Triangles with Integer Sides, Revisited

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Introduction Our attention has once more been drawn (see [**2**] and its references) to the problem of determining the number T_n of triangles with integer sides and perimeter n. The solution of this problem can be written neatly as

$$T_n = \begin{cases} \left\langle \dfrac{(n+3)^2}{48} \right\rangle & \text{for } n \text{ odd;} \\ \left\langle \dfrac{n^2}{48} \right\rangle & \text{for } n \text{ even,} \end{cases}$$

where $\langle x \rangle$ is the integer closest to x. Our proof comes in two stages. First we show by a direct combinatorial argument that

$$T_n = \begin{cases} p_3\left(\dfrac{n-3}{2}\right) & \text{for } n \text{ odd;} \\ p_3\left(\dfrac{n-6}{2}\right) & \text{for } n \text{ even,} \end{cases}$$

where $p_3(n)$ is the number of partitions of n into at most three parts. Then we show using a novel partial fractions technique that

$$p_3(n) = \left\langle \frac{(n+3)^2}{12} \right\rangle .$$

The proofs For a triangle with integer sides $a \le b \le c$ and odd perimeter n,

$$a+b-c, \quad b+c-a, \quad c+a-b$$

are odd and positive,

$$a+b-c-1, \quad b+c-a-1, \quad c+a-b-1$$

are even and nonnegative, and if

$$p = \frac{1}{2}(b+c-a-1), \quad q = \frac{1}{2}(c+a-b-1), \quad \text{and} \quad r = \frac{1}{2}(a+b-c-1),$$

then $p \ge q \ge r$ are nonnegative integers, and $p+q+r = \dfrac{n-3}{2}$.

Conversely, if $n \ge 3$ is odd and $p \ge q \ge r$ are nonnegative integers with $p+q+r = \dfrac{n-3}{2}$ and if

$$a = q+r+1, \quad b = p+r+1, \quad \text{and} \quad c = p+q+1,$$

then $a \le b \le c$ are the sides of a triangle with perimeter n.

Similarly, given a triangle with integer sides $a \le b \le c$ and even perimeter n,

$$a+b-c, \quad b+c-a, \quad \text{and} \quad c+a-b$$

are even and positive,

$$a+b-c-2, \quad b+c-a-2, \quad \text{and} \quad c+a-b-2$$

are even and nonnegative, and if

$$p = \frac{1}{2}(b+c-a-2), \quad q = \frac{1}{2}(c+a-b-2), \quad \text{and} \quad r = \frac{1}{2}(a+b-c-2),$$

then $p \ge q \ge r$ are nonnegative integers, and $p+q+r = \dfrac{n-6}{2}$.

Conversely, if $n \ge 6$ is even and $p \ge q \ge r$ are nonnegative integers with $p+q+r = \dfrac{n-6}{2}$ and if

$$a = q+r+2, \quad b = p+r+2, \quad \text{and} \quad c = p+q+2$$

then $a \le b \le c$ are the sides of a triangle with perimeter n. ■

To show that

$$p_3(n) = \left\langle \frac{(n+3)^2}{12} \right\rangle,$$

we start with the generating function

$$\sum_{n \ge 0} p_3(n) q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)}.$$

To see that this is indeed the generating function for partitions into at most three parts, we note that partitions into at most three parts are equinumerous with partitions into parts no greater than three ([**1**, Theorem 1.4]) and the generating function for partitions into parts no greater than three is easily seen to be

$$(1+q^3+q^{3+3}+\ \cdots)(1+q^2+q^{2+2}+\ \cdots)(1+q^1+q^{1+1}+\ \cdots)$$

$$= \frac{1}{(1-q)(1-q^2)(1-q^3)}.$$

In order to extract an explicit formula for $p_3(n)$ from the generating function, it is usual to use partial fractions. I have found a method (other than by using a computer package, such as MAPLE) by which we can avoid the horrors of finding the complete partial fractions expansion of the generating function. I will demonstrate the method below, and follow it with an explanation of the various steps. Before I do so, let me first make the following observations.

The denominator of the generating function can be factored as follows.

$$(1-q)^3(1+q)(1-\omega q)(1-\overline{\omega} q)$$

where ω is a cube root of unity.

Thus the partial fractions expansion of the generating function takes the form

$$\sum_{n\geq 0} p_3(n)q^n = \frac{A}{(1-q)^3} + \frac{B}{(1-q)^2} + \frac{C}{1-q} + \frac{D}{1+q} + \frac{E}{1-\omega q} + \frac{F}{1-\overline{\omega} q}.$$

It follows that

$$p_3(n) = A\binom{n+2}{2} + B\binom{n+1}{1} + C + D(-1)^n + E\omega^n + F\overline{\omega}^n.$$

Observe that the expression $C + D(-1)^n + E\omega^n + F\overline{\omega}^n$ is periodic with period 6, so takes values $c_i,\ i = 0, \cdots, 5$ according to the residue of n modulo 6. It follows that the generating function can be written

$$\sum_{n\geq 0} p_3(n)q^n = \frac{A}{(1-q)^3} + \frac{B}{(1-q)^2} + \frac{c_0 + c_1 q + c_2 q^2 + c_3 q^3 + c_4 q^4 + c_5 q^5}{1-q^6}.$$

With this in mind, we have

$$\begin{aligned}
\sum_{n\geq 0} p_3(n)q^n &= \frac{1}{(1-q)(1-q^2)(1-q^3)} \\
&= \frac{1}{(1-q)^3}\cdot\frac{1}{(1+q)(1+q+q^2)} \\
&= \frac{1}{(1-q)^3}\cdot\left(\frac{1}{6} + \frac{(1-q)(5+3q+q^2)}{6(1+q)(1+q+q^2)}\right) \\
&= \frac{1}{6}\frac{1}{(1-q)^3} + \frac{1}{(1-q)^2}\cdot\frac{5+3q+q^2}{6(1+q)(1+q+q^2)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}\frac{1}{(1-q)^3} + \frac{1}{(1-q)^2}\cdot\left(\frac{1}{4} + \frac{(1-q)(7+7q+3q^2)}{12(1+q)(1+q+q^2)}\right) \\
&= \frac{1}{6}\frac{1}{(1-q)^3} + \frac{1}{4}\frac{1}{(1-q)^2} + \frac{7+7q+3q^2}{12(1-q)(1+q)(1+q+q^2)} \\
&= \frac{1}{6}\frac{1}{(1-q)^3} + \frac{1}{4}\frac{1}{(1-q)^2} + \frac{(7+7q+3q^2)(1-q+q^2)}{12(1-q^6)} \\
&= \frac{1}{6}\frac{1}{(1-q)^3} + \frac{1}{4}\frac{1}{(1-q)^2} + \frac{7+3q^2+4q^3+3q^4}{12(1-q^6)} \\
&= \frac{1}{6}\sum_{n\geq 0}\binom{n+2}{2}q^n + \frac{1}{4}\sum_{n\geq 0}(n+1)q^n + \frac{1}{12}(7+3q^2+4q^3+3q^4)\sum_{n\geq 0}q^{6n} \\
&= \sum_{n\geq 0}\frac{1}{12}(n^2+6n+5)q^n + \frac{1}{12}\left(7+3q^2+4q^3+3q^4\right)\sum_{n\geq 0}q^{6n} \\
&= \sum_{n\geq 0}\frac{(n+3)^2}{12}q^n + \frac{1}{12}(3-4q-q^2-q^4-4q^5)\sum_{n\geq 0}q^{6n}.
\end{aligned}
$$

The result follows.

The explanation of the various steps above is the following. Since we know the major contribution to $p_3(n)$ comes from the term $(1-q)^3$ in the denominator, we attempt to separate this term from the generating function. We write

$$\sum_{n\geq 0} p_3(n)q^n = \frac{1}{(1-q)^3}f(q) \quad \text{where} \quad f(q) = \frac{1}{(1+q)(1+q+q^2)}.$$

Then we replace $f(q)$ by its Taylor series about 1, at least to the extent of writing

$$f(q) = f(1) + g(q) = \frac{1}{6} + g(q).$$

An easy calculation gives

$$g(q) = f(q) - f(1) = \frac{1}{(1+q)(1+q+q^2)} - \frac{1}{6} = \frac{6-(1+q)(1+q+q^2)}{6(1+q)(1+q+q^2)} = \frac{(1-q)(5+3q+q^2)}{6(1+q)(1+q+q^2)}.$$

Thus we obtain the fourth line above.

We then apply the same procedure to the function $\dfrac{5+3q+q^2}{6(1+q)(1+q+q^2)}$ and arrive at the sixth line, where the denominator is

$$12(1-q)(1+q)(1+q+q^2).$$

Now we note that $(1-q)(1+q)(1+q+q^2)$ divides $1-q^6$, since

$$1-q^6 = (1-q^3)(1+q^3) = (1-q)(1+q)(1+q+q^2)(1-q+q^2).$$

So we multiply top and bottom by $1 - q + q^2$ to obtain the eighth line above. The rest is straightforward.

References

[1] George E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and its Applications, Reading, Mass., Addison–Wesley, 1976.

[2] Nicholas Krier and Bennet Manvel, Counting integer triangles, this Magazine 71 (1998), 291–295.