\pm5,\pm5), (\pm7,\pm7,\pm7) \text{ or } (\pm1,\pm5,\pm7) \pmod{18}.$$

Conversely, if $(k,l,m) \equiv (\pm1,\pm1,\pm1), (\pm5,\pm5,\pm5), (\pm7,\pm7,\pm7)$ or $(\pm1,\pm5,\pm7) \pmod{18}$ then $k^2+l^2+m^2 \equiv 3 \pmod{72}$. Thus,

$$\sum_{n\geq 0} t(9n)q^{72n+3} = \sum_{n\geq 0} r(72n+3)q^{72n+3}$$

$$
\begin{aligned}
&= \sum q^{(18k+1)^2+(18l+1)^2+(18m+1)^2} \\
&\quad + \sum q^{(18k-5)^2+(18l-5)^2+(18m-5)^2} \\
&\quad + \sum q^{(18k+7)^2+(18l+7)^2+(18m+7)^2} \\
&\quad + 6 \sum q^{(18k+1)^2+(18l-5)^2+(18m+7)^2} \\
&= \sum_{k+l+m \equiv 0 \pmod 3} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}.
\end{aligned}
$$

If we set $k+l+m = 3u,\ 2k-l-m = 3v,\ l-m = w$ then $v \equiv w \pmod 2$,

$$
\begin{aligned}
&(6k+1)^2 + (6l+1)^2 + (6m+1)^2 \\
&\qquad = 12(k+l+m)^2 + 6(2k-l-m)^2 + 18(l-m)^2 + 12(k+l+m) + 3 \\
&\qquad = 12(3u)^2 + 6(3v)^2 + 18w^2 + 12(3u) + 3 \\
&\qquad = 108u^2 + 54v^2 + 18w^2 + 36u + 3
\end{aligned}
$$

and

$$
\sum_{n \geq 0} t(9n) q^{72n+3} = \sum_{v \equiv w \pmod 2} q^{108u^2+54v^2+18w^2+36u+3}.
$$

That is,

$$
\begin{aligned}
\sum_{n \geq 0} t(9n) q^n &= \sum q^{(3u^2+u)/2} \sum_{v \equiv w \pmod 2} q^{(3v^2+w^2)/4} \\
&= \sum q^{(3u^2+u)/2} \sum q^{(3(s+t)^2+(s-t)^2)/4} \\
&= \sum q^{(3u^2+u)/2} \sum q^{s^2+st+t^2} \\
&= (-q;q^3)_\infty (-q^2;q^3)_\infty (q^3;q^3)_\infty \left\{ 1 + 6 \sum_{n \geq 1} \left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}} \right) \right\} \\
&= \frac{(q^3;q^3)_\infty}{(q;q^6)_\infty (q^5;q^6)_\infty} \left\{ 1 + 6 \sum_{n \geq 1} \left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^3 n - 1}{1-q^{3n-1}} \right) \right\}.
\end{aligned}
$$

(For a proof that

$$
\sum q^{s^2+st+t^2} = 1 + 6 \sum_{n \geq 1} \left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}} \right)
$$

see the Appendix.)

In similar fashion,

$$\begin{aligned}\sum_{n\geq 0} t(9n+6)q^{72n+51} &= \sum_{n\geq 0} r(72n+51)q^{72n+51}\\ &= 3\sum q^{(18k+1)^2+(18l-5)^2+(18m-5)^2}\\ &\quad + 3\sum q^{(18k-5)^2+(18l+7)^2+(18m+7)^2}\\ &\quad + 3\sum q^{(18k+7)^2+(18l+1)^2+(18m+1)^2}\\ &= \sum_{k+l+m\equiv 1 \pmod 3} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}.\end{aligned}$$

If we set $k+l+m=3u+1,\ 2k-l-m=3v-1,\ l-m=w$ then $v\not\equiv w \pmod 2$,

$$\begin{aligned}&(6k+1)^2+(6l+1)^2+(6m+1)^2\\ &\qquad = 12(k+l+m)^2+6(2k-l-m)^2+18(l-m)^2+12(k+l+m)+3\\ &\qquad = 12(3u+1)^2+6(3v-1)^2+18w^2+12(3u+1)+3\\ &\qquad = 108u^2+54v^2+18w^2+108u-36v+33\end{aligned}$$

and

$$\sum_{n\geq 0} t(9n+6)q^{72n+51} = \sum_{v\not\equiv w \pmod 2} q^{108u^2+54v^2+18w^2+108u-36v+33}.$$

That is,

$$\begin{aligned}\sum_{n\geq 0} t(9n+6)q^n &= \sum q^{3((u^2+u)/2)} \sum_{v\not\equiv w \pmod 2} q^{(3v^2+w^2-2v-1)/4}\\ &= \sum q^{3((u^2+u)/2)} \sum q^{(3(s+t+1)^2+(s-t)^2-2(s+t+1)-1)/4}\\ &= \sum q^{3((u^2+u)/2)} \sum q^{s^2+st+t^2+s+t}\\ &= 2\frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty}\times 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\\ &= 6\times\frac{(q^3;q^6)_\infty^2(q^6;q^6)_\infty^4}{(q;q)_\infty}.\end{aligned}$$

(For a proof that

$$\sum q^{s^2+st+t^2+s+t} = 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}$$

see the Appendix.)

4. Proof of the main result in the case $\lambda = 1$.

We want to show that

$$\begin{aligned} t(27n+12) &= 3 \times t(3n+1), \\ t(27n+21) &= 5 \times t(3n+2), \\ t(81n+3) &= 4 \times t(9n) \end{aligned}$$

and

$$t(81n+57) = 4 \times t(9n+6).$$

As we shall see, it is crucial for us to prove

$$(*) \qquad \sum_{n\geq 0} t(27n+3)q^n = 4\sum_{n\geq 0} t(3n)q^n - 3\sum_{n\geq 0} t(n)q^{3n+1}.$$

The third and fourth of the above relations follow from (*) on comparing coefficients of q^{3n} and q^{3n+2} respectively. Also, we shall require (*) in Section 5 to prove the main result for $\lambda > 1$.

We shall prove (*) in full detail, and outline the proofs of the first and second relations above.

First,

$$\begin{aligned} \sum_{n\geq 0} t(3n)q^{24n+3} &= \sum_{n\geq 0} r(24n+3)q^{24n+3} \\ &= \sum q^{(6k+1)^2+(6l+1)^2+(6m+1)^2} \\ &\quad + \sum_{k,l,m\geq 0} q^{(6k+3)^2+(6l+3)^2+(6m+3)^2} \\ &= q^3\sum q^{36k^2+12k+36l^2+12l+36m^2+12m} \\ &\quad + q^{27}\sum_{k,l,m\geq 0} q^{36k^2+36k+36l^2+36l+36m^2+36m} \end{aligned}$$

so

$$\sum_{n\geq 0} t(3n)q^n = \sum q^{(3k^2+k)/2+(3l^2+l)/2+(3m^2+m)/2}$$

$$+\,q\sum_{k,l,m\geq 0} q^{3((k^2+k)/2)+3((l^2+l)/2)+3((m^2+m)/2)}$$

$$=\sum q^{(3k^2+k)/2+(3l^2+l)/2+(3m^2+m)/2}+\sum_{n\geq 0} t(n)q^{3n+1}.$$

Next, $t(27n+3)=r(216n+27)$, so we need to consider

$$216n+27=k^2+l^2+m^2$$

with k,l,m odd and positive. Modulo 54, this becomes

$$k^2+l^2+m^2\equiv 27 \pmod{54}$$

and we find that

$$\sum_{n\geq 0} t(27n+3)q^{216n+27}=\sum_{n\geq 0} r(216n+27)q^{216n+27}$$
$$=3\Big\{\sum q^{(54k-23)^2+(54l-17)^2+(54m-17)^2}+\sum q^{(54k-17)^2+(54l+25)^2+(54m+25)^2}$$
$$+\sum q^{(54k-11)^2+(54l+13)^2+(54m+13)^2}+\sum q^{(54k-5)^2+(54l+1)^2+(54m+1)^2}$$
$$+\sum q^{(54k+1)^2+(54l-11)^2+(54m-11)^2}+\sum q^{(54k+7)^2+(54l-23)^2+(54m-23)^2}$$
$$+\sum q^{(54k+13)^2+(54l+19)^2+(54m+19)^2}+\sum q^{(54k+19)^2+(54l+7)^2+(54m+7)^2}$$
$$+\sum q^{(54k+25)^2+(54l-5)^2+(54m-5)^2}\Big\}$$
$$+6\Big\{\sum q^{(54k-23)^2+(54l+19)^2+(54m+1)^2}+\sum q^{(54k-17)^2+(54l+7)^2+(54m-11)^2}$$
$$+\sum q^{(54k-11)^2+(54l-5)^2+(54m-23)^2}+\sum q^{(54k-5)^2+(54l-17)^2+(54m+19)^2}$$
$$+\sum q^{(54l+1)^2+(54l+25)^2+(54m+7)^2}+\sum q^{(54k+7)^2+(54l+13)^2+(54m-5)^2}$$
$$+\sum q^{(54k+13)^2+(54l+1)^2+(54m-17)^2}+\sum q^{(54k+19)^2+(54l-11)^2+(54m+25)^2}$$
$$+\sum q^{(54k+25)^2+(54l-23)^2+(54m+13)^2}\Big\}$$
$$+\sum q^{(18k+3)^2+(18l+3)^2+(18m+3)^2}$$
$$+\sum_{k,l,m\geq 0} q^{(18k+9)^2+(18l+9)^2+(18m+9)^2}.$$

The last two terms constitute

$$\sum_{n\geq 0} t(3n)q^{216n+27}$$

so we have

$$\sum_{n\geq 0} t(27n+3)q^{216n+27} - \sum_{n\geq 0} t(3n)q^{216n+27}$$
$$= 3 \sum_{\substack{4k+l+m\equiv -4 \pmod 9 \\ l-m\equiv 0 \pmod 3}} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}.$$

If we set $4k+l+m = 9u-4,\ -k+2l+2m = 9v+1,\ l-m = 3w$ then $u \equiv w \pmod 2$,

$$(6k+1)^2 + (6l+1)^2 + (6m+1)^2$$
$$= 2(4k+l+m)^2 + 4(-k+2l+2m)^2 + 18(l-m)^2 + 4(4k+l+m) + 4(-k+2l+2m) + 3$$
$$= 2(9u-4)^2 + 4(9v+1)^2 + 18(3w)^2 + 4(9u-4) + 4(9v+1) + 3$$
$$= 162u^2 + 324v^2 + 162w^2 - 108u + 108v + 27$$

and

$$\sum_{n\geq 0} t(27n+3)q^{216n+27} - \sum_{n\geq 0} t(3n)q^{216n+27}$$

$$= 3 \sum_{u\equiv w \pmod 2} q^{162u^2+324v^2+162w^2-108u+108v+27}.$$

That is,

$$\sum_{n\geq 0} t(27n+3)q^n - \sum_{n\geq 0} t(3n)q^n$$

$$= 3 \sum q^{(3v^2+v)/2} \sum_{u\equiv w \pmod 2} q^{(3u^2+3w^2-2u)/4}$$

$$= 3 \sum q^{(3v^2+v)/2} \sum q^{(3(s+t)^2+3(s-t)^2-2(s+t))/4}$$

$$= 3 \sum q^{(3v^2+v)/2+(3s^3-s)/2+(3t^2-t)/2}$$

$$= 3 \left(\sum_{n\geq 0} t(3n)q^n - \sum_{n\geq 0} t(n)q^{3n+1} \right),$$

so

$$\sum_{n\geq 0} t(27n+3)q^n = 4 \sum_{n\geq 0} t(3n)q^n - 3 \sum_{n\geq 0} t(n)q^{3n+1}.$$

In the same way, we can show

$$\sum_{n\geq 0} t(27n+12)q^{216n+99} - \sum_{n\geq 0} t(3n+1)q^{216n+99}$$

$$= 3 \sum_{\substack{4k+l+m\equiv 2 \pmod 9 \\ l-m\equiv 0 \pmod 3}} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}$$

from which it follows that

$$\sum_{n\geq 0} t(27n+12)q^{n} - \sum_{n\geq 0} t(3n+1)q^{n}$$

$$= 3 \sum q^{3((v^2+v)/2)} \sum_{u\equiv w \pmod 2} q^{(3u^2+3w^2+2u)/4}$$

$$= 6 \sum_{v\geq 0} q^{3((v^2+v)/2)+(3s^2+s)/2+(3t^2+t)/2}$$

$$= 2 \sum_{n\geq 0} t(3n+1)q^{n}$$

and

$$\sum_{n\geq 0} t(27n+21)q^{216n+171} - \sum_{n\geq 0} t(3n+2)q^{216n+171}$$

$$= 3 \sum_{\substack{4k+l+m\equiv -1 \pmod 9 \\ l-m\equiv 0 \pmod 3}} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}$$

from which it follows that

$$\sum_{n\geq 0} t(27n+21)q^{n} - \sum_{n\geq 0} t(3n+2)q^{n}$$

$$= 3 \sum q^{(3v^2+v)/2} \sum_{u\not\equiv w \pmod 2} q^{(3u^2+3w^2-3)/4}$$

$$= 3 \sum q^{(3v^2+v)/2+3((s^2+s)/2)+3((t^2+t)/2)}$$

$$= 12 \sum_{s,t\geq 0} q^{(3v^2+v)/2+3((s^2+s)/2)+3((t^2+t)/2)}$$

$$= 4 \sum_{n\geq 0} t(3n+2)q^n.$$

Thus the first and second relations hold.

5. Proof of the main result for $\lambda > 1$.

We have shown that

$$r(9^\lambda n) = \begin{cases} 3^\lambda \times r(n) & \text{if } n \equiv 11 \pmod{24} \\ (2\times 3^\lambda - 1) \times r(n) & \text{if } n \equiv 19 \pmod{24} \\ ((3^{\lambda+1}-1)/2) \times r(n) & \text{if } n \equiv 3 \text{ or } 51 \pmod{72} \end{cases}$$

for $\lambda = 1$, and it is trivially true for $\lambda = 0$.

Suppose now that $\lambda \geq 2$, and that the result is true for $\lambda - 1$ and $\lambda - 2$.

We have

$$\sum_{n\geq 0} t(27n+3)q^n = 4\sum_{n\geq 0} t(3n)q^n - 3\sum_{n\geq 0} t(n)q^{3n+1}.$$

If we consider the coefficient of q^{3n+1} we obtain

$$t(81n+30) = 4\times t(9n+3) - 3\times t(n)$$

or,

$$r(648n+243) = 4\times r(72n+27) - 3\times r(8n+3),$$

or,

$$r(81(8n+3)) = 4\times r(9(8n+3)) - 3\times r(8n+3).$$

It follows that for all n,

$$r(81n) = 4\times r(9n) - 3\times r(n),$$

for if $n \not\equiv 3 \pmod 8$ then $81n \equiv 9n \equiv n \not\equiv 3 \pmod 8$ so $r(81n) = r(9n) = r(n) = 0$ and the result is trivially true.

It follows that for every $\lambda \geq 2$,

$$r(9^\lambda n) = 4\times r(9^{\lambda-1}n) - 3\times r(9^{\lambda-2}n).$$

Thus, if $n \equiv 11 \pmod{24}$,

$$\begin{aligned} r(9^\lambda n) &= 4\times r(9^{\lambda-1}n) - 3\times r(9^{\lambda-2}n) \\ &= 4\times 3^{\lambda-1}r(n) - 3\times 3^{\lambda-2}r(n) \end{aligned}$$

$$= (4 \times 3^{\lambda-1} - 3 \times 3^{\lambda-2})r(n)$$
$$= 3^{\lambda} r(n),$$

if $n \equiv 19 \pmod{24}$,

$$\begin{aligned} r(9^{\lambda} n) &= 4 \times r(9^{\lambda-1} n) - 3 \times r(9^{\lambda-2} n) \\ &= 4 \times (2 \times 3^{\lambda-1} - 1) r(n) - 3 \times (2 \times 3^{\lambda-2} - 1) r(n) \\ &= (4 \times (2 \times 3^{\lambda-1} - 1) - 3 \times (2 \times 3^{\lambda-2} - 1)) r(n) \\ &= (2 \times 3^{\lambda} - 1) r(n), \end{aligned}$$

while if $n \equiv 3$ or $51 \pmod{72}$,

$$\begin{aligned} r(9^{\lambda} n) &= 4 \times r(9^{\lambda-1} n) - 3 \times r(9^{\lambda-2} n) \\ &= 4 \times ((3^{\lambda} - 1)/2) r(n) - 3 \times ((3^{\lambda-1} - 1)/2) r(n) \\ &= (4 \times ((3^{\lambda} - 1)/2) - 3 \times ((3^{\lambda-1} - 1)/2)) r(n) \\ &= ((3^{\lambda+1} - 1)/2) r(n), \end{aligned}$$

and our result is proved.

Appendix.

In [3], a generalisation of the following identity was established:

$$\sum a^{s+t+u} q^{(s^2+t^2+u^2)/2}$$
$$= (-aq^{\frac{1}{2}}; q)_\infty^3 (-a^{-1} q^{\frac{1}{2}}; q)_\infty^3 (q; q)_\infty^3 =$$
$$= c_0 (-a^3 q^{\frac{3}{2}}; q^3)_\infty (-a^{-3} q^{\frac{3}{2}}; q^3)_\infty (q^3; q^3)_\infty$$
$$+ c_1 \left\{ a(-a^3 q^{\frac{5}{2}}; q^3)_\infty (-a^{-3} q^{\frac{1}{2}}; q^3)_\infty (q^3; q^3)_\infty + a^{-1} (-a^3 q^{\frac{1}{2}}; q^3)_\infty (-a^{-3} q^{\frac{5}{2}}; q^3)_\infty (q^3; q^3)_\infty \right\}$$

where

$$c_0 = 1 + 6 \sum_{n \geq 1} \left(\frac{q^{3n-2}}{1 - q^{3n-2}} - \frac{q^{3n-1}}{1 - q^{3n-1}} \right) \text{ and } c_1 = 3q^{\frac{1}{2}} \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty}.$$

Thus we have

$$\sum q^{s^2+st+t^2} = \sum q^{(s^2+t^2+(-s-t)^2)/2}$$

$$= \sum_{s+t+u=0} q^{(s^2+t^2+u^2)/2}$$

$$
\begin{aligned}
&= \text{the constant term in } \sum a^{s+t+u} q^{(s^2+t^2+u^2)/2} \\
&= c_0 \\
&= 1 + 6 \sum_{n \geq 1} \left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum q^{s^2+st+t^2+s+t} &= \sum q^{(s^2+t^2+(-s-t-1)^2-1)/2} \\
&= q^{-\frac{1}{2}} \sum_{s+t+u=-1} q^{(s^2+t^2+u^2)/2} \\
&= q^{-\frac{1}{2}} \times \text{the coefficient of } a^{-1} \text{ in } \sum a^{s+t+u} q^{(s^2+t^2+u^2)/2} \\
&= q^{-\frac{1}{2}} c_1 \\
&= 3 \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.
\end{aligned}
$$

References

[**1**] George E. Andrews, EΥPHKA! num = Δ + Δ + Δ, J. Number Theory **23** (1986), 285-293.

[**2**] C.F.Gauss, “Werke”, Bd. 10, p. 497, Teubner, Leipzig, 1917.

[**3**] Michael Hirschhorn, Frank Garvan and Jon Borwein, Cubic analogues of the Jacobian theta function $\theta(z,q)$, *Can. J. Math.* **45** (1993), 673–694.