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ANOTHER SHORT PROOF OF RAMANUJAN'S MOD 5 PARTITION CONGRUENCE, AND MORE

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We present another novel short proof of Ramanujan's partition congruence

$$p(5n+4) \equiv 0 \pmod 5$$

in addition to that given by John L. Drost [2], and go on to prove rather more.

Ramanujan made the remarkable observation from a table of values of $p(n)$, the number of partitions of n, that $p(5n+4)$ is divisible by 5. He observed and conjectured much more, and his conjectures turned out in the main to be correct. He gave a simple proof, based on identities of Euler and Jacobi, of the above conjecture, and his proof is essentially the one reproduced in Hardy and Wright [3] and referred to by Drost. Ramanujan's proof relies on manipulating power series, and considering coefficients modulo 5. It is my intention to give a proof of a similar sort, using only the identity of Jacobi mentioned above, and which is more transparent than that of Ramanujan. And further, with a little extra work including the use of Jacobi's triple-product identity, we prove remarkable congruences for the partition function due to Atkin and Swinnerton-Dyer.

As is usual, write $(q)_\infty = \prod_{n\geq 1}(1-q^n)$. Then

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q)_\infty}.$$

We begin with Jacobi's identity [3, Theorem 357],

$$(q)_\infty^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)/2}.$$

Each coefficient is congruent modulo 5 to 0, ± 1 or ± 2. Specifically, the coefficient is congruent to 1 when $n \equiv 0$ or $9 \pmod{10}$, -1 when $n \equiv 4$ or $5 \pmod{10}$, $+2$ when $n \equiv 1$

or 8 (mod 10), -2 when $n \equiv 3$ or 6 (mod 10) and 0 when $n \equiv 2$ or 7 (mod 10). Thus we find that, modulo 5,

$$(q)_\infty^3 \equiv \sum_{n\ge 0} q^{10n(10n+1)/2} - \sum_{n\ge 0} q^{(10n+4)(10n+5)/2} - \sum_{n\ge 0} q^{(10n+5)(10n+6)/2} + \sum_{n\ge 0} q^{(10n+9)(10n+10)/2}$$
$$+ 2\sum_{n\ge 0} q^{(10n+1)(10n+2)/2} - 2\sum_{n\ge 0} q^{(10n+3)(10n+4)/2} - 2\sum_{n\ge 0} q^{(10n+6)(10n+7)/2} + 2\sum_{n\ge 0} q^{(10n+8)(10n+9)/2}$$

$$\equiv \sum_{n\ge 0} q^{50n^2+5n} - \sum_{n\ge 0} q^{50n^2+45n+10} - \sum_{n\ge 0} q^{50n^2+55n+15} + \sum_{n\ge 0} q^{50n^2+95n+45}$$
$$+ 2\sum_{n\ge 0} q^{50n^2+15n+1} - 2\sum_{n\ge 0} q^{50n^2+35n+6} - 2\sum_{n\ge 0} q^{50n^2+65n+21} + 2\sum_{n\ge 0} q^{50n^2+85n+36}.$$

Observe that in the first four sums the powers of q are congruent to 0 (mod 5) while in the latter four sums the powers of q are congruent to 1 (mod 5). Thus we have

$$(q)_\infty^3 \equiv X + 2qY$$

where each of $X,\ Y$ is a series in powers of q^5.

Also

$$(q)_\infty^5 = \prod_{n\ge 1}(1-q^n)^5 = \prod_{n\ge 1}(1-5q^n+10q^{2n}-10q^{3n}+5q^{4n}-q^{5n}) \equiv \prod_{n\ge 1}(1-q^{5n}) \equiv (q^5)_\infty.$$

Thus

$$\sum_{n\ge 0} p(n)q^n = \frac{1}{(q)_\infty} = \frac{(q)_\infty^9}{(q)_\infty^{10}} = \frac{((q)_\infty^3)^3}{((q)_\infty^5)^2} \equiv \frac{((q)_\infty^3)^3}{(q^5)_\infty^2} \equiv \frac{(X+2qY)^3}{(q^5)_\infty^2}$$
$$\equiv \frac{X^3+6qX^2Y+12q^2XY^2+8q^3Y^3}{(q^5)_\infty^2}$$
$$\equiv \frac{X^3+qX^2Y+2q^2XY^2+3q^3Y^3}{(q^5)_\infty^2}.$$

Comparing terms containing powers of q congruent to 4 modulo 5 on both sides, we see that

$$\sum_{n\ge 0} p(5n+4)q^{5n+4} \equiv 0 \pmod 5. \quad \blacksquare$$

Notice that, at no extra cost, we obtain the congruences

$$\sum_{n\ge 0} p(5n)q^{5n} \equiv X^3/(q^5)_\infty^2, \qquad \sum_{n\ge 0} p(5n+1)q^{5n+1} \equiv qX^2Y/(q^5)_\infty^2,$$
$$\sum_{n\ge 0} p(5n+2)q^{5n+2} \equiv 2q^2XY^2/(q^5)_\infty^2 \quad\text{and}\quad \sum_{n\ge 0} p(5n+3)q^{5n+3} \equiv 3q^3Y^3/(q^5)_\infty^2.$$

It is not hard to show that each of $X,\ Y$ is an infinite product. Indeed, as we shall see,

$$X = \prod_{n\ge 1}(1-q^{25n-15})(1-q^{25n-10})(1-q^{25n}),$$
$$Y = \prod_{n\ge 1}(1-q^{25n-20})(1-q^{25n-5})(1-q^{25n}).$$

It follows that

$$\sum_{n\ge 0} p(5n)q^n \equiv \prod_{n\ge 1} \frac{(1-q^{5n-3})(1-q^{5n-2})(1-q^{5n})}{(1-q^{5n-4})^2(1-q^{5n-1})^2},$$

$$\sum_{n\ge 0} p(5n{+}1)q^n \equiv \prod_{n\ge 1} \frac{(1-q^{5n})}{(1-q^{5n-4})(1-q^{5n-1})}, \quad \sum_{n\ge 0} p(5n{+}2)q^n \equiv 2\prod_{n\ge 1} \frac{(1-q^{5n})}{(1-q^{5n-3})(1-q^{5n-2})}$$

$$\text{and} \quad \sum_{n\ge 0} p(5n+3)q^n \equiv 3\prod_{n\ge 1} \frac{(1-q^{5n-4})(1-q^{5n-1})(1-q^{5n})}{(1-q^{5n-3})^2(1-q^{5n-2})^2}.$$

These remarkable results are due to Atkin and Swinnerton-Dyer [1, Theorem 1].

We now show that $X,\ Y$ are infinite products as claimed.
We have

$$X = \sum_{n\ge 0} q^{50n^2+5n} - \sum_{n\ge 0} q^{50n^2+45n+10} - \sum_{n\ge 0} q^{50n^2+55n+15} + \sum_{n\ge 0} q^{50n^2+95n+45}.$$

In the first sum, replace n by $-n$, in the third replace n by $-n-1$ and in the fourth replace n by $n-1$. Then we find

$$\begin{aligned} X &= \sum_{n\le 0} q^{50n^2-5n} - \sum_{n\ge 0} q^{50n^2+45n+10} - \sum_{n\le -1} q^{50n^2+45n+10} + \sum_{n\ge 1} q^{50n^2-5n} \\ &= \sum_{-\infty}^{\infty} q^{50n^2-5n} - \sum_{-\infty}^{\infty} q^{50n^2+45n+10} \\ &= \sum_{-\infty}^{\infty} (-1)^n q^{(25n^2-5n)/2}. \end{aligned}$$

(The terms for n even in the final sum correspond to the first sum on the line above, the terms for n odd to the second sum.)
In the same way, we find

$$\begin{aligned}
Y &= \sum_{n\ge 0} q^{50n^2+15n} - \sum_{n\ge 0} q^{50n^2+35n+5} - \sum_{n\ge 0} q^{50n^2+65n+20} + \sum_{n\ge 0} q^{50n^2+85n+35} \\
&= \sum_{n\le 0} q^{50n^2-15n} - \sum_{n\ge 0} q^{50n^2+35n+5} - \sum_{n\le -1} q^{50n^2+35n+5} + \sum_{n\ge 1} q^{50n^2-15n} \\
&= \sum_{-\infty}^{\infty} q^{50n^2-15n} - \sum_{-\infty}^{\infty} q^{50n^2+35n+5} \\
&= \sum_{-\infty}^{\infty} (-1)^n q^{(25n^2-15n)/2}.
\end{aligned}$$

To complete the proof, we now invoke Jacobi's triple product identity [3, Theorem 352], in the form

$$\sum_{-\infty}^{\infty} (-1)^n a^n q^{(n^2-n)/2} = \prod_{n\ge 1} (1-aq^{n-1})(1-a^{-1}q^n)(1-q^n)$$

with q replaced by q^{25} and a replaced by q^{10} and q^5 respectively.

References

1. A. O. L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, *Proc. London Math. Soc.* 4 (1954), 84-106.

2. John L. Drost, A shorter proof of the Ramanujan congruence mod 5, *This MONTHLY* 104 (1997), 963-964.

3. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th edition, Clarendon Press, Oxford 1979.

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