ON REPRESENTATIONS OF A NUMBER AS A SUM OF THREE SQUARES

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Abstract. We give a variety of results involving $s(n)$, the number of representations of n as a sum of three squares. Using elementary techniques we prove that if $9 \nmid n$,

$$s(9^\lambda n) = \left(\frac{3^{\lambda+1}-1}{2} - \left(\frac{-n}{3}\right) \frac{3^\lambda - 1}{2} \right) s(n);$$

via the theory of modular forms of half integer weight we prove the corresponding result with 3 replaced by p, an odd prime. This leads to a formula for $s(n)$ in terms of $s(n')$, where n' is the square-free part of n.

We also find generating function formulae for various subsequences of $\{s(n)\}$, for instance

$$\sum_{n \geq 0} s(3n+2) q^n = 12 \prod_{n \geq 1} (1+q^{2n-1})^2 (1-q^{6n})^3.$$

1. Introduction

In three recent papers [1],[2],[3] we studied $a_4(n)$, the number of 4-cores of n, and $t(n)$, the number of representations of n as a sum of three triangular numbers. We showed that each of $a_4(n)$ and $t(n)$ satisfies many arithmetic relations. Now we turn our attention to $s(n)$, the number of representations of n as a sum of three integer squares.

Much can be said about $s(n)$. For instance, as we shall show,

$$s(4n) = s(n) \quad \text{and} \quad s(8n+7) = 0,$$

from which it follows that

$$s(4^\lambda(8n+7)) = 0,$$

a well-known result due to Fermat.

Our main result, which we prove using only Jacobi's triple product identity, is the following:
For $\lambda \geq 1$,

$$\begin{aligned} s(9^\lambda(3n+1)) &= (2\times 3^\lambda - 1)\times s(3n+1),\\ s(9^\lambda(3n+2)) &= 3^\lambda \times s(3n+2),\\ s(9^\lambda(9n+3)) &= ((3^{\lambda+1}-1)/2)\times s(9n+3),\\ s(9^\lambda(9n+6)) &= ((3^{\lambda+1}-1)/2)\times s(9n+6). \end{aligned} \tag{1}$$

We shall also establish the generating function formulae

$$\sum_{n\geq 0} s(3n+1)q^n = 6\begin{pmatrix} q^2, q^6, q^6, q^6, q^6, q^6, q^6, q^6, q^{10}, q^{12}, q^{12}, q^{12} \\ q, q^3, q^3, q^3, q^3, q^5, q^7, q^9, q^9, q^9, q^9, q^{11} \end{pmatrix}; q^{12}\Bigg)_\infty,$$

$$\sum_{n\geq 0} s(3n+2)q^n = 12\begin{pmatrix} q^2, q^2, q^6, q^6, q^6, q^6, q^6, q^{10}, q^{10}, q^{12}, q^{12}, q^{12} \\ q, q, q^3, q^3, q^5, q^5, q^7, q^7, q^9, q^9, q^{11}, q^{11} \end{pmatrix}; q^{12}\Bigg)_\infty,$$

$$\sum_{n\geq 0} s(9n+3)q^n = 8\left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^5, q^7, q^{11} \end{matrix}; q^{12}\right)_\infty \left\{1 + 6\sum_{n\geq 1}\left(\frac{q^{6n-4}}{1-q^{6n-4}} - \frac{q^{6n-2}}{1-q^{6n-2}}\right)\right\}$$

and

$$\sum_{n\geq 0} s(9n+6)q^n = 24\left(\begin{matrix} q^6, q^6, q^6, q^6, q^6, q^{12}, q^{12}, q^{12} \\ q^2, q^3, q^3, q^4, q^8, q^9, q^9, q^{10} \end{matrix}; q^{12}\right)_\infty.$$

Here we introduce the notation

$$\left(\begin{matrix} q^{a_1}, q^{a_2}, \cdots, q^{a_l} \\ q^{b_1}, q^{b_2}, \cdots, q^{b_l} \end{matrix}; q^k\right)_\infty = \prod_{n\geq 0} \frac{(1-q^{kn+a_1})(1-q^{kn+a_2})\cdots(1-q^{kn+a_l})}{(1-q^{kn+b_1})(1-q^{kn+b_2})\cdots(1-q^{kn+b_l})}.$$

We shall further show that

$$s(8n+3) = 8t(n) \quad\text{and}\quad s(8n+5) = 24a_4(n).$$

As a consequence of these relations, the main results of [2] and [3] follow from those of this paper.

Upon proving (1), we were led to make the following conjecture. In what follows, p is an odd prime, λ is a positive integer and $\left(\frac{n}{p}\right) = 0$ if $p|n$.

If $p^2 \nmid n$ then

$$s(p^{2\lambda}n) = \left(\frac{p^{\lambda+1}-1}{p-1} - \left(\frac{-n}{p}\right)\frac{p^{\lambda}-1}{p-1}\right)s(n). \tag{2}$$

This together with $s(4n) = s(n)$ yields the formula

$$s(n) = s(n') \prod_{\substack{p|n \\ p \text{ odd}}} \left(\frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-n'}{p}\right)\frac{p^{\lambda_p}-1}{p-1}\right), \tag{3}$$

where n' is the square-free part of n and $p^{2\lambda_p}||n$ or $p^{2\lambda_p+1}||n$.

The referee of an earlier version of this paper has shown that more is true. Indeed,

$$s(p^2n) = \left((p+1) - \left(\frac{-n}{p}\right)\right)s(n) - p\ s(n/p^2), \tag{4}$$

from which it follows by induction that

$$s(p^{2\lambda}n) = \left(\frac{p^{\lambda+1}-1}{p-1} - \left(\frac{-n}{p}\right)\frac{p^{\lambda}-1}{p-1}\right)s(n) - p\frac{p^{\lambda}-1}{p-1}s(n/p^2). \tag{5}$$

The truth of (2) is immediate. The referee has kindly let us reproduce his proof of (4), which he describes as "a very routine application of the theory of modular forms of half integer weight".

In Section 2 we shall give proofs of the relations between $a_4(n)$, $t(n)$ and $s(n)$, and more. In Section 3 we shall prove Fermat's theorem. In Section 4 we prove the case $\lambda = 1$ of (1), and in Section 5 we complete the proof. In Section 6 we obtain our generating function formulae. In the final section we give the proof of the result for general odd prime p.

2. Relations between $a_4(n)$, $t(n)$ and $s(n)$

We start with the observation that

$$\sum_{n=-\infty}^{\infty} q^{n^2} = T_0(q^8) + 2qT_1(q^8) + 2q^4T_4(q^8)$$

where

$$T_0(q) = \sum_{n=-\infty}^{\infty} q^{2n^2} = \left(\begin{matrix} q^4, q^4, q^4, q^8 \\ q^2, q^2, q^6, q^6 \end{matrix} ; q^8 \right)_\infty ,$$

$$T_1(q) = \sum_{n\geq 0} q^{\frac{1}{2}(n^2+n)} = \left(\begin{matrix} q^2 \\ q \end{matrix} ; q^2 \right)_\infty$$

and

$$T_4(q) = \sum_{n\geq 0} q^{2(n^2+n)} = \left(\begin{matrix} q^8 \\ q^4 \end{matrix} ; q^8 \right)_\infty .$$

For, $n^2 \equiv 0, 1$ or $4 \pmod 8$; $n^2 \equiv 0 \pmod 8$ if $n \equiv 0 \pmod 4$, $n^2 \equiv 1 \pmod 8$ if $n \equiv 1 \pmod 2$, $n^2 \equiv 4 \pmod 8$ if $n \equiv 2 \pmod 4$.
Thus,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{n^2} &= \sum_{n=-\infty}^{\infty} q^{(4n)^2} + \sum_{n=-\infty}^{\infty} q^{(2n+1)^2} + \sum_{-\infty}^{\infty} q^{(4n+2)^2} \\ &= \sum_{n=-\infty}^{\infty} q^{16n^2} + q \sum_{n=-\infty}^{\infty} q^{4n^2+4n} + q^4 \sum_{n=-\infty}^{\infty} q^{16n^2+16n} \\ &= \sum_{n=-\infty}^{\infty} q^{16n^2} + 2q \sum_{n\geq 0} q^{4n^2+4n} + 2q^4 \sum_{n\geq 0} q^{16n^2+16n}, \end{aligned}$$

as stated. (The product forms are obtained via Jacobi's triple product.)
Thus,

$$\sum_{n\geq 0} s(n)q^n = \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^3$$

$$= (T_0(q^8) + 2qT_1(q^8) + 2q^4T_4(q^8))^3$$

$$\begin{aligned}
&= (T_0(q^8)^3 + 12q^8T_0(q^8)T_4(q^8)^2) + (6qT_0(q^8)^2T_1(q^8) + 24q^9T_1(q^8)T_4(q^8)^2) \\
&\quad + 12q^2T_0(q^8)T_1(q^8)^2 + 8q^3T_1(q^8)^3 + (6q^4T_0(q^8)^2T_4(q^8) + 8q^{12}T_4(q^8)^3) \\
&\quad + 24q^5T_0(q^8)T_1(q^8)T_4(q^8) + 24q^6T_1(q^8)^2T_4(q^8).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{n\geq 0} s(8n+3)q^n &= 8T_1(q)^3 \\
&= 8(\sum_{n\geq 0} q^{\frac{1}{2}(n^2+n)})^3 \\
&= 8\sum_{n\geq 0} t(n)q^n,
\end{aligned}$$

$$\begin{aligned}
\sum_{n\geq 0} s(8n+5)q^n &= 24T_0(q)T_1(q)T_4(q) \\
&= 24\left(\begin{matrix} q^4, q^4, q^4, q^8 \\ q^2, q^2, q^6, q^6 \end{matrix}; q^8\right)_\infty \left(\begin{matrix} q^2, q^4, q^6, q^8 \\ q,\ q^3, q^5, q^7 \end{matrix}; q^8\right)_\infty \left(\begin{matrix} q^8 \\ q^4 \end{matrix}; q^8\right)_\infty \\
&= 24\left(\begin{matrix} q^4, q^4, q^4, q^8, q^8, q^8 \\ q,\ q^2, q^3, q^5, q^6, q^7 \end{matrix}; q^8\right)_\infty \\
&= 24\left(\begin{matrix} q^4, q^4, q^4 \\ q,\ q^2, q^3 \end{matrix}; q^4\right)_\infty \\
&= 24\frac{(q^4)_\infty^4}{(q)_\infty} \\
&= 24\sum_{n\geq 0} a_4(n)q^n
\end{aligned}$$

and

$$\sum_{n\geq 0} s(8n+7)q^n = 0.$$

We observe also that

$$\begin{aligned}\sum_{n\ge 0} s(8n+2)q^n &= 12T_0(q)T_1(q)^2\\ &= 12\left(\begin{matrix}q^4,q^4,q^4,q^8\\ q^2,q^2,q^6,q^6\end{matrix};q^8\right)_\infty \left(\begin{matrix}q^2,q^4,q^6,q^8\\ q,\ q^3,q^5,q^7\end{matrix};q^8\right)_\infty^2\\ &= 12\left(\begin{matrix}q^4,q^4,q^4,q^4,q^4,q^8,q^8,q^8\\ q,\ q,\ q^3,q^3,q^5,q^5,q^7,q^7\end{matrix};q^8\right)_\infty\end{aligned}$$

and

$$\begin{aligned}\sum_{n\ge 0} s(8n+6)q^n &= 24T_1(q)^2T_4(q)\\ &= 24\left(\begin{matrix}q^2,q^4,q^6,q^8\\ q,\ q^3,q^5,q^7\end{matrix};q^8\right)_\infty^2 \left(\begin{matrix}q^8\\ q^4\end{matrix};q^8\right)_\infty\\ &= 24\left(\begin{matrix}q^2,q^2,q^4,q^6,q^6,q^8,q^8,q^8\\ q,\ q,\ q^3,q^3,q^5,q^5,q^7,q^7\end{matrix};q^8\right)_\infty .\end{aligned}$$

3. Proof that $s(4^\lambda(8n+7))=0$

We saw in Section 2 that $s(8n+7)=0$. In order to prove Fermat's result, we need only prove that $s(4n)=s(n)$.

We can write

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n^2} &= \sum_{n=-\infty}^{\infty} q^{(2n)^2} + \sum_{n=-\infty}^{\infty} q^{(2n+1)^2}\\ &= T_0(q^4)+2qT_1(q^4)\end{aligned}$$

where

$$T_0(q)=\sum_{n=-\infty}^{\infty} q^{n^2} = \left(\begin{matrix}q^2,q^2,q^2,q^4\\ q,\ q,\ q^3,q^3\end{matrix};q^4\right)_\infty$$

and

$$T_1(q)=\sum_{n\ge 0} q^{n^2+n} = \left(\begin{matrix}q^4\\ q^2\end{matrix};q^4\right)_\infty .$$

Thus,

$$\begin{aligned}\sum_{n\geq 0} s(n)q^n &= (T_0(q^4)+2qT_1(q^4))^3\\ &= T_0(q^4)^3+6qT_0(q^4)^2T_1(q^4)+12q^2T_0(q^4)T_1(q^4)^2+8q^3T_1(q^4)^3.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0} s(4n)q^n &= T_0(q)^3\\ &= (\sum_{n=-\infty}^{\infty} q^{n^2})^3\\ &= \sum_{n\geq 0} s(n)q^n,\end{aligned}$$

so $s(4n)=s(n)$, as claimed.

We observe also that

$$\begin{aligned}\sum_{n\geq 0} s(4n+1)q^n &= 6T_0(q)^2T_1(q) = 6\begin{pmatrix} q^2,q^2,q^2,q^2,q^2,q^4,q^4,q^4 \\ q,\ q,\ q,\ q,\ q^3,q^3,q^3,q^3 \end{pmatrix};q^4\Big)_\infty,\\ \sum_{n\geq 0} s(4n+2)q^n &= 12T_0(q)T_1(q)^2 = 12\begin{pmatrix} q^2,q^4,q^4,q^4 \\ q,\ q,\ q^3,q^3 \end{pmatrix};q^4\Big)_\infty\end{aligned}$$

and

$$\sum_{n\geq 0} s(4n+3)q^n = 8T_1(q)^3 = 8\left(\begin{matrix} q^4,q^4,q^4 \\ q^2,q^2,q^2 \end{matrix};q^4\right)_\infty .$$

4. Proof of the case $\lambda=1$ of (1)

We want to show that

$$\begin{aligned}s(27n+9) &= 5\times s(3n+1),\\ s(27n+18) &= 3\times s(3n+2),\\ s(81n+27) &= 4\times s(9n+3)\end{aligned}$$

and

$$s(81n+54) = 4 \times s(9n+6).$$

We start with

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n^2} &= \sum_{n=-\infty}^{\infty} q^{(3n)^2} + \sum_{n=-\infty}^{\infty} q^{(3n+1)^2} + \sum_{n=-\infty}^{\infty} q^{(3n-1)^2} \\ &= T_0(q^3) + 2qT_1(q^3)\end{aligned}$$

where

$$T_0(q) = \sum_{n=-\infty}^{\infty} q^{3n^2} = \left(\begin{matrix} q^6, q^6, q^6, q^{12} \\ q^3, q^3, q^9, q^9 \end{matrix} ; q^{12} \right)_\infty$$

and

$$T_1(q) = \sum_{n=-\infty}^{\infty} q^{3n^2-2n} = \left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q,\ q^5, q^7,\ q^{11} \end{matrix} ; q^{12} \right)_\infty .$$

Hence,

$$\begin{aligned}\sum_{n\geq 0} s(n)q^n &= (T_0(q^3) + 2qT_1(q^3))^3 \\ &= (T_0(q^3)^3 + 8q^3T_1(q^3)^3) + 6qT_0(q^3)^2T_1(q^3) + 12q^2T_0(q^3)T_1(q^3)^2.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0} s(3n)q^n &= T_0(q)^3 + 8qT_1(q)^3 \\ &= (\sum_{n=-\infty}^{\infty} q^{3n^2})^3 + 8q(\sum_{n=-\infty}^{\infty} q^{3n^2-2n})^3 \\ &= \sum_{n\geq 0} s(n)q^{3n} + 8q \left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q,\ q^5, q^7,\ q^{11} \end{matrix} ; q^{12} \right)_\infty^3 ,\end{aligned}$$

or,

$$\sum_{n\geq 0} s(3n)q^n - \sum_{n\geq 0} s(n)q^{3n} = 8q \left(\begin{matrix} q^2, q^2, q^2, q^6, q^6, q^6, q^{10}, q^{10}, q^{10}, q^{12}, q^{12}, q^{12} \\ q,\ q,\ q,\ q^5, q^5, q^5, q^7,\ q^7,\ q^7,\ q^{11}, q^{11}, q^{11} \end{matrix} ; q^{12} \right)_\infty . \quad (6)$$

We shall require this result shortly.

We note also that

$$\sum_{n\geq 0} s(3n+1)q^n = 6T_0(q)^2T_1(q) = 6\left(\begin{matrix} q^2, q^6, q^6, q^6, q^6, q^6, q^6, q^6, q^{10}, q^{12}, q^{12}, q^{12} \\ q,\ q^3, q^3, q^3, q^3, q^5, q^7, q^9, q^9,\ q^9,\ q^9,\ q^{11} \end{matrix}; q^{12}\right)_\infty$$

and

$$\sum_{n\geq 0} s(3n+2)q^n = 12T_0(q)T_1(q)^2 = 12\left(\begin{matrix} q^2, q^2, q^6, q^6, q^6, q^6, q^6, q^{10}, q^{10}, q^{12}, q^{12}, q^{12} \\ q,\ q,\ q^3, q^3, q^5, q^5, q^7, q^7,\ q^9,\ q^9,\ q^{11}, q^{11} \end{matrix}; q^{12}\right)_\infty.$$

In the usual way it can be shown that

$$\sum_{n=-\infty}^{\infty} q^{n^2} = T_0(q^{27}) + 2qT_1(q^{27}) + 2q^4T_4(q^{27}) + 2q^{169}T_7(q^{27}) + 2q^9T_9(q^{27}) + 2q^{64}T_{10}(q^{27})$$
$$+ 2q^{121}T_{13}(q^{27}) + 2q^{16}T_{16}(q^{27}) + 2q^{100}T_{19}(q^{27}) + 2q^{49}T_{22}(q^{27}) + 2q^{25}T_{25}(q^{27})$$

where

$$T_0(q) = \sum_{n=-\infty}^{\infty} q^{3n^2}, \qquad T_1(q) = \sum_{n=-\infty}^{\infty} q^{27n^2-2n}, \qquad T_4(q) = \sum_{n=-\infty}^{\infty} q^{27n^2-4n},$$

$$T_7(q) = \sum_{n=-\infty}^{\infty} q^{27n^2-26n}, \qquad T_9(q) = \sum_{n=-\infty}^{\infty} q^{3n^2-2n}, \qquad T_{10}(q) = \sum_{n=-\infty}^{\infty} q^{27n^2-16n},$$

$$T_{13}(q) = \sum_{n=-\infty}^{\infty} q^{27n^2-22n}, \qquad T_{16}(q) = \sum_{n=-\infty}^{\infty} q^{27n^2-8n}, \qquad T_{19}(q) = \sum_{n=-\infty}^{\infty} q^{27n^2-20n},$$

$$T_{22}(q) = \sum_{n=-\infty}^{\infty} q^{27n^2-14n} \text{ and } T_{25}(q) = \sum_{n=-\infty}^{\infty} q^{27n^2-10n}.$$

Here we have used

$$2q^9T_9(q^{27}) = \sum_{n=-\infty}^{\infty} q^{(27n+3)^2} + \sum_{n=-\infty}^{\infty} q^{(27n+6)^2} + \sum_{n=-\infty}^{\infty} q^{(27n+12)^2}$$
$$+ \sum_{n=-\infty}^{\infty} q^{(27n-12)^2} + \sum_{n=-\infty}^{\infty} q^{(27n-6)^2} + \sum_{n=-\infty}^{\infty} q^{(27n-3)^2}$$

$$= \sum_{n=-\infty}^{\infty} q^{9(3n+1)^2} + \sum_{n=-\infty}^{\infty} q^{9(3n-1)^2}$$
$$= 2q^9 \sum_{n=-\infty}^{\infty} q^{81n^2-54n}.$$

We have

$$\sum_{n\geq 0} s(n)q^n = (T_0(q^{27}) + 2qT_1(q^{27}) + 2q^4T_4(q^{27}) + 2q^{169}T_7(q^{27}) + 2q^9T_9(q^{27}) + 2q^{64}T_{10}(q^{27})$$
$$+ 2q^{121}T_{13}(q^{27}) + 2q^{16}T_{16}(q^{27}) + 2q^{100}T_{19}(q^{27}) + 2q^{49}T_{22}(q^{27}) + 2q^{25}T_{25}(q^{27}))^3.$$

It follows that

$$\sum_{n\geq 0} s(27n)q^n = T_0(q)^3 + 8qT_9(q)^3 + 24qE(q)$$
$$= (\sum_{n=-\infty}^{\infty} q^{3n^2})^3 + 8q(\sum_{n=-\infty}^{\infty} q^{3n^2-2n})^3 + 24qE(q)$$
$$= \sum_{n\geq 0} s(3n)q^n + 24qE(q),$$

where

$$E(q) = T_1(q)^2T_{25}(q) + q^3T_4(q)^2T_{19}(q) + q^{16}T_7(q)^2T_{13}(q) + q^{10}T_{10}(q)^2T_7(q) + q^8T_{13}(q)^2T_1(q)$$
$$+ q^2T_{16}(q)^2T_{22}(q) + q^7T_{19}(q)^2T_{16}(q) + q^5T_{22}(q)^2T_{10}(q) + qT_{25}(q)^2T_4(q)$$
$$+ 2qT_1(q)T_4(q)T_{22}(q) + 2q^9T_1(q)T_7(q)T_{19}(q) + 2q^2T_1(q)T_{10}(q)T_{16}(q)$$
$$+ 2q^6T_4(q)T_7(q)T_{16}(q) + 2q^6T_4(q)T_{10}(q)T_{13}(q) + 2q^8T_7(q)T_{22}(q)T_{25}(q)$$
$$+ 2q^6T_{10}(q)T_{19}(q)T_{25}(q) + 2q^5T_{13}(q)T_{16}(q)T_{25}(q) + 2q^9T_{13}(q)T_{19}(q)T_{22}(q).$$

Now,

$$q^{27}E(q^{27}) = \{\sum q^{(27k+1)^2+(27l+1)^2+(27m-5)^2} + \sum q^{(27k-2)^2+(27l-2)^2+(27m+10)^2}$$
$$+ \sum q^{(27k+13)^2+(27l+13)^2+(27m-11)^2} + \sum q^{(27k-8)^2+(27l-8)^2+(27m+13)^2}$$
$$+ \sum q^{(27k-11)^2+(27l-11)^2+(27m+1)^2} + \sum q^{(27k+4)^2+(27l+4)^2+(27m+7)^2}$$

$$
\begin{aligned}
&+\sum q^{(27k+10)^2+(27l+10)^2+(27m+4)^2}+\sum q^{(27k+7)^2+(27l+7)^2+(27m-8)^2}\\
&+\sum q^{(27k-5)^2+(27l-5)^2+(27m-2)^2}\}\\
+2\{&\sum q^{(27k+1)^2+(27l-2)^2+(27m+7)^2}+\sum q^{(27k+1)^2+(27l+13)^2+(27m+10)^2}\\
&+\sum q^{(27k+1)^2+(27l-8)^2+(27m+4)^2}+\sum q^{(27k-2)^2+(27l+13)^2+(27m+4)^2}\\
&+\sum q^{(27k-2)^2+(27l-8)^2+(27m-11)^2}+\sum q^{(27k+13)^2+(27l+7)^2+(27m-5)^2}\\
&+\sum q^{(27k-8)^2+(27l+10)^2+(27m-5)^2}+\sum q^{(27k-11)^2+(27l+4)^2+(27m-5)^2}\\
&+\sum q^{(27k-11)^2+(27l+10)^2+(27m+7)^2}\}
\end{aligned}
$$

$$
=\sum q^{(3u+1)^2+(3v+1)^2+(3w+1)^2},
$$

where the sum is taken over all triples (u, v, w) which satisfy

$$
2u+2v-w\equiv 2,\quad 2u-v+2w\equiv -4,\quad -u+2v+2w\equiv -4 \pmod 9.
$$

Let

$$
r=\frac{1}{9}(2u+2v-w-2),\quad s=\frac{1}{9}(2u-v+2w+4),\quad t=\frac{1}{9}(-u+2v+2w+4).
$$

Conversely,

$$
u=2r+2s-t,\quad v=2r-s+2t,\quad w=-r+2s+2t-2.
$$

Then

$$
\begin{aligned}
&(3u+1)^2+(3v+1)^2+(3w+1)^2\\
&\qquad=(6r+6s-3t+1)^2+(6r-3s+6t+1)^2+(-3r+6s+6t-5)^2\\
&\qquad=81r^2+81s^2+81t^2+54r-54s-54t+27.
\end{aligned}
$$

Thus,

$$
q^{27}E(q^{27})=q^{27}\sum_{r,s,t=-\infty}^{\infty} q^{81r^2+54r+81s^2-54s+81t^2-54t},
$$

$$\begin{aligned}
E(q) &= \sum_{r,s,t=-\infty}^{\infty} q^{3r^2+2r+3s^2-2s+3t^2-2t} \\
&= (\sum_{n=-\infty}^{\infty} q^{3n^2-2n})^3 \\
&= \left(\begin{matrix} q^2, q^2, q^2, q^6, q^6, q^6, q^{10}, q^{10}, q^{10}, q^{12}, q^{12}, q^{12} \\ q,\ q,\ q,\ q^5, q^5, q^5, q^7,\ q^7,\ q^7,\ q^{11}, q^{11}, q^{11} \end{matrix} ; q^{12} \right)_\infty
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n\geq 0} s(27n)q^n - \sum_{n\geq 0} s(3n)q^n &= 24qE(q) \\
&= 24q \left(\begin{matrix} q^2, q^2, q^2, q^6, q^6, q^6, q^{10}, q^{10}, q^{10}, q^{12}, q^{12}, q^{12} \\ q,\ q,\ q,\ q^5, q^5, q^5, q^7,\ q^7,\ q^7,\ q^{11}, q^{11}, q^{11} \end{matrix} ; q^{12} \right)_\infty \\
&= 3(\sum_{n\geq 0} s(3n)q^n - \sum_{n\geq 0} s(n)q^{3n}) \text{ from (6).}
\end{aligned}$$

Comparing powers congruent to 1 (mod 3) and congruent to 2 (mod 3), we obtain

$$s(81n+27) = 4 \times s(9n+3)$$

and

$$s(81n+54) = 4 \times s(9n+6).$$

Comparing powers congruent to 0 (mod 3), we find

$$s(81n) = 4 \times s(9n) - 3 \times s(n), \tag{7}$$

a result we shall require later.

Similarly, we find

$$\begin{aligned}
\sum_{n\geq 0} s(27n+9)q^n &= 6T_0(q)^2T_9(q) + 24F(q) \\
&= 6(\sum_{n=-\infty}^{\infty} q^{3n^2})^2(\sum_{n=-\infty}^{\infty} q^{3n^2-2n}) + 24F(q) \\
&= \sum_{n\geq 0} s(3n+1)q^n + 24F(q),
\end{aligned}$$

where

$$\begin{aligned}F(q) = q^6T_1(q)^2T_7(q) + T_4(q)^2T_1(q) + q^{14}T_7(q)^2T_{22}(q) + q^5T_{10}(q)^2T_{16}(q) + q^{11}T_{13}(q)^2T_{10}(q)\\ + qT_{16}(q)^2T_4(q) + q^8T_{19}(q)^2T_{25}(q) + q^7T_{22}(q)^2T_{19}(q) + q^6T_{25}(q)^2T_{13}(q)\\ + 2q^3T_1(q)T_{10}(q)T_{25}(q) + 2q^6T_1(q)T_{13}(q)T_{22}(q) + 2q^4T_1(q)T_{16}(q)T_{19}(q)\\ + 2q^7T_4(q)T_7(q)T_{25}(q) + 2q^4T_4(q)T_{10}(q)T_{22}(q) + 2q^8T_4(q)T_{13}(q)T_{19}(q)\\ + 2q^{12}T_7(q)T_{10}(q)T_{19}(q) + 2q^{11}T_7(q)T_{13}(q)T_{16}(q) + 2q^3T_{16}(q)T_{22}(q)T_{25}(q).\end{aligned}$$

As before, we find that

$$q^9F(q^{27}) = \sum q^{(3u+1)^2+(3v+1)^2+(3w+1)^2},$$

where the sum is taken over all triples (u, v, w) which satisfy

$$2u + 2v - w \equiv -4, \quad 2u - v + 2w \equiv -1, \quad -u + 2v + 2w \equiv -1 \pmod 9.$$

Let

$$r = \frac{1}{9}(2u + 2v - w + 4), \quad s = \frac{1}{9}(2u - v + 2w + 1), \quad t = \frac{1}{9}(-u + 2v + 2w + 1).$$

Conversely,

$$u = 2r + 2s - t - 1, \quad v = 2r - s + 2t - 1, \quad w = -r + 2s + 2t.$$

Then

$$\begin{aligned}(3u + 1)^2 &+ (3v + 1)^2 + (3w + 1)^2\\ &= (6r + 6s - 3t - 2)^2 + (6r - 3s + 6t - 2)^2 + (-3r + 6s + 6t + 1)^2\\ &= 81r^2 + 81s^2 + 81t^2 - 54r + 9.\end{aligned}$$

Thus,

$$q^9F(q^{27}) = q^9 \sum_{r,s,t=-\infty}^{\infty} q^{81r^2-54r+81s^2+81t^2},$$

$$\begin{aligned}F(q) &= \sum_{r,s,t=-\infty}^{\infty} q^{3r^2-2r+3s^2+3t^2}\\ &= (\sum_{n=-\infty}^{\infty} q^{3n^2})^2 (\sum_{n=-\infty}^{\infty} q^{3n^2-2n})\\ &= \left(\begin{matrix} q^2, q^6, q^6, q^6, q^6, q^6, q^6, q^6, q^{10}, q^{12}, q^{12}, q^{12} \\ q,\ q^3, q^3, q^3, q^3, q^5, q^7, q^9, q^9,\ q^9,\ q^9,\ q^{11} \end{matrix} ; q^{12}\right)_\infty\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 0} s(27n+9)q^n - \sum_{n\geq 0} s(3n+1)q^n &= 24F(q)\\ &= 24\left(\begin{matrix} q^2, q^6, q^6, q^6, q^6, q^6, q^6, q^6, q^{10}, q^{12}, q^{12}, q^{12} \\ q,\ q^3, q^3, q^3, q^3, q^5, q^7, q^9, q^9,\ q^9,\ q^9,\ q^{11} \end{matrix} ; q^{12}\right)_\infty\\ &= 4\sum_{n\geq 0} s(3n+1)q^n.\end{aligned}$$

Comparing coefficients of q^n yields

$$s(27n+9) = 5 \times s(3n+1).$$

Finally we find that

$$\sum_{n\geq 0} s(27n+18)q^n - \sum_{n\geq 0} s(3n+2)q^n = 24G(q)$$

where

$$q^{18}G(q^{27}) = \sum q^{(3u+1)^2+(3v+1)^2+(3w+1)^2},$$

the sum taken over all triples (u, v, w) which satisfy

$$2u + 2v - w \equiv -1, \quad 2u - v + 2w \equiv 2, \quad -u + 2v + 2w \equiv 2 \pmod 9.$$

Let

$$r = \frac{1}{9}(2u + 2v - w + 1), \quad s = \frac{1}{9}(2u - v + 2w - 2), \quad t = \frac{1}{9}(-u + 2v + 2w - 2).$$

Conversely,

$$u = 2r + 2s - t, \quad v = 2r - s + 2t, \quad w = -r + 2s + 2t + 1.$$

Also,

$$\begin{aligned}(3u+1)^2+(3v+1)^2+(3w+1)^2 \\ &= (6r+6s-3t+1)^2+(6r-3s+6t+1)^2+(-3r+6s+6t+4)^2 \\ &= 81r^2+81s^2+81t^2+54s+54t+18.\end{aligned}$$

Thus

$$q^{18}G(q^{27}) = q^{18}\sum_{r,s,t=-\infty}^{\infty} q^{81r^2+81s^2+54s+81t^2+54t},$$

$$\begin{aligned}G(q) &= \sum_{r,s,t=-\infty}^{\infty} q^{3r^2+3s^2+2s+3t^2+2t} \\ &= (\sum_{n=-\infty}^{\infty} q^{3n^2})(\sum_{n=-\infty}^{\infty} q^{3n^2-2n})^2 \\ &= 24\begin{pmatrix} q^2,q^2,q^6,q^6,q^6,q^6,q^6,q^{10},q^{10},q^{12},q^{12},q^{12} \\ q,\ q,\ q^3,q^3,q^5,q^5,q^7,q^7,\ q^9,\ q^9,\ q^{11},q^{11} \end{pmatrix}_\infty,\end{aligned}$$

$$\begin{aligned}\sum_{n\geq 0} s(27n+18)q^n - \sum_{n\geq 0} s(3n+2)q^n &= 24G(q) \\ &= 24\begin{pmatrix} q^2,q^2,q^6,q^6,q^6,q^6,q^6,q^{10},q^{10},q^{12},q^{12},q^{12} \\ q,\ q,\ q^3,q^3,q^5,q^5,q^7,q^7,\ q^9,\ q^9,\ q^{11},q^{11} \end{pmatrix}_\infty \\ &= 2\sum_{n\geq 0} s(3n+2)q^n\end{aligned}$$

and

$$s(27n+18) = 3\times s(3n+2).$$

5. Proof of (1)

We have shown that

$$s(9^\lambda n) = \begin{cases} (2\times 3^\lambda - 1)\times s(n) & \text{if } n\equiv 1 \pmod 3 \\ 3^\lambda \times s(n) & \text{if } n \equiv 2 \pmod 3 \\ ((3^{\lambda+1}-1)/2)\times s(n) & \text{if } n\equiv 3 \text{ or } 6 \pmod 9 \end{cases}$$

for $\lambda = 1$, and it is trivially true for $\lambda = 0$.

We have also shown (7) that

$$s(81n) = 4 \times s(9n) - 3 \times s(n).$$

(1) now follows by induction on λ.

6. Generating functions for $s(9n+3)$ and $s(9n+6)$

We start with

$$\sum_{n=-\infty}^{\infty} q^{n^2} = T_0(q^9) + 2qT_1(q^9) + 2q^4T_4(q^9) + 2q^{16}T_7(q^9),$$

where

$$T_0(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left(\begin{matrix} q^2, q^2, q^2, q^4 \\ q, \; q, \; q^3, q^3 \end{matrix} ; q^4 \right)_\infty ,$$

$$T_1(q) = \sum_{n=-\infty}^{\infty} q^{9n^2-2n} = \left(\begin{matrix} q^{14}, q^{18}, q^{22}, q^{36} \\ q^7, \; q^{11}, q^{25}, q^{29} \end{matrix} ; q^{36} \right)_\infty ,$$

$$T_4(q) = \sum_{n=-\infty}^{\infty} q^{9n^2-4n} = \left(\begin{matrix} q^{10}, q^{18}, q^{26}, q^{36} \\ q^5, \; q^{13}, q^{23}, q^{31} \end{matrix} ; q^{36} \right)_\infty$$

and

$$T_7(q) = \sum_{n=-\infty}^{\infty} q^{9n^2-8n} = \left(\begin{matrix} q^2, q^{18}, q^{34}, q^{36} \\ q, \; q^{17}, q^{19}, q^{35} \end{matrix} ; q^{36} \right)_\infty .$$

Thus,

$$\sum_{n \geq 0} s(n)q^n = (T_0(q^9) + 2qT_1(q^9) + 2q^4T_4(q^9) + 2q^{16}T_7(q^9))^3.$$

It follows that

$$\sum_{n \geq 0} s(9n)q^n = T_0(q)^3 + 24qT_1(q)T_4(q)^2 + 24q^2T_1(q)^2T_7(q) + 24q^4T_4(q)T_7(q)^2,$$

$$\sum_{n\geq 0} s(9n+1)q^n = 6T_0(q)^2T_1(q),$$

$$\sum_{n\geq 0} s(9n+2)q^n = 12T_0(q)T_1(q)^2 + 24q^2T_0(q)T_4(q)T_7(q),$$

$$\sum_{n\geq 0} s(9n+3)q^n = 8T_1(q)^3 + 8qT_4(q)^3 + 8q^5T_7(q)^3 + 48q^2T_1(q)T_4(q)T_7(q),$$

$$\sum_{n\geq 0} s(9n+4)q^n = 6T_0(q)^2T_4(q),$$

$$\sum_{n\geq 0} s(9n+5)q^n = 12q^3T_0(q)T_7(q)^2 + 24T_0(q)T_1(q)T_4(q),$$

$$\sum_{n\geq 0} s(9n+6)q^n = 24T_1(q)^2T_4(q) + 24q^3T_1(q)T_7(q)^2 + 24q^2T_4(q)^2T_7(q),$$

$$\sum_{n\geq 0} s(9n+7)q^n = 6qT_0(q)^2T_7(q)$$

and

$$\sum_{n\geq 0} s(9n+8)q^n = 12T_0(q)T_4(q)^2 + 24qT_0(q)T_1(q)T_7(q).$$

Thus, trivially,

$$\sum_{n\geq 0} s(9n+1)q^n = 6\begin{pmatrix} q^2, q^2, q^2, q^4 \\ q,\ q,\ q^3, q^3 \end{pmatrix};q^4\Bigg)_\infty^2 \begin{pmatrix} q^{14}, q^{18}, q^{22}, q^{36} \\ q^7,\ q^{11}, q^{25}, q^{29} \end{pmatrix};q^{36}\Bigg)_\infty,$$

$$\sum_{n\geq 0} s(9n+4)q^n = 6\begin{pmatrix} q^2, q^2, q^2, q^4 \\ q,\ q,\ q^3, q^3 \end{pmatrix};q^4\Bigg)_\infty^2 \begin{pmatrix} q^{10}, q^{18}, q^{26}, q^{36} \\ q^5,\ q^{13}, q^{23}, q^{31} \end{pmatrix};q^{36}\Bigg)_\infty$$

and

$$\sum_{n\geq 0} s(9n+7)q^n = 6q\begin{pmatrix} q^2, q^2, q^2, q^4 \\ q,\ q,\ q^3, q^3 \end{pmatrix};q^4\Bigg)_\infty^2 \begin{pmatrix} q^2, q^{18}, q^{34}, q^{36} \\ q,\ q^{17}, q^{19}, q^{35} \end{pmatrix};q^{36}\Bigg)_\infty.$$

Also,

$$\sum_{n\geq 0} s(9n+3)q^{9n+3} = 8\{\sum q^{(9k+1)^2+(9l+1)^2+(9m+1)^2} + \sum q^{(9k-2)^2+(9l-2)^2+(9m-2)^2}$$

$$+\sum q^{(9k+4)^2+(9l+4)^2+(9m+4)^2}+6\sum q^{(9k+1)^2+(9l-2)^2+(9m+4)^2}\}$$

$$=8\sum q^{(3u+1)^2+(3v+1)^2+(3w+1)^2},$$

where the sum is taken over all triples (u,v,w) which satisfy

$$u+v+w\equiv 0,\quad 2u-v-w\equiv 0,\quad -u+2v-w\equiv 0,\quad -u-v+2w\equiv 0 \pmod 3.$$

Write

$$p=\frac{1}{3}(u+v+w),\quad r=\frac{1}{3}(2u-v-w),\quad s=\frac{1}{3}(-u+2v-w),\quad t=\frac{1}{3}(-u-v+2w).$$

Then

$$r+s+t=0.$$

Conversely, given $p,\ r,\ s,\ t$ with $r+s+t=0$,

$$u=p+r,\quad v=p+s,\quad w=p+t.$$

Also,

$$\begin{aligned}(3u+1)^2&+(3v+1)^2+(3w+1)^2\\&=(3p+3r+1)^2+(3p+3s+1)^2+(3p+3t+1)^2\\&=27p^2+18p+3+9r^2+9s^2+9t^2.\end{aligned}$$

Thus,

$$\sum_{n\geq 0}s(9n+3)q^{9n+3}=8q^3\sum_{p=-\infty}^{\infty}q^{27p^2+18p}\sum_{r+s+t=0}q^{9r^2+9s^2+9t^2},$$

or,

$$\begin{aligned}\sum_{n\geq 0}s(9n+3)q^n&=8\sum_{p=-\infty}^{\infty}q^{3p^2+2p}\sum_{r+s+t=0}q^{r^2+s^2+t^2}\\&=\left(\begin{matrix}q^2,q^6,q^{10},q^{12}\\q,\ q^5,q^7,\ q^{11}\end{matrix};q^{12}\right)_\infty\left\{1+6\sum_{n\geq 1}\left(\frac{q^{6n-4}}{1-q^{6n-4}}-\frac{q^{6n-2}}{1-q^{6n-2}}\right)\right\}.\end{aligned}$$

Here we have used a result from the Appendix of [3].

Similarly, we find

$$\sum_{n\geq 0} s(9n+6)q^{9n+6} = 8\sum q^{(3u+1)^2+(3v+1)^2+(3w+1)^2}$$

where the sum is taken over all triples (u,v,w) which satisfy

$$u+v+w\equiv -1,\quad 2u-v-w\equiv 1,\quad -u+2v-w\equiv 1,\quad -u-v+2w\equiv 1 \pmod 3.$$

Write

$$p=\frac{1}{3}(u+v+w+1),\;\; r=\frac{1}{3}(2u-v-w-1),\;\; s=\frac{1}{3}(-u+2v-w-1),\;\; t=\frac{1}{3}(-u-v+2w-1).$$

Then

$$r+s+t=-1.$$

Conversely, given $p,\ r,\ s,\ t$ with $r+s+t=-1$,

$$u=p+r,\quad v=p+s,\quad w=p+t.$$

Also,

$$\begin{aligned}(3u+1)^2&+(3v+1)^2+(3w+1)^2\\ &=(3p+3r+1)^2+(3p+3s+1)^2+(3p+3t+1)^2\\ &=27p^2+9r^2+9s^2+9t^2-3.\end{aligned}$$

Thus,

$$\sum_{n\geq 0} s(9n+6)q^{9n+6} = 8\sum_{p=-\infty}^{\infty} q^{27p^2}\sum_{r+s+t=-1} q^{9r^2+9s^2+9t^2-3},$$

or,

$$\begin{aligned}\sum_{n\geq 0} s(9n+6)q^{n} &= 8\sum_{p=-\infty}^{\infty} q^{3p^2}\sum_{r+s+t=-1} q^{r^2+s^2+t^2-1}\\ &=24\left(\begin{matrix}q^6,q^6,q^6,q^{12}\\ q^3,q^3,q^9,q^9,\end{matrix};q^{12}\right)_\infty\left(\begin{matrix}q^6,q^6,q^{12},q^{12}\\ q^2,q^4,q^8,\ q^{10}\end{matrix};q^{12}\right)_\infty\\ &=24\left(\begin{matrix}q^6,q^6,q^6,q^6,q^6,q^{12},q^{12},q^{12}\\ q^2,q^3,q^3,q^4,q^8,q^9,\ q^9,\ q^{10}\end{matrix};q^{12}\right)_\infty.\end{aligned}$$

Once again we have used a result from the Appendix of [3].

7. Proof of the result for general odd prime p

We now give a proof of the result

$$s(p^2n) = \left((p+1) - \left(\frac{-n}{p}\right)\right) s(n) - p\ s(n/p^2).$$

It is well-known that

$$\Theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q = \exp(2\pi i\tau)$$

is a modular form of weight 1/2. Therefore

$$\sum_{n\geq 0} s(n)q^n = \Theta^3(\tau)$$

is a modular form of weight 3/2, and in fact $\Theta^3 \in M_{3/2}(\widetilde{\Gamma}_0(4))$. (See [4], p.184 for notation). From [4], p.184, Dim $M_{3/2}(\widetilde{\Gamma}_0(4)) = 1$, and thus the linear space $M_{3/2}(\widetilde{\Gamma}_0(4))$ is generated by Θ^3. When p is an odd prime the Hecke operator T_{p^2} preserves this space and is given explicitly by [4], Prop.13, p.207. An easy calculation gives

$$T_{p^2}\Theta^3 = (p+1)\Theta^3.$$

The result follows immediately by [4], Prop.13, p.207.

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