JACOBI'S TWO-SQUARE THEOREM AND RELATED IDENTITIES

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(RAMA83-97)

Abstract. We show that Jacobi's two–square theorem is an almost immediate consequence of a famous identity of his, and draw combinatorial conclusions from two identities of Ramanujan.

Keywords: Jacobi's two–square theorem, identities of Ramanujan
Classification: 11E25

1. Introduction

Jacobi's two-square theorem states that

The number of representations of n as a sum of two squares is 4 times the difference between the number of divisors of n congruent to 1 modulo 4 and the number of divisors of n congruent to 3 modulo 4.

This theorem is equivalent to the q-series identity

$$\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^2 = 1 + 4\sum_{n\geq 0}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right). \tag{1}$$

It is possible to give a proof of (1) directly from Jacobi's triple product identity [2], Theorem 352,

$$\prod_{n\geq 1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1-q^{2n}) = \sum_{-\infty}^{\infty} a^n q^{n^2}. \tag{2}$$

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Indeed, I gave such a proof in [4]. The proof there consists of two parts. First it is shown that

(3)
$$\prod_{n\geq 1}(1-q^n)^3=\prod_{n\geq 1}(1+q^{4n-3})(1+q^{4n-1})(1-q^{4n})\left\{1-4\sum_{n\geq 0}\left(\frac{q^{4n+1}}{1+q^{4n+1}}-\frac{q^{4n+3}}{1+q^{4n+3}}\right)\right\}$$

and then it is shown that (3) can be transformed to give (1).

In this note I intend to do several things. I show how (3) can be obtained from Jacobi's celebrated identity [3], Theorem 357

$$\prod_{n\geq 1}(1-q^n)^3=\sum_{n\geq 0}(-1)^n(2n+1)q^{\frac{1}{2}(n^2+n)}, \tag{4}$$

(which itself, of course, follows from (2)), and I streamline the derivation of (1) from (3). Thus we obtain an extremely short proof of Jacobi's two-square theorem.

In the same manner I show that two identities of Ramanujan [1], pp. 114-115 which can be cast as

$$\prod_{n\geq 1}(1-q^{2n-1})^5(1-q^{2n})^3=\sum_{-\infty}^{\infty}(6n+1)q^{(3n^2+n)/2} \tag{5}$$

and

$$\prod_{n\geq 1}(1-q^{4n-3})^2(1-q^{4n-2})(1-q^{4n-1})^2(1-q^{4n})^3=\sum_{-\infty}^{\infty}(3n+1)q^{3n^2+2n} \tag{6}$$

lead to companion identities to (1), namely

$$\left(\sum_{-\infty}^{\infty}q^{n^2}\right)^3\Big/\sum_{-\infty}^{\infty}q^{3n^2}=1+6\sum_{n\geq 0}(-1)^n\left(\frac{q^{3n+1}}{1-(-1)^nq^{3n+1}}+\frac{q^{3n+2}}{1+(-1)^nq^{3n+2}}\right) \tag{7}$$

and

$$\left(\sum_{n\geq 0}q^{(n^2+n)/2}\right)^3\Big/\sum_{n\geq 0}q^{3(n^2+n)/2}=1+3\sum_{n\geq 0}\left(\frac{q^{6n+1}}{1-q^{6n+1}}-\frac{q^{6n+5}}{1-q^{6n+5}}\right). \tag{8}$$

Identities (7) and (8) have the following combinatorial interpretations.

Theorem 1. The number of representations of n as a sum of three squares is

$$6\left(d^*_{1,3}(n) - d^*_{-1,3}(n)\right) + 12\sum_{k>0}\left(d^*_{1,3}(n-3k^2) - d^*_{-1,3}(n-3k^2)\right) + 2\delta$$

where for $i = \pm 1$ and $n > 0$,

$$d^*_{i,3}(n) = \sum_{\substack{d|n \\ d\equiv i \pmod 3}} (-1)^{\frac{n}{d}\left(\frac{d-i}{3}\right)}$$

and $\delta = 0$ unless n is three times a square, in which case $\delta = 1$.

Theorem 2. The number of representations of n as a sum of three triangular numbers is

$$3\sum_{k\geq 0}\left(d_{1,6}(n-3(k^2+k)/2) - d_{-1,6}(n-3(k^2+k)/2)\right) + \delta$$

where for $i = \pm 1$ and $n > 0$

$$d_{i,6}(n) = \sum_{\substack{d|n \\ d\equiv i \pmod 6}} 1$$

and $\delta = 0$ unless n is three times a triangle, in which case $\delta = 1$.

Theorem 2 was found by John A. Ewell [2].

2. Proofs

We have

$$\prod_{n\geq 1}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2},$$

which we cunningly write

$$\prod_{n\geq 1}(1-q^n)^3 = \sum_{-\infty}^{\infty}(4n+1)q^{2n^2+n}.$$

Now, it follows from (2) that

$$\sum_{-\infty}^{\infty} a^{4n+1}q^{2n^2+n} = a\prod_{n\geq 1}(1+a^4q^{4n-1})(1+a^{-4}q^{4n-3})(1-q^{4n}).$$

If we differentiate this with respect to a, then set $a = 1$, we find

$$\sum_{-\infty}^{\infty}(4n+1)q^{2n^2+n} = \prod_{n\geq 1}(1+q^{4n-3})(1+q^{4n-1})(1-q^{4n})\left\{1-4\sum_{n\geq 0}\left(\frac{q^{4n+1}}{1+q^{4n+1}}-\frac{q^{4n+3}}{1+q^{4n+3}}\right)\right\}.$$

Thus we have

$$\prod_{n\geq 1}(1-q^n)^3 = \prod_{n\geq 1}(1+q^{4n-3})(1+q^{4n-1})(1-q^{4n})\left\{1-4\sum_{n\geq 0}\left(\frac{q^{4n+1}}{1+q^{4n+1}}-\frac{q^{4n+3}}{1+q^{4n+3}}\right)\right\},$$

which is (3).

We now proceed to prove (1) from (3).
The product on the left of (3) can be written

$$\prod_{n\geq 1}(1-q^{2n-1})^3(1-q^{2n})^3$$

and the product on the right can be written

$$\prod_{n\geq 1}(1+q^{2n-1})(1-q^{4n}) = \prod_{n\geq 1}(1-q^{4n-2})(1-q^{4n})/(1-q^{2n-1}) = \prod_{n\geq 1}(1-q^{2n})/(1-q^{2n-1}),$$

so we have

$$\prod_{n\geq 1}(1-q^{2n-1})^3(1-q^{2n})^3 = \prod_{n\geq 1}\frac{(1-q^{2n})}{(1-q^{2n-1})}\left\{1-4\sum_{n\geq 0}\left(\frac{q^{4n+1}}{1+q^{4n+1}}-\frac{q^{4n+3}}{1+q^{4n+3}}\right)\right\}.$$

Thus we find

$$\prod_{n\geq 1}(1-q^{2n-1})^4(1-q^{2n})^2 = 1-4\sum_{n\geq 0}\left(\frac{q^{4n+1}}{1+q^{4n+1}}-\frac{q^{4n+3}}{1+q^{4n+3}}\right)$$

or,

$$\left(\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right)^2 = 1-4\sum_{n\geq 0}\left(\frac{q^{4n+1}}{1+q^{4n+1}}-\frac{q^{4n+3}}{1+q^{4n+3}}\right).$$

If we put $-q$ for q, we obtain

$$\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^2 = 1 + 4\sum_{n\geq 0}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right),$$

which is (1), and the proof of Jacobi's two-square theorem is complete!

We now show that (5) leads to (7), (6) to (8). We have from (2) that

$$\sum_{-\infty}^{\infty} a^{6n+1} q^{(3n^2+n)/2} = a\prod_{n\geq 1}(1+a^6q^{3n-1})(1+a^{-6}q^{3n-2})(1-q^{3n}).$$

If we differentiate this with respect to a, then set $a=1$, we find

$$\sum_{-\infty}^{\infty}(6n+1)q^{(3n^2+n)/2} = \prod_{n\geq 1}(1+q^{3n-2})(1+q^{3n-1})(1-q^{3n})\left\{1 - 6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1+q^{3n+1}} - \frac{q^{3n+2}}{1+q^{3n+2}}\right)\right\}.$$

Thus (5) becomes

(9)

$$\prod_{n\geq 1}(1-q^{2n-1})^5(1-q^{2n})^3$$
$$= \prod_{n\geq 1}(1+q^{3n-2})(1+q^{3n-1})(1-q^{3n})\left\{1 - 6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1+q^{3n+1}} - \frac{q^{3n+2}}{1+q^{3n+2}}\right)\right\}.$$

Now, the product on the right of (9) can be written

$$\prod_{n\geq 1}\frac{(1+q^n)(1-q^{3n})}{(1+q^{3n})} = \prod_{n\geq 1}\frac{(1-q^{6n-3})(1-q^{3n})}{(1-q^{2n-1})} = \prod_{n\geq 1}(1-q^{6n-3})^2(1-q^{6n})/\prod_{n\geq 1}(1-q^{2n-1}),$$

so (9) becomes

$$\prod_{n\geq 1}(1-q^{2n-1})^6(1-q^{2n})^3/\prod_{n\geq 1}(1-q^{6n-3})^2(1-q^{6n}) = 1 - 6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1+q^{3n+1}} - \frac{q^{3n+2}}{1+q^{3n+2}}\right),$$

or,

$$\left(\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right)^3 / \sum_{-\infty}^{\infty}(-1)^n q^{3n^2} = 1 - 6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1+q^{3n+1}} - \frac{q^{3n+2}}{1+q^{3n+2}}\right).$$

If we now replace q by $-q$, we obtain (7).

Similarly, it follows from (2) that

$$\sum_{-\infty}^{\infty} a^{3n+1} q^{3n^2+2n} = a \prod_{n\geq 1} (1+a^3 q^{6n-1})(1+a^{-3} q^{6n-5})(1-q^{6n}).$$

If we differentiate this with respect to a, then set $a = 1$, we obtain

$$\sum_{-\infty}^{\infty} (3n+1) q^{3n^2+2n} = \prod_{n\geq 1} (1+q^{6n-5})(1+q^{6n-1})(1-q^{6n}) \left\{ 1 - 3 \sum_{n\geq 0} \left(\frac{q^{6n+1}}{1+q^{6n+1}} - \frac{q^{6n+5}}{1+q^{6n+5}} \right) \right\}.$$

Thus (6) becomes

(10)

$$\prod_{n\geq 1} (1-q^{4n-3})^2 (1-q^{4n-2})(1-q^{4n-1})^2 (1-q^{4n})^3$$
$$= \prod_{n\geq 1} (1+q^{6n-5})(1+q^{6n-1})(1-q^{6n}) \left\{ 1 - 3 \sum_{n\geq 0} \left(\frac{q^{6n+1}}{1+q^{6n+1}} - \frac{q^{6n+5}}{1+q^{6n+5}} \right) \right\}.$$

The product on the left of (10) can be written

$$\prod_{n\geq 1} \frac{(1-q^{2n-1})^2 (1-q^{2n})^3}{(1-q^{4n-2})^2} = \prod_{n\geq 1} \frac{(1-q^{2n})^3}{(1+q^{2n-1})^2},$$

while the product on the right can be written

$$\prod_{n\geq 1} \frac{(1+q^{2n-1})(1-q^{6n})}{(1+q^{6n-3})}$$

so (10) becomes

$$\prod_{n\geq 1} \frac{(1-q^{2n})^3}{(1+q^{2n-1})^3} \Big/ \prod_{n\geq 1} \frac{(1-q^{6n})}{(1+q^{6n-3})} = 1 - 3 \sum_{n\geq 0} \left(\frac{q^{6n+1}}{1+q^{6n+1}} - \frac{q^{6n+5}}{1+q^{6n+5}} \right).$$

If we now put $-q$ for q and use the fact that

$$\prod_{n\geq 1} \frac{(1-q^{2n})}{(1-q^{2n-1})} = \sum_{n\geq 0} q^{(n^2+n)/2},$$

we obtain (8).

References

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