A NEW IDENTITY FOR $(q;q)_\infty^{10}$ WITH AN APPLICATION TO RAMANUJAN'S PARTITION CONGRUENCE MODULO 11

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1. Introduction

The first primary purpose of this paper is to prove a new representation for $(q;q)_\infty^{10}$, given in Theorem 2.2 below, where

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots, \qquad |q|<1.$$

Our principal second goal is to show that this identity leads to a short proof of Ramanujan's famous congruence $p(11n+6) \equiv 0 \pmod{11}$, where $p(n)$ denotes the number of unrestricted partitions of the positive integer n. Our proof of Theorem 2.2 is short but depends upon some results of Ramanujan from his notebooks [18]. In Section 4, we give a more elementary proof, based upon Ramanujan's ${}_1\psi_1$ summation formula, of Ramanujan's principal result, Lemma 2.1, which is employed in our proof of Theorem 2.2. In Section 5, we give an entirely different and direct proof of Theorem 2.2 based on several elementary identities for theta functions due to Ramanujan. In fact, during our first proof of Theorem 2.2, we establish a special case of a general result on Eisenstein series found in Ramanujan's lost notebook [19, p. 369]. More precisely, Ramanujan asserts that every member of a certain class of infinite series can be expressed in terms of Ramanujan's Eisenstein series P, Q, and R (to be defined in Section 6). A less precise version of this claim appears in Ramanujan's notebooks [18], [2, p. 65, Entry 35(i)], but we prove the better version in Section 6. D. Stanton empirically discovered an analogue of Theorem 2.2, and in Section 7 we give a proof of Stanton's identity. In Section 8, we prove that $(q;q)_\infty^{10}$ is lacunary; in this connection, see a result of J.–P. Serre [20].

2. A New Representation for $(q;q)_\infty^{10}$

To establish this new representation, we need a lemma that is easily derivable from two results in Ramanujan's notebooks [18], [2, p. 345, Entry 1(iv); p. 475, Entry 7(i)]. See also [7, p. 109] for a similar proof.

Lemma 2.1. *We have*

$$1+3\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-27\sum_{n=1}^{\infty}\frac{nq^{9n}}{1-q^{9n}}=\frac{(q^3;q^3)_\infty^{10}}{(q;q)_\infty^3(q^9;q^9)_\infty^3}. \tag{2.1}$$

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Proof. From [2, p. 475, Entry 7(i)],

$$1+3\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-27\sum_{n=1}^{\infty}\frac{nq^{9n}}{1-q^{9n}}$$
$$=\frac{(q^3;q^3)_\infty^6}{(q;q)_\infty^2(q^9;q^9)_\infty^2}\left\{(q;q)_\infty^6+9q(q;q)_\infty^3(q^9;q^9)_\infty^3+27q^2(q^9;q^9)_\infty^6\right\}^{1/3}. \tag{2.2}$$

In comparing (2.2) with (2.1), we see that it remains to show that

$$\left\{(q;q)_\infty^6+9q(q;q)_\infty^3(q^9;q^9)_\infty^3+27q^2(q^9;q^9)_\infty^6\right\}^{1/3}=\frac{(q^3;q^3)_\infty^4}{(q;q)_\infty(q^9;q^9)_\infty}. \tag{2.3}$$

However, by [2, p. 345, Entry 1(iv)],

$$\left(3+\frac{(q;q)_\infty^3}{q(q^9;q^9)_\infty^3}\right)^3=27+\frac{(q^3;q^3)_\infty^{12}}{q^3(q^9;q^9)_\infty^{12}}. \tag{2.4}$$

By cubing both sides of (2.3) and rearranging terms, we easily see that (2.3) is equivalent to (2.4), and so this completes the proof of Lemma 2.1. □

Theorem 2.2. *For* $|q|<1$,

$$32(q;q)_\infty^{10}=9\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{3n(n+1)/2}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/6}\right)$$
$$-\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{3n(n+1)/2}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)/6}\right). \tag{2.5}$$

Proof. Recall Jacobi's identity [2, p. 39, Entry 24(ii)]

$$(q;q)_\infty^3=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{2.6}$$

Differentiating both sides of (2.6) with respect to q, we find that

$$-3(q;q)_\infty^3\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^n}=\tfrac{1}{4}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)(n^2+n)q^{n(n+1)/2-1}$$
$$=\tfrac{1}{16}\sum_{n=-\infty}^{\infty}(-1)^n\left((2n+1)^3-(2n+1)\right)q^{n(n+1)/2-1}. \tag{2.7}$$

Upon the rearrangement of (2.7) with the help of (2.6), we find that

$$2(q;q)_\infty^3\left(1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)=\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)/2}. \tag{2.8}$$

Hence, by (2.6) and (2.8),

$$\begin{aligned}
&9\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{3n(n+1)/2}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/6}\right)\\
&-\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{3n(n+1)/2}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)/6}\right)\\
=&4(q^3;q^3)_\infty^3(q^{1/3};q^{1/3})_\infty^3\left(9-9\cdot 24\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}-1+24\sum_{n=1}^{\infty}\frac{nq^{n/3}}{1-q^{n/3}}\right)\\
=&32(q^3;q^3)_\infty^3(q^{1/3};q^{1/3})_\infty^3\left(1+3\sum_{n=1}^{\infty}\frac{nq^{n/3}}{1-q^{n/3}}-27\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}}\right)\\
=&32(q^3;q^3)_\infty^3(q^{1/3};q^{1/3})_\infty^3\frac{(q;q)_\infty^{10}}{(q^3;q^3)_\infty^3(q^{1/3};q^{1/3})_\infty^3}\\
=&32(q;q)_\infty^{10},
\end{aligned}$$

by Lemma 2.1. This completes the proof. □

A completely different proof of Theorem 2.2 has been given by Liu [13].

The equality (2.8) is a special case of a general theorem of Ramanujan [18], [2, p. 61, Entry 35(i)] giving an identity for

$$\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^mq^{n(n+1)/2},$$

where m is any nonnegative integer. We prove the better formulation in Ramanujan's lost notebook [19] in Section 6.

3. A New Proof of Ramanujan's Congruence for $p(n)$ Modulo 11

Perhaps the simplest, most elementary proof of Ramanujan's congruence modulo 11 depends upon Winquist's identity [22]. Our proof is much different but in the same elementary spirit.

Theorem 3.1. *For each nonnegative integer n,*

$$p(11n+6)\equiv 0 \pmod{11}. \tag{3.1}$$

Proof. We begin by rewriting (2.5) in the form

$$32(q;q)_\infty^{10}=\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}\left\{9(2m+1)^3(2n+1)-(2m+1)(2n+1)^3\right\}q^{(9m^2+9m+n^2+n)/6}. \tag{3.2}$$

Let $u=2m+1$ and $v=2n+1$. Then (3.2) becomes

$$32(q;q)_\infty^{10}=\sum_{\substack{u,v=-\infty\\ u,v\equiv 1 \pmod 2}}^{\infty}(-1)^{(u+v-2)/2}uv(9u^2-v^2)q^{(9u^2+v^2-10)/24}. \tag{3.3}$$

If we write

$$(q;q)_\infty^{10} = \sum_{n=0}^{\infty} \alpha(n) q^n, \tag{3.4}$$

then equating coefficients of $q^n, n \geq 1$, on both sides of (3.3), we find that

$$\alpha(n) = \tfrac{1}{32} \sum_{\substack{u,v=-\infty \\ u,v\equiv 1 \,(\mathrm{mod}\, 2) \\ 9u^2+v^2-10=24n}}^{\infty} (-1)^{(u+v-2)/2} uv(9u^2 - v^2). \tag{3.5}$$

If $n \equiv 6 \,(\mathrm{mod}\, 11)$, then $9u^2 + v^2 - 10 \equiv 1 \,(\mathrm{mod}\, 11)$, or $9u^2 + v^2 \equiv 0 \,(\mathrm{mod}\, 11)$. By examining all cases modulo 11, we see that both $u, v \equiv 0 \,(\mathrm{mod}\, 11)$. It follows from (3.5) that

$$\alpha(11n + 6) \equiv 0 \,(\mathrm{mod}\, 11^4). \tag{3.6}$$

Now, from (3.4),

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty} = \frac{(q;q)_\infty^{10}}{(q;q)_\infty^{11}} \equiv \frac{(q;q)_\infty^{10}}{(q^{11};q^{11})_\infty} = \frac{\sum_{n=0}^{\infty} \alpha(n) q^n}{(q^{11};q^{11})_\infty} \,(\mathrm{mod}\, 11).$$

Extracting those terms with indices of the form $11n + 6$ and employing (3.4), we conclude that

$$\sum_{n=0}^{\infty} p(11n+6) q^{11n+6} \equiv \frac{\sum_{n=0}^{\infty} \alpha(11n+6) q^{11n+6}}{(q^{11};q^{11})_\infty} \equiv 0 \,(\mathrm{mod}\, 11). \tag{3.7}$$

The congruence (3.1) is now immediate from (3.7) and (3.6). □

The congruence (3.6) also follows from a result of M. Newman [14, p. 489], [15, p. 70]. See also Winquist's paper [22, p. 58].

4. A New Proof of Lemma 2.1

In this section, we give a new proof of Lemma 2.1, which uses less sophisticated machinery than our first proof. Our proof is in the spirit of that of L.–C. Shen [21]. Other proofs of Lemma 4.1 can be found in the papers [11, eq. (3.5)], [12, eq. (7.19)] by Liu. We frequently use without comment the elementary transformation

$$\sum_{n=0}^{\infty} \frac{x^n}{1 - yq^n} = \sum_{n=0}^{\infty} \frac{y^n}{1 - xq^n}. \tag{4.1}$$

Lemma 4.1. *If*

$$a(q) := 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right), \tag{4.2}$$

then

$$a^2(q) = 1 + 12 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} - 36 \sum_{n=1}^{\infty} \frac{nq^{3n}}{1 - q^{3n}}. \tag{4.3}$$

This is part of Entry 3(i) in Chapter 21 of Ramanujan's second notebook [18], [2, p. 460], but we provide a much different, simpler proof here.

Proof. Define

$$\phi(x) := x(q/x^6;q^3)_\infty(x^6q^2;q^3)_\infty(q^3;q^3)_\infty \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty}(-1)^n x^{6n+1}q^{n(3n+1)/2}. \tag{4.5}$$

Then

$$\phi''(x) = 6\sum_{n=-\infty}^{\infty}(-1)^n(6n^2+n)x^{6n-1}q^{n(3n+1)/2}. \tag{4.6}$$

By logarithmic differentiation and the use of (4.1),

$$\frac{\phi'}{\phi}(x) = \frac{1}{x} + 6\sum_{n=0}^{\infty}\left(\frac{x^{-7}q^{3n+1}}{1-x^{-6}q^{3n+1}} - \frac{x^5q^{3n+2}}{1-x^6q^{3n+2}}\right) \tag{4.7}$$

$$= \frac{1}{x} + 6\sum_{n=0}^{\infty}\left(\frac{x^{-6n-7}q^{n+1}}{1-q^{3n+3}} - \frac{x^{6n+5}q^{2n+2}}{1-q^{3n+3}}\right). \tag{4.8}$$

Next, from (4.8),

$$\left(\frac{\phi'}{\phi}\right)'(x) = -\frac{1}{x^2} - 6\sum_{n=0}^{\infty}\left(\frac{(6n+7)x^{-6n-8}q^{n+1}}{1-q^{3n+3}} + \frac{(6n+5)x^{6n+4}q^{2n+2}}{1-q^{3n+3}}\right). \tag{4.9}$$

Therefore from (4.7) and (4.2),

$$\frac{\phi'}{\phi}(1) = a(q), \tag{4.10}$$

From (4.6) and (4.9),

$$\begin{aligned}
\frac{\phi''}{\phi}(1) =& \frac{6}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n(6n^2+n)q^{n(3n+1)/2} \\
=& \frac{6}{(q;q)_\infty}\left\{2\sum_{n=-\infty}^{\infty}(-1)^n(3n^2+n)q^{n(3n+1)/2} - \sum_{n=-\infty}^{\infty}(-1)^n nq^{n(3n+1)/2}\right\} \\
=& \frac{6}{(q;q)_\infty}\left\{4q\frac{d}{dq}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}\right) - \frac{d}{dx}\left(\sum_{n=-\infty}^{\infty}(-1)^n x^n q^{n(3n+1)/2}\right)_{x=1}\right\} \\
=& \frac{6}{(q;q)_\infty}\left\{4q\frac{d}{dq}\left((q;q)_\infty\right) - \frac{d}{dx}\left((q/x;q^3)_\infty(xq^2;q^3)_\infty(q^3;q^3)_\infty\right)_{x=1}\right\} \\
=& \frac{6}{(q;q)_\infty}\left\{4(q;q)_\infty\sum_{n=0}^{\infty}\frac{-nq^n}{1-q^n} - (q;q)_\infty\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)\right\} \\
=& -24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right).
\end{aligned} \tag{4.11}$$

Lastly, by (4.9),

$$\left(\frac{\phi'}{\phi}\right)'(1) = -1 - 6\sum_{n=0}^{\infty}\left(\frac{(6n+7)q^{n+1}}{1-q^{3n+3}} - \frac{(6n+5)q^{2n+2}}{1-q^{3n+3}}\right). \tag{4.12}$$

Relating (4.10)–(4.12) by the elementary differential equation

$$\left(\frac{\phi'}{\phi}\right)^2 = \frac{\phi''}{\phi} - \left(\frac{\phi'}{\phi}\right)',$$

we find that

$$\begin{aligned}
a^2(q) =& 1 + 6\sum_{n=0}^{\infty}\left(\frac{(6n+7)q^{n+1}}{1-q^{3n+3}} + \frac{(6n+5)q^{2n+2}}{1-q^{3n+3}}\right)\\
&- 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)\\
=& 1 + 36\sum_{n=0}^{\infty}\left(\frac{(n+1)q^{n+1}}{1-q^{3n+3}} + \frac{(n+1)q^{2n+2}}{1-q^{3n+3}}\right) - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\\
=& 1 + 36\sum_{n=0}^{\infty}\frac{(n+1)(q^{n+1}+q^{2n+2})}{1-q^{3n+3}} - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\\
=& 1 + 36\sum_{n=1}^{\infty}\frac{n(q^n+q^{2n})}{1-q^{3n}} - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\\
=& 1 + 36\sum_{n=1}^{\infty}\frac{nq^n(1+q^n+q^{2n})}{1-q^{3n}} - 36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}} - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\\
=& 1 + 12\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 36\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{3n}},
\end{aligned}$$

and this completes the proof of Lemma 4.1. □

We state without proof Entry 1(v) in [2, p. 346] in a slightly different form.

Lemma 4.2.

$$a(q^3) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty} + 3q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}. \tag{4.13}$$

Lemma 4.3.

$$\text{(i)} \qquad a(q) - a(q^3) = 6q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}, \tag{4.14}$$

$$\text{(ii)} \qquad 3a(q^3) - a(q) = 2\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}. \tag{4.15}$$

These equalities follow from [5, pp. 93–94, eqs. (2.8), (2.9); p. 109, eq. Lemma 5.1], but we provide a much different proof using a corollary of Ramanujan's famous ${}_1\psi_1$-summation formula (4.16).

Proof. Referring to Ramanujan's ${}_1\psi_1$ summation formula as given in [2, p. 34, eq. (17.6)], we set $z = x$, $a = y$, and $b = qy$ to deduce that

$$\sum_{n=-\infty}^{\infty} \frac{x^n}{1-yq^n} = \frac{(xy;q)_\infty(\frac{q}{xy};q)_\infty(q;q)_\infty^2}{(x;q)_\infty(\frac{q}{x};q)_\infty(y;q)_\infty(\frac{q}{y};q)_\infty}, \qquad |q|<|x|<1. \tag{4.16}$$

By (4.2),

$$\begin{aligned}
a(q)-a(q^3) =&6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}-\frac{q^{9n+3}}{1-q^{9n+3}}+\frac{q^{9n+6}}{1-q^{9n+6}}\right)\\
=&6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}+q^{6n+2}}{1-q^{9n+3}}-\frac{q^{3n+2}+q^{6n+4}}{1-q^{9n+6}}\right)\\
=&6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{9n+3}}-\frac{q^{6n+4}}{1-q^{9n+6}}\right)\\
=&6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{9n+3}}+\frac{q^{-3n-2}}{1-q^{-9n-6}}\right)\\
=&6q\frac{(q^9;q^9)_\infty^2}{(q^3;q^9)_\infty(q^6;q^9)_\infty},
\end{aligned}$$

where we applied (4.16) with q replaced by q^9 and $x = y = q^3$. This completes the proof of (4.14).

By (4.14) and (4.13),

$$\begin{aligned}
3a(q^3)-a(q) =&2a(q^3)-\left\{a(q)-a(q^3)\right\}\\
=&2\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}+6q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}-6q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}=2\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty},
\end{aligned}$$

and this completes the proof of (4.15). □

We are now set to complete our new proof of Lemma 2.1.

Proof. By Lemmas 4.1–4.2,

$$\begin{aligned}
1+3\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-27\sum_{n=1}^{\infty}\frac{nq^{9n}}{1-q^{9n}} &= \frac{1}{4}a^2(q)+\frac{3}{4}a^2(q^3)\\
=&\frac{1}{4}\left\{a(q)-a(q^3)\right\}^2+\frac{1}{2}a(q^3)\left\{a(q)-3a(q^3)\right\}+2a^2(q^3)\\
=&9q^2\frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2}-\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}\left\{\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}+3q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}\right\}\\
&+2\left\{\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}+3q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}\right\}^2\\
=&\frac{1}{(q^3;q^3)^2}\left\{(q;q)_\infty^6+9q(q;q)_\infty^3(q^9;q^9)_\infty^3+27q^2(q^9;q^9)_\infty^6\right\}
\end{aligned}$$

$$=\frac{(q^3;q^3)_\infty^{10}}{(q;q)_\infty^3(q^9;q^9)_\infty^3},$$

where in the last equality we applied [2, p. 345, Entry 1(iv)]

$$\left(3+\frac{(q;q)_\infty^3}{q(q^9;q^9)_\infty^3}\right)^3=27+\frac{(q^3;q^3)_\infty^{12}}{q^3(q^9;q^9)_\infty^{12}}. \tag{4.17}$$

□

5. Another Proof of Theorem 2.2

Returning to (2.6), set

$$J(q):=(q;q)_\infty^3=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} \tag{5.1}$$

and

$$F(q):=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)/2}. \tag{5.2}$$

Proceeding as in (2.7), we easily find that

$$q\frac{d}{dq}J(q^3)=\frac{3}{8}F(q^3)-\frac{3}{8}J(q^3) \tag{5.3}$$

and

$$q\frac{d}{dq}J(q^{1/3})=\frac{1}{24}F(q^{1/3})-\frac{1}{24}J(q^{1/3}). \tag{5.4}$$

Thus, (2.5) can be written in the form

$$\begin{aligned}32(q;q)_\infty^{10}=&36\left\{\frac{8}{3}q\frac{d}{dq}J(q^3)+J(q^3)\right\}J(q^{1/3})\\&-4\left\{24q\frac{d}{dq}J(q^{1/3})+J(q^{1/3})\right\}J(q^3),\end{aligned}$$

or, upon simplification,

$$(q;q)_\infty^{10}=3q\frac{d}{dq}\{J(q^3)\}J(q^{1/3})-3q\frac{d}{dq}\{J(q^{1/3})\}J(q^3)+J(q^3)J(q^{1/3}). \tag{5.5}$$

If we divide both sides of (5.5) by $J^2(q^3)$, we find that

$$\frac{(q;q)_\infty^{10}}{J^2(q^3)}=-3q\frac{d}{dq}\left\{\frac{J(q^{1/3})}{J(q^3)}\right\}+\frac{J(q^{1/3})}{J(q^3)}. \tag{5.6}$$

We will prove (5.6) by using several elementary facts about theta functions from Ramanujan's notebooks [2].

First, let us introduce two of Ramanujan's theta functions. Define, for $|q|<1$,

$$\varphi(q):=\sum_{n=-\infty}^{\infty}q^{n^2}=(-q;q^2)_\infty^2(q^2;q^2)_\infty \tag{5.7}$$

and

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{5.8}$$

where the product representations in (5.7) and (5.8) follow from the Jacobi triple product identity [2, pp. 36–38]. Also, after Ramanujan, define

$$\chi(q) := (-q;q^2)_\infty \qquad \text{and} \qquad f(-q) := (q;q)_\infty. \tag{5.9}$$

Some basic properties of the functions φ, ψ, f, and χ are [2, p. 39, Entry 24]

$$\frac{\psi^2(q)}{\psi^2(-q)} = \frac{\varphi(q)}{\varphi(-q)}, \tag{5.10}$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \sqrt[3]{\frac{\varphi(q)}{\psi(-q)}} = \frac{\varphi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)}, \tag{5.11}$$

$$f^3(-q^2) = \varphi(-q)\psi^2(q). \tag{5.12}$$

By (5.11) and (5.12), we find that

$$f^3(-q) = \psi(q)\varphi^2(-q). \tag{5.13}$$

We now use two identities [2, p. 345, Entry 1]

$$1 - 8\nu^3 = \frac{\varphi^4(-q)}{\varphi^4(-q^3)}, \tag{5.14}$$

$$3 + \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)} = \frac{1}{\nu} + 4\nu^2, \tag{5.15}$$

where

$$\nu(q) = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)}. \tag{5.16}$$

Differentiating both sides of (5.15) with respect to q and then multiplying both sides by $-3q^{4/3}$, we deduce that

$$\frac{f^3(-q^{1/3})}{f^3(-q^3)} - 3q\frac{d}{dq}\left\{\frac{f^3(-q^{1/3})}{f^3(-q^3)}\right\} = 3q^{4/3}\frac{1-8\nu^3}{\nu^2}\nu'. \tag{5.17}$$

Note that $J(q) = f^3(-q)$ in the notation of (5.9). Comparing (5.17) to (5.6), we see that it remains to show that

$$3q^{4/3}\frac{1-8\nu^3}{\nu^2}\nu' = \frac{f^{10}(-q)}{f^6(-q^3)}. \tag{5.18}$$

Next, we evaluate ν'. To do this, we use (5.21) below which relates ν with the *multiplier* m defined by (5.20). We then derive a formula for m'. By [2, p. 223, Entry 3(ii)] and [2, pp. 226–227, Entry 4(iii), (iv)], we can verify that

$$4\frac{\psi^3(q)}{\psi(q^3)} - \frac{\varphi^3(q)}{\varphi(q^3)} = 3\varphi(q)\varphi(q^3). \tag{5.19}$$

(See also [3, p. 16, Thm. 5.8; p. 18, Thm. 5.15] for better proofs than those in [2].) For convenience, let us define

$$m := m(q) := \frac{\varphi^2(q)}{\varphi^2(q^3)}. \tag{5.20}$$

By (5.14),

$$1 - 8\nu^3 = m^2(-q). \tag{5.21}$$

Dividing both sides of (5.19) by $\varphi^3(q)/\varphi(q^3)$, we find that

$$4\frac{\psi^3(q)\varphi(q^3)}{\psi(q^3)\varphi^3(q)} = 1 + 3\frac{\varphi^2(q^3)}{\varphi^2(q)} = \frac{m+3}{m}, \tag{5.22}$$

or

$$\frac{\varphi^3(q)\psi(q^3)}{\psi^3(q)\varphi(q^3)} = \frac{4m}{m+3}. \tag{5.23}$$

Taking logarithmic derivatives of both sides of (5.23) with respect to q, we arrive at

$$3\Big(\frac{\varphi'(q)}{\varphi(q)} - \frac{\psi'(q)}{\psi(q)}\Big) - 3q^2\Big(\frac{\varphi'(q^3)}{\varphi(q^3)} - \frac{\psi'(q^3)}{\psi(q^3)}\Big) = \frac{3m'}{m(m+3)}. \tag{5.24}$$

However, by [2, p. 51, Entry 32(i)],

$$\frac{\varphi'(q)}{\varphi(q)} - \frac{\psi'(q)}{\psi(q)} = \frac{1-\varphi^4(-q)}{8q}. \tag{5.25}$$

Employing (5.25) in (5.24) with q replaced by q and q^3, respectively, we obtain

$$\frac{1-\varphi^4(-q)}{8q} - q^2\frac{1-\varphi^4(-q^3)}{8q^3} = \frac{m'}{m(m+3)}. \tag{5.26}$$

In (5.26), we solve for m' and use (5.14) and (5.16) to obtain

$$\begin{aligned}
m' &= \frac{1}{8q}m(m+3)(\varphi^4(-q^3) - \varphi^4(-q)) \\
&= \frac{1}{8q}m(m+3)\varphi^4(-q^3)\left(1 - \frac{\varphi^4(-q)}{\varphi^4(-q^3)}\right) \\
&= \frac{1}{8q}m(m+3)\varphi^4(-q^3)8\nu^3 \\
&= m(m+3)\varphi^4(-q^3)\frac{\chi^3(-q)}{\chi^9(-q^3)}.
\end{aligned} \tag{5.27}$$

Using (5.22), (5.20), and the equality of the first and third expressions in (5.11), we conclude from (5.27) that

$$m' = 4\frac{\varphi(q)}{\varphi^3(q^3)}\varphi(-q)\varphi(-q^3)\psi^2(q)\psi^2(q^3). \tag{5.28}$$

Equality (5.28), upon the use of (5.10) with q replaced by q and q^3, respectively, simplifies to

$$m' = 4\frac{\varphi^2(q)}{\varphi^2(q^3)}\psi^2(-q)\psi^2(-q^3). \tag{5.29}$$

Differentiating both sides of (5.21) with respect to q, we find that

$$-24\nu^2\nu' = -2m(-q)m'(-q). \tag{5.30}$$

Now, we use (5.20) and (5.29), with q replaced by $-q$, in (5.30) to conclude that

$$\nu' = \frac{1}{3\nu^2}\frac{\varphi^4(-q)}{\varphi^4(-q^3)}\psi^2(q)\psi^2(q^3). \tag{5.31}$$

We return to (5.18) and use (5.31). It thus suffices to show that

$$3q^{4/3}\frac{1-8\nu^3}{\nu^2}\frac{1}{3\nu^2}\frac{\varphi^4(-q)}{\varphi^4(-q^3)}\psi^2(q)\psi^2(q^3) = \frac{f^{10}(-q)}{f^6(-q^3)}. \tag{5.32}$$

After simplification with the help of (5.13) and (5.14), (5.32) can be reduced to

$$q^{-4/3}\nu^4 = \frac{\varphi^2(-q)\psi^4(q^3)}{f(-q)\psi(q)\varphi^4(-q^3)}, \tag{5.33}$$

which we now easily prove. From (5.11), we deduce that

$$f(-q)\psi(q) = f^2(-q^2) \tag{5.34}$$

and that

$$\chi(q) = \sqrt{\frac{\varphi(q)}{f(-q^2)}} = \sqrt[3]{\frac{\varphi(q)}{\psi(-q)}}. \tag{5.35}$$

Using first (5.34) and then (5.35) in (5.33), we find that it suffices to prove that

$$q^{-4/3}\nu^4 = \frac{\chi^4(-q)}{\chi^{12}(-q^3)}, \tag{5.36}$$

which follows from the definition (5.16) of ν. Hence, the proof of (5.6) is complete.

6. A Class of Infinite Series Representable in Terms of Eisenstein Series

Recall the definitions of Ramanujan's Eisenstein series, P, Q, and R,

$$P(q) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \tag{6.1}$$

$$Q(q) := 1 + 240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}, \tag{6.2}$$

and

$$R(q) := 1 - 504\sum_{k=1}^{\infty}\frac{k^5q^k}{1-q^k}, \tag{6.3}$$

where $|q| < 1$. On page 369 of his lost notebook [19], Ramanujan briefly considers two classes of infinite series. Each member of each class can be represented as a polynomial in P, Q, and R. One of the classes is considered in more detail on page 188 of his lost notebook, and this page was extensively examined by Berndt and A. J. Yee [6]. Ramanujan briefly considered the second class in Entry 35(i) of Chapter 16 in his second notebook [18], [2, pp. 61–62], where a recurrence relation is given in terms of members of yet a third class of infinite series. The approach indicated by Ramanujan on page 369 of his lost notebook, however, is neater and more direct, with the aforementioned third class of series not arising. Our purpose in this section is to prove the claims about that class of series in the lost notebook not examined by Berndt and Yee in their paper [6], namely, the series $U_n(q)$ below.

Define, for each nonnegative integer n,

$$U_n(q) := \frac{1}{(q;q)_\infty^3}\sum_{j=1}^{\infty}(-1)^j(2j-1)^{n+1}q^{j(j-1)/2} := \frac{F_n(q)}{(q;q)_\infty^3}. \tag{6.4}$$

It is easy to show that

$$P(q) = 1 + 24q\frac{\frac{d}{dq}(q;q)_\infty}{(q;q)_\infty}. \tag{6.5}$$

Recall also Ramanujan's famous differential equations [16], [17, p. 142],

$$q\frac{dP}{dq} = \frac{P^2-Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ-R}{3}, \qquad \text{and} \qquad q\frac{dR}{dq} = \frac{PR-Q^2}{2}. \tag{6.6}$$

The key to Ramanujan's work on $U_n(q)$ is the following differential-recurrence relation [19, p. 369].

Lemma 6.1. *For each nonnegative integer n,*

$$U_{n+2}(q) = P(q)U_n(q) + 8qU_n'(q). \tag{6.7}$$

Proof. By the definition of $U_n(q)$, (6.4),

$$U_n'(q) = \frac{F_n'(q)(q;q)_\infty - 3F_n(q)\frac{d}{dq}(q;q)_\infty}{(q;q)_\infty^4},$$

so that, by (6.5),

$$\begin{aligned} P(q)U_n(q) + 8qU_n'(q) &= \left(1 + 24q\frac{\frac{d}{dq}(q;q)_\infty}{(q;q)_\infty}\right)\frac{F_n(q)}{(q;q)_\infty^3} + \frac{8qF_n'(q)(q;q)_\infty - 24F_n(q)q\frac{d}{dq}(q;q)_\infty}{(q;q)_\infty^4} \\ &= \frac{F_n(q) + 8qF_n'(q)}{(q;q)_\infty^3}. \end{aligned} \tag{6.8}$$

On the other hand, by a simple calculation

$$\begin{aligned} 8qF_n'(q) &= \sum_{j=1}^{\infty}(-1)^j(2j-1)^{n+1}\left((4j^2-4j+1)-1\right)q^{j(j-1)/2} \\ &= F_{n+2}(q) - F_n(q). \end{aligned} \tag{6.9}$$

Substituting (6.9) into (6.8) and simplifying, we complete the proof. ∎

Theorem 6.2. *If* $U_n(q)$ *is defined by* (6.4)*, then*

$$U_0(q) = 1, \tag{6.10}$$

$$U_2(q) = P, \tag{6.11}$$

$$U_4(q) = \frac{1}{3}\left(5P^2 - 2Q\right), \tag{6.12}$$

$$U_6(q) = \frac{1}{9}\left(35P^3 - 42PQ + 16R\right), \tag{6.13}$$

$$U_8(q) = \frac{1}{3}\left(35P^4 - 84P^2Q - 12Q^2 + 64PR\right), \tag{6.14}$$

$$U_{10}(q) = \frac{1}{9}\left(385P^5 - 1540P^3Q - 660PQ^2 + 1760P^2R + 64QR\right). \tag{6.15}$$

Proof. The trivial equality (6.10) follows immediately from (6.4) and Jacobi's identity (2.6).

Setting $n = 0$ in (6.7) and using (6.10), we deduce (6.11), which is the same as (2.8).

Next, setting $n = 2$ in (6.7), employing (6.11), and then using the first equation in (6.6), we easily complete the proof of (6.12).

Fourthly, apply the differential operator $q\frac{d}{dq}$ to (6.12), use (6.7), and then employ the first two equations of (6.6) to find that

$$U_6 - PU_4 = \frac{40}{3} \cdot 2P\left(\frac{P^2 - Q}{12}\right) - \frac{16}{3}\frac{PQ - R}{3}.$$

The desired result (6.13) now follows from (6.12) and simplification.

Fifthly, apply the differential operator $q\frac{d}{dq}$ to (6.13), use (6.7), and then employ all the equations of (6.6) to find that

$$U_8 - PU_6 = \frac{8}{9}\left(105P^2\frac{P^2 - Q}{12} - 42Q\frac{P^2 - Q}{12} - 42P\frac{PQ - R}{3} + 16\frac{PR - Q^2}{2}\right).$$

If we use (6.13) on the left side above, collect terms with like powers, and simplify, we obtain (6.14).

Lastly, apply the differential operator $q\frac{d}{dq}$ to (6.14), use (6.7), and then employ all the equations of (6.6) to find that

$$\begin{aligned} U_{10} - PU_8 = &\frac{8}{3}\left(140P^3\frac{P^2 - Q}{12} - 168PQ\frac{P^2 - Q}{12} - 84P^2\frac{PQ - R}{3}\right. \\ &\left. -24Q\frac{PQ - R}{3} + 64R\frac{P^2 - Q}{12} + 64P\frac{PR - Q^2}{2}\right). \end{aligned}$$

Using (6.14) on the left side above and then simplifying, we arrive at (6.15) to complete the proof. □

It is easy to see from our calculations above that we can deduce the following general theorem stated by Ramanujan [19, p. 369].

Theorem 6.3. *For any positive integer* s*,*

$$U_{2s} = \sum K_{\ell,m,n} P^\ell Q^m R^n, \tag{6.16}$$

where the sum is over all nonnegative triples of integers ℓ, m, n such that $\ell+2m+3n = s$.

Although one can find formulas for some of the coefficients $K_{\ell,m,n}$ in (6.16), it seems extremely difficult to find a general formula for all $K_{\ell,m,n}$. Along these lines, see the paper by Berndt and Yee [6] concerning Ramanujan's attempt to find a general formula for the other series on page 369 of [19].

7. A related identity

When we showed Theorem 2.2 to Dennis Stanton, he experimented on his laptop computer and discovered a similar identity wherein, roughly speaking, "9" is replaced by "4." Our purpose in this section is to establish Stanton's analogue of (2.5).

Theorem 7.1. *For $|q| < 1$,*

$$\begin{aligned}12\varphi^5(-q)\psi^5(\sqrt{q}) &= 12\frac{(q;q)_\infty^{20}}{(\sqrt{q};\sqrt{q})^5(q^2;q^2)_\infty^5} \qquad (7.1)\\ &=4\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/4}\right)\\ &\quad-\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)/4}\right).\end{aligned}$$

We require the following lemma in our proof of Theorem 7.1. This lemma is, in fact, due to Ramanujan [18], [5, p. 377, Entry 38], but our proof here is completely different from that in [5].

Lemma 7.2. *We have*

$$1+8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-32\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}}=\frac{(q^2;q^2)_\infty^{20}}{(q;q)_\infty^8(q^4;q^4)_\infty^8}=\varphi^4(q). \qquad (7.2)$$

Proof. First, note that the second equality of (7.2) follows from (5.7).

Next, recall a representation for $\varphi^4(q)$, useful in deriving a famous identity of Jacobi for the number of ways a positive integer can be represented as a sum of four squares, namely [2, p. 54, eq. (33.5), with $-q^2$ replaced by q],

$$\varphi^4(q)=1+8\sum_{n=1}^{\infty}\frac{q^n}{(1+(-q)^n)^2}. \qquad (7.3)$$

Applying the elementary transformation

$$\sum_{n=0}^{\infty}\frac{xy^nq^n}{(1-xq^n)^2}=x\frac{d}{dx}\sum_{n=0}^{\infty}\frac{y^n}{1-xq^n}=x\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{1-yq^n}=\sum_{n=1}^{\infty}\frac{nx^n}{1-yq^n},$$

with q replaced by $-q$, $y=-1$, and $x=q$, on the right side of (7.3), we find that

$$\begin{aligned}\varphi^4(q) =&1+8\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n}\\ =&1+8\left\{\sum_{n=1}^{\infty}\frac{2nq^{2n}(1-q^{2n})}{1-q^{4n}}+\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1-q^{2n+1}}\right\}\\ =&1+8\left\{\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}-\sum_{n=1}^{\infty}\frac{4nq^{4n}}{1-q^{4n}}+\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1-q^{2n+1}}\right\}\\ =&1+8\left\{\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{4nq^{4n}}{1-q^{4n}}\right\},\end{aligned}$$

and this completes the proof of Lemma 7.2. □

Proof of Theorem 7.1. First note that the first equality of (7.1) follows readily from (5.7) and (5.8).

By (2.6) and (2.8),

$$\begin{aligned}&4\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/4}\right)\\ &-\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^3q^{n(n+1)/4}\right)\\ =&4(q^2;q^2)_\infty^3(\sqrt{q};\sqrt{q})_\infty^3\left(4-4\cdot 24\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}-1+24\sum_{n=1}^{\infty}\frac{nq^{n/2}}{1-q^{n/2}}\right)\\ =&12(q^2;q^2)_\infty^3(\sqrt{q};\sqrt{q})_\infty^3\left(1+8\sum_{n=1}^{\infty}\frac{nq^{n/2}}{1-q^{n/2}}-32\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right)\\ =&12\frac{(q;q)_\infty^{20}}{(\sqrt{q};\sqrt{q})_\infty^5(q^2;q^2)_\infty^5},\end{aligned}$$

by Lemma 7.2 and (5.7). This completes the proof of Theorem 7.1. □

8. Further Remarks

J.–P. Serre [20] found a different representation for $(q;q)_\infty^{10}$ in the course of proving that $(q;q)_\infty^r$ is lacunary for even r if and only if $r=2,4,6,8,10,14,26$. We now show that Theorem 2.2 can be utilized to prove that $(q;q)_\infty^{10}$ is lacunary, i.e., the density of non-zero coefficients is 0.

Theorem 8.1. *The function $(q;q)_\infty^{10}$ is lacunary.*

Proof. Recall that the coefficients of $(q;q)_\infty^{10}$ are given by (3.5). Thus, the number of non-zero coefficients up to x, say, does not exceed the number of integers that can be represented as a sum of two squares $(3u)^2+v^2$. However, by a theorem of E. Landau [9], [10, pp. 59–66], [1] that was rediscovered by Ramanujan [8, pp. 62–63], [4, pp. 61–66], the number of such integers is asymptotic to $bx/\sqrt{\log x}$, for some positive constant b. The desired result now follows. □

Originally, we had hoped to find an elementary proof of (5.5), a proof possibly in the spirit of proofs of Jacobi's identity (2.6). This would then yield a more elementary proof of (2.5), but we have been unable to find such a proof. Perhaps this goal was unrealistic.

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