Determinants, Paths, and Plane Partitions

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Ira M. Gessel[1]
Department of Mathematics
Brandeis University
Waltham, MA 02254

X. G. Viennot
Département de Mathématiques
Université de Bordeaux I
33405 Talence, France

1. Introduction

In studying representability of matroids, Lindström [42] gave a combinatorial interpretation to certain determinants in terms of disjoint paths in digraphs. In a previous paper [25], the authors applied this theorem to determinants of binomial coefficients. Here we develop further applications. As in [25], the paths under consideration are lattice paths in the plane. Our applications may be divided into two classes: first are those in which a determinant is shown to count some objects of combinatorial interest, and second are those which give a combinatorial interpretation to some numbers which are of independent interest. In the first class are formulas for various types of plane partitions, and in the second class are combinatorial interpretations for Fibonomial coefficients, Bernoulli numbers, and the less-known Salié and Faulhaber numbers (which arise in formulas for sums of powers, and are closely related to Genocchi and Bernoulli numbers).

Other enumerative applications of disjoint paths and related methods can be found in [14], [26], [19], [51–54], [57], and [67].

2. Lindström's theorem

Let D be an acyclic digraph. D need not be finite, but we assume that there are only finitely many paths between any two vertices. Let k be a fixed positive integer. A *k-vertex* is a k-tuple of vertices of D. If $\mathbf{u} = (u_1, \ldots, u_k)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_k)$ are k-vertices of D, a *k-path* from $\mathbf{u}$ to $\mathbf{v}$ is a k-tuple $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ such that A_i is a path from u_i to v_i. The k-path $\mathbf{A}$ is *disjoint* if the paths A_i are vertex-disjoint. Let S_k be the set of permutations of $\{1, 2, \ldots, k\}$. Then for $\pi \in S_k$, by $\pi(\mathbf{v})$ we mean the k-vertex $\big(v_{\pi(1)}, \ldots, v_{\pi(k)}\big)$.

Let us assign a weight to every edge of D. We define the weight of a path to be the product of the weights of its edges and the weight of a k-path to be the product of the weights of its components. Let $\mathsf{P}(u_i, v_j)$ be the set of paths from u_i to v_j and let $P(u_i, v_j)$ be the sum of their weights. Define $\mathsf{P}(\mathbf{u}, \mathbf{v})$ and $P(\mathbf{u}, \mathbf{v})$ analogously for k-paths from $\mathbf{u}$ to $\mathbf{v}$. Let $\mathsf{N}(\mathbf{u}, \mathbf{v})$ be the subset of $\mathsf{P}(\mathbf{u}, \mathbf{v})$ of disjoint paths and let $N(\mathbf{u}, \mathbf{v})$ be the sum of their weights. It is clear that for any permutation π of $\{1, 2, \ldots, k\}$,

$$P\big(\mathbf{u}, \pi(\mathbf{v})\big) = \prod_{i=1}^{k} P\big(u_i, v_{\pi(i)}\big) \tag{2.1}$$

We use the notation $|m_{ij}|_r^s$ to denote the determinant of the matrix $(m_{ij})_{i,j=r,\ldots,s}$.

Theorem 1. (Lindström [42])

$$\sum_{\pi \in S_k} (\operatorname{sgn} \pi) N\big(\mathbf{u}, \pi(\mathbf{v})\big) = |P(u_i, v_j)|_1^k\,.$$

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Proof. By (2.1) and the definition of a determinant, the formula is equivalent to

$$\sum_{\pi \in S_k} (\operatorname{sgn} \pi) N\big(\mathbf{u}, \pi(\mathbf{v})\big) = \sum_{\pi \in S_k} (\operatorname{sgn} \pi) P\big(\mathbf{u}, \pi(\mathbf{v})\big). \tag{2.2}$$

To prove (2.2) we construct a bijection $\mathbf{A} \mapsto \mathbf{A}^*$ from

$$\bigcup_{\pi \in S_k} \big[\mathsf{P}\big(\mathbf{u}, \pi(\mathbf{v})\big) - \mathsf{N}\big(\mathbf{u}, \pi(\mathbf{v})\big)\big]$$

to itself with the following properties:

(i) $\mathbf{A}^{**} = \mathbf{A}$.

(ii) The weight of $\mathbf{A}^*$ equals the weight of $\mathbf{A}$.

(iii) If $\mathbf{A} \in P\big(\mathbf{u}, \pi(\mathbf{v})\big)$ and $\mathbf{A}^* \in P\big(\mathbf{u}, \sigma(\mathbf{v})\big)$, then $\operatorname{sgn} \sigma = -\operatorname{sgn} \pi$.

We can then group together terms on the right side of (2.2) corresponding to pairs $\{\mathbf{A}, \mathbf{A}^*\}$ of nondisjoint k-paths, and all terms cancel except those on the left.

To construct the bijection, let $\mathbf{A} = (A_1, \dots, A_k)$ be a nondisjoint k-path. Let i be the least integer for which A_i intersects another path. Let x be the first point of intersection of A_i with another path and let j be the least integer greater than i for which A_j meets x. Construct A_i^* by following A_i to x and then following A_j to its end, and construct A_j^* similarly from A_j and A_i. For $l \neq i, j$, let $A_l^* = A_l$. Then properties (i), (ii), and (iii) are easily verified and the theorem is proved.

It is interesting to note that Lindström's applications of Theorem 1 are totally different from ours.

Let us say that a pair $(\mathbf{u}, \mathbf{v})$ of k-vertices is *nonpermutable* if $N(\mathbf{u}, \pi(\mathbf{v}))$ is empty whenever π is not the identity permutation. Then we have the following important corollary of Theorem 1:

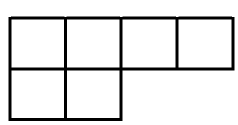
Corollary 2. *If* $(\mathbf{u}, \mathbf{v})$ *is nonpermutable, then* $N(\mathbf{u}, \mathbf{v}) = |P(u_i, v_j)|_1^k$.

An argument similar to that of Theorem 1 was apparently first given by Chaundy [11] in his work on plane partitions. (Thanks to David Bressoud for this reference.) Another related argument was given by Karlin and MacGregor [36]. We thank Joseph Kung for bringing Lindström's paper to our attention.

3. Plane partitions and tableaux

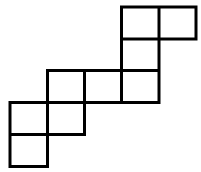
First we give some definitions. A *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a nonincreasing sequence of nonnegative integers, called the *parts* of λ. The sum of the parts of λ is denoted by $|\lambda|$. It is convenient to identify two partitions which differ only in the number of zeros. (All our formulas will remain valid under this identification.)

The *diagram* (or *Ferrers diagram*) of λ is an arrangements of squares with λ_i squares, left justified, in the ith row. (Zero parts are ignored.) We follow Macdonald [43] and draw the first (largest) part at the top, so that the diagram of (42) is



The *conjugate* λ' of λ is the partition whose diagram is the transpose of that of λ. We write $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for each i. If $\lambda \geq \mu$, then the diagram of $\lambda - \mu$ is obtained from the diagram of λ by removing the diagram of μ.

If $\lambda_i = \mu_{i-1} + 1$ for $i > 1$, then the diagram of $\lambda - \mu$ is called a *skew hook* (also called *rim hook* or *border strip*). For example,



is a skew hook of shape $(54421) - (331)$.

By an *array* we mean an array (p_{ij}) of integers defined for some values of i and j. A *(skew) plane partition* of *shape* $\lambda - \mu$ is a filling of the diagram of $\lambda - \mu$ with integers which are weakly decreasing in every row and column, or equivalently (if λ and μ have k parts), an array (p_{ij}) of integers defined for $1 \le i \le k$ and $\mu_i < j \le \lambda_i$ satisfying

$$p_{ij} \ge p_{i,j+1} \tag{3.1}$$

and

$$p_{ij} \ge p_{i+1,j} \tag{3.2}$$

whenever these entries are defined. The integers p_{ij} are called the *parts* of the plane partition. For example, a plane partition of shape $(431) - (110)$ is

	3	3	1
	3	0	
4			

If μ has no nonzero parts, it is omitted. The plane partition (p_{ij}) is *row-strict* if (3.2) is replaced by $p_{ij} > p_{i,j+1}$ and column-strictness is defined similarly.

Reverse plane partitions are defined by reversing all inequalities in the above definitions.

A *tableau* is a column-strict reverse plane partition. A *row-strict tableau* is a row-strict (but not necessarily column-strict) reverse plane partition. A *standard tableau* is a reverse plane partition in which the parts are $1, 2, \ldots, n$, without repetitions, for some n.

We shall first apply Theorem 1 to the digraph in which the vertices are lattice points in the plane and the edges go from (i, j) to $(i, j+1)$ and $(i+1, j)$. Thus paths in this digraph are ordinary lattice paths with unit horizontal and vertical steps. Later, we shall consider some modifications of this digraph.

Correspondences between k-paths and arrays are important in what follows. We restrict ourselves to k-paths from $\mathbf{u}$ to $\mathbf{v}$ where $u_i = (a_i, b_i)$ and $v_i = (c_i, d_i)$, and the parameters satisfy $a_{i+1} < a_i$, $b_{i+1} \ge b_i$, $c_{i+1} < c_i$, and $d_{i+1} \ge d_i$ for all i. These conditions imply that $(\mathbf{u}, \mathbf{v})$ is nonpermutable. The strictness conditions, which are not necessary for nonpermutability, allow shapes to be parametrized by partitions, and allow a simpler translation of the disjointness condition on k-paths into a condition on arrays.

The correspondences are determined by first choosing a labeling of all the horizontal steps in the digraph of lattice points. Then we associate to a k-path an array in which row i consists of the labels of the horizontal steps of path i, with each row shifted one place to the right in relation to the previous row. As an example,

Figure 1

in Figure 1 the horizontal step from (l, h) to $(l+1, h)$ is assigned the label h and the corresponding array is

	1	1
1	2	2
2	4	

With this labeling, a disjoint k-path corresponds to a tableau. If instead we assign to the horizontal step

Figure 2

from (l,h) to $(l+1,h)$ the label $l+h$, as in Figure 2, the corresponding array is a row-strict tableau:

	2	3
0	2	3
0	3	

To construct the most general labeling, it is convenient to start with the first labeling described above, and then "relabel."

The first correspondence sketched above, which assigns to the k-path $\mathbf{A}$ an array T, may be described more formally as follows: If there is a horizontal step from (l,h) to $(l+1,h)$ in path i, then we define $T_{i,l+i}$ to be h. In other words, T_{ij} is the height of the horizontal step in path i from $x = j-i$ to $x = j-i+1$ if such a step exists, and is undefined otherwise. The essential fact about this correspondence is that $\mathbf{A}$ is disjoint if and only if T is a tableau. (As a technicality, in order to be consistent with the requirement that $j \geq 1$ for all parts T_{ij}, we need that $a_k > -k$.) Note that a tableau does not uniquely determine a k-path since the endpoints are not determined. However the correspondence does give a bijection between tableaux of shape $\lambda - \mu$ satisfying $b_i \leq T_{ij} \leq d_i$, where $\mu_i = a_i + i - 1$ and $\lambda_i = c_i + i - 1$, and k-paths with initial points (a_i, b_i) and endpoints (c_i, d_i). Theorem 1 allows us then to count these tableaux.

We now "relabel" the tableau. Let L be a set of "labels" (which will usually be integers). For each $p \in L$ we have a weight $w(p)$, usually an indeterminate. A *relabeling function* is a sequence $\mathbf{f} = \{f_i\}_{i=-\infty}^{\infty}$ of injective functions from the integers to L. Given a relabeling function $\mathbf{f}$ we define the weight of a vertical step to be 1 and the weight of a horizontal step from (r,s) to $(r+1,s)$ to be $w(f_r(s))$. Then the sum of the weights of all paths from (a,b) to (c,d) is

$$H_{\mathbf{f}}(a,b,c,d) = \sum w(f_a(n_a))w(f_{a+1}(n_{a+1}))\cdots w(f_{c-1}(n_{c-1})), \tag{3.3}$$

where the sum is over all sequences $n_a, \ldots, n_{c-1}$ satisfying

$$b \leq n_a \leq n_{a+1} \leq \cdots \leq n_{c-1} \leq d.$$

(If $a = c$ then $H_{\mathbf{f}}(a,b,c,d)$ is 1 if $b \leq d$ and 0 if $b > d$.)

Now let $T = (T_{ij})$ be a tableau. We define $U = \mathbf{f}(T)$ to be an array of the same shape as T with $U_{ij} = f_{j-i}(T_{ij})$. Each f_i applies to a diagonal of the tableau since horizontal steps with the same abscissa correspond to tableau entries lying on the same diagonal. Thus the involution in the proof of Theorem 1 moves labels along a diagonal of the array, and so the same weighting must apply to a label in all positions on a diagonal

The condition that T_{ij} be a tableau satisfying $b_i \leq T_{ij} \leq d_i$ is equivalent to the conditions

$$\begin{gathered} f_{j-i}^{-1}(U_{ij}) \leq f_{j-i+1}^{-1}(U_{i,j+1}) \\ f_{j-i}^{-1}(U_{ij}) < f_{j-i-1}^{-1}(U_{i+1,j}) \\ b_i \leq f_{j-i}^{-1}(U_{ij}) \leq d_i. \end{gathered} \tag{3.4}$$

The reader may check, for example, that if we take $f_i(n) = n + i$, then $\mathbf{f}(T)$ is a row-strict tableau and if we take $f_i(n) = -n$, then $\mathbf{f}(T)$ is a column-strict plane partition. Let us define the weight of $U = \mathbf{f}(T)$ to be $\prod_u w(u)$ over all parts u of U. Then by Theorem 1, the sum of the weights of all arrays of the form $\mathbf{f}(T)$, where T is a tableau of shape $\lambda - \mu$, and $b_i \leq T_{ij} \leq d_i$, is the determinant $|P(u_i, v_j)|_1^k$,where $u_i = (a_i, b_i)$, $v_i = (c_i, d_i)$, $\mu_i = a_i + i - 1$, and $\lambda_i = c_i + i - 1$. Thus we have

Theorem 3. *Let $\mathbf{f}$ be a relabeling function, let λ and μ be partitions with k parts, and let the integers b_i and d_i satisfy $b_{i+1} \geq b_i$ and $d_{i+1} \geq d_i$. Then the sum of the weights of $\mathbf{f}(t)$ over all tableaux T of shape $\lambda - \mu$ satisfying $b_i \leq T_{ij} \leq d_i$ is*

$$|H_{\mathbf{f}}(\mu_i - i + 1, b_i, \lambda_j - j + 1, d_j)|_1^k, \tag{3.5}$$

where $H_{\mathbf{f}}$ is defined by (3.3).

In Section 5, we will use Theorem 3 to count tableaux in which an entry l in position (i, j) is assigned the weight x_{i-j}^l. For now, we consider the case in which $f_i(n) = n + ti$ for some number t, with $w(l) = x_l$, where the x_l are indeterminates. Here $H_{\mathbf{f}}(a, b, c, d)$ becomes

$$\sum x_{n_a+ta} x_{n_{a+1}+t(a+1)} \cdots x_{n_{c-1}+t(c-1)}, \tag{3.6}$$

where the sum is over $b \leq n_a \leq n_{a+1} \leq \cdots \leq n_{c-1} \leq d$. If we set $n_j + tj = m_{j-a+1}$, we may rewrite (3.6) as

$$\sum_{m_1, m_2, \cdots, m_{c-a}} x_{m_1} \cdots x_{m_{c-a}},$$

where the sum is over all sequences $m_1, \ldots, m_{c-a}$ satisfying $m_1 \geq b + ta$, $m_{c-a} \leq d + t(c-1)$, and $m_{j+1} \geq m_j + t$.

Let us now write $H_n^{(t)}(a, b)$ for $\sum x_{i_1} x_{i_2} \cdots x_{i_n}$ over all $i_1, \ldots, i_n$ satisfying $i_1 \geq a, i_n \leq b$, and $i_{l+1} \geq i_l + t$ for all l. Note that for $t = 0$, $H_n^{(t)}(a, b)$ is the complete symmetric function of degree n in $x_a, \ldots, x_b$, and for $t = 1$ it is the elementary symmetric function of degree n in these variables. For other values of t, $H_n^{(t)}(a, b)$ is not symmetric.

We find that (3.5) reduces to

$$H^{(t)}_{\lambda_j - \mu_i + i - j}\big(b_i + t(\mu_i - i + 1), d_j + t(\lambda_j - j)\big),$$

and simplifying (3.4) we obtain the following:

Corollary 4. *Let λ and μ be partitions with k parts and let the integers b_i and d_i satisfy b_{i+1} and $d_{i+1} \geq d_i$. Then the sum of the weights of all arrays U of shape $\lambda - \mu$ satisfying*

$$U_{ij} \leq U_{i,j+1} - t \tag{3.7}$$

$$U_{ij} < U_{i+1,j} + t \tag{3.8}$$

$$b_i + t(j - i) \leq U_{ij} \leq d_i + t(j - i) \tag{3.9}$$

is

$$\left| H^{(t)}_{\lambda_j - \mu_i + i - j}\left(b_i + t(\mu_i - i + 1), d_j + t(\lambda_j - j)\right) \right|_1^k.$$

Note that (3.9) may be replaced by inequalities on only the first and last element of each row:

$$b_i + t(\mu_i + 1 - i) \leq U_{i,\mu_i+1}$$

and

$$U_{i,\lambda_i} \leq d_i + t(\lambda_i - i).$$

If we set $A_i = b_i + t(\mu_i - i + 1)$ and $B_i = d_i + t(\lambda_i - i)$ we may restate this result as follows:

Let λ and μ be partitions with k parts and let the integers A_i and B_i satisfy $A_{i+1} - A_i \geq t(\mu_{i+1} - \mu_i - 1)$ and $B_{i+1} - B_i \geq t(\lambda_{i+1} - \lambda_i - 1)$. Then the sum of the weights of all arrays U of shape $\lambda - \mu$ satisfying (3.7), (3.8), and

$$A_i \leq U_{i,\mu_i+1}, \quad U_{i,\lambda_i} \leq B_i$$

is $\left|H^{(t)}_{\lambda_j-\mu_i+i-j}(A_i,B_j)\right|_1^k$.

We obtain further interesting results by substituting for the variables, or removing the part restrictions. In the cases $t=0$ and $t=1$, these generating functions are called flagged Schur functions [72].

First we remove the part restrictions in Corollary 4. We can do this most easily by taking the limit as b_i goes to $-\infty$ and d_i goes to $+\infty$. (It is easily verified that this is legitimate.) Let us define $h_n^{(t)}$ by

$$h_n^{(t)} = \lim_{\substack{a\to-\infty\\ b\to\infty}} H_n^{(t)}(a,b) = \sum x_{i_1}\cdots x_{i_n}$$

where the sum is over all $i_1,\dots,i_n$ satisfying $i_{l+1} \ge i_l + t$ for all l.

Thus $h_n^{(0)}$ is the ordinary complete symmetric function h_n and $h_n^{(1)}$ is the elementary symmetric function e_n. Let us write $s^{(t)}_{\lambda/\mu}$ for the determinant $|h^{(t)}_{\lambda_i-\mu_j+j-i}|_1^k$. (To agree with usual practice we have transposed the determinant in Corollary 4.) Thus for $t=0$, $s^{(t)}_{\lambda/\mu}$ is the ordinary Schur function. Then we have

Corollary 5. *$s^{(t)}_{\lambda/\mu}$ is the sum of the weights of all arrays p of shape $\lambda-\mu$ satisfying*

$$\begin{aligned} p_{ij} &\le p_{i+1,j} - t \\ = p_{ij} &< p_{i,j+1} + t. \end{aligned}$$

Let us call these tableaux *t-tableaux*. Note that the conjugate of a t-tableau is a $(1-t)$-tableau. It follows that $s^{(1-t)}_{\lambda/\mu} = s^{(t)}_{\lambda'/\mu'}$. If we set $e_n^{(t)} = h_n^{(1-t)}$ then we also have

$$s^{(t)}_{\lambda/\mu} = |e^{(t)}_{\lambda'_i-\mu'_j+j-i}|_1^k, \tag{3.10}$$

which reduces to a well-known formula in the case $t=0$.

There is a homomorphism θ from symmetric functions to exponential generating functions which is useful in counting standard tableaux. If f is any symmetric function we define

$$\theta(f) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!},$$

where f_n is the coefficient of $x_1x_2\cdots x_n$ in f. It is easily verified that θ is a homomorphism and that $\theta(h_n) = z^n/n!$. (See, for example, [24].) In particular, from the symmetric generating function $|h_{\lambda_i-\mu_j+j-i}|_1^k$ for tableaux of shape $\lambda-\mu$, we obtain the exponential generating function

$$\left|\frac{z^{\lambda_i-\mu_j+j-i}}{(\lambda_i-\mu_j+j-i)!}\right|_1^k$$

for standard tableaux of shape $\lambda-\mu$. Since each standard tableau contributes a term $z^n/n!$, where $n = \sum_i(\lambda_i-\mu_i)$, the number of standard tableaux of shape $\lambda-\mu$ is

$$n!\left|\frac{1}{(\lambda_i-\mu_j+j-i)!}\right|_1^k.$$

An interesting question is whether there is any reasonable expression for the coefficient of $x_1x_2\cdots x_n$ in $s^{(t)}_{\lambda/\mu}$.

4. The dimer problem

In the case $t=-1$ we can derive some particularly nice formulas which yield a surprising connection between tableaux and the dimer problem. Let us evaluate $e_n^{(-1)} = h_n^{(2)}$ at $x_1 = \cdots = x_{m-1} = 1$, $x_i = 0$ for other i. This is the number of sequences $a_1, \dots, a_n$ satisfying $a_1 \geq 1$, $a_n \leq m-1$, and $a_{i+1} \geq a_i + 2$. We may rewrite this condition as

$$1 \leq a_1 < a_2 - 1 < a_3 - 2 < \cdots < a_n - n + 1 \leq m - n,$$

so the number of such sequences is $\binom{m-n}{n}$. Let us set

$$P_m(u) = \sum_n e_n^{(-1)}(\underbrace{1,\dots,1}_{m-1})u^n = \sum_{n=0}^{\lfloor m/2 \rfloor} \binom{m-n}{n} u^n.$$

Lemma 6.

$$P_m(u) = \prod_{j=1}^{\lfloor m/2 \rfloor} (1 + 4u \cos^2 \frac{j\pi}{m+1}).$$

Proof. By Riordan [55, pp. 75–76] we have

$$P_m(u) = (-i)^m u^{m/2} U_m\left(\frac{i}{2\sqrt{u}}\right),$$

where $i = \sqrt{-1}$ and $U_m(x)$ is the Chebyshev polynomial determined by $U_m(\cos\theta) = \sin(m+1)\theta/\sin\theta$. Since $U_m(x)$ is a polynomial of degree m with leading coefficient 2^m which vanishes at $x = \cos(j\pi/(m+1))$ for $j = 1, \dots, m$, we have

$$U_m(x) = 2^m \prod_{j=1}^{m} \left(x - \cos\frac{j\pi}{m+1}\right).$$

Since

$$\cos\frac{j\pi}{m+1} = -\cos\left(\pi - \frac{j\pi}{m+1}\right) = -\cos\frac{(m+1-j)\pi}{m+1},$$

we have

$$U_m(x) = \begin{cases} 2^m \displaystyle\prod_{j=1}^{m/2} \left(x^2 - \cos^2\frac{j\pi}{m+1}\right), & m \text{ even;} \\ 2^m x \displaystyle\prod_{j=1}^{(m-1)/2} \left(x^2 - \cos^2\frac{j\pi}{m+1}\right), & m \text{ odd.} \end{cases}$$

Thus for m even we have

$$P_m(u) = (-u)^{m/2} 2^m \prod_{j=1}^{m/2} \left(-\frac{1}{4u} - \cos^2\frac{j\pi}{m+1}\right) = \prod_{j=1}^{m/2} \left(1 + 4u\cos^2\frac{j\pi}{m+1}\right)$$

and for m odd we have

$$P_m(u) = (-i)^m u^{m/2} 2^m \frac{i}{2\sqrt{u}} \prod_{j=1}^{(m-1)/2} \left(-\frac{1}{4u} - \cos^2\frac{j\pi}{m+1}\right) = \prod_{j=1}^{(m-1)/2} \left(1 + 4\cos^2\frac{j\pi}{m+1}\right).$$

Theorem 7.

$$s^{(-1)}_{\lambda/\mu}(\underbrace{1,\ldots,1}_{m-1}) = s_{\lambda/\mu}\left(4\cos^2\frac{\pi}{m+1},\ldots,4\cos^2\frac{\lfloor m/2\rfloor\pi}{m+1}\right).$$

Proof. By Lemma 6, we have

$$e^{(-1)}_n(\underbrace{1,\ldots,1}_{m-1}) = e_n\left(4\cos^2\frac{\pi}{m+1},\ldots,4\cos^2\frac{\lfloor m/2\rfloor\pi}{m+1}\right).$$

Then the result follows from the cases $t=-1$ and $t=0$ of (3.10).

Let us call a rectangle of height m and width m an $m\times n$ rectangle. The *dimer problem* is concerned with the number of ways of covering a rectangle with 1×2 (horizontal) and 2×1 (vertical) dominoes. We shall consider here only the case of an $m\times n$ rectangle in which both m and n are even. (Similar formulas exist when m and n are of opposite parity.) In this case it is easy to see that any covering must contain an even number of horizontal dominoes and an even number of vertical dominoes. Now let us assign to a covering with i vertical dominoes and j horizontal dominoes the weight $u^{i/2}v^{j/2}$. Kasteleyn [37] showed that the sum of the weights of all coverings of an $m\times n$ rectangle (which he calls an $n\times m$ rectangle) is

$$\prod_{i=1}^{m/2}\prod_{j=1}^{n/2}\left(4u\cos^2\frac{i\pi}{m+1}+4v\cos^2\frac{j\pi}{n+1}\right). \tag{4.1}$$

We can also interpret the product in (4.1) in terms of tableaux. Macdonald [43, Ex. 5, p. 37] gives the formula

$$\prod_{i=1}^{r}\prod_{j=1}^{s}(x_i+y_j) = \sum_\lambda s_\lambda(x)s_{\tilde\lambda'}(y),$$

where the sum is over all partitions λ with at most r parts and largest part at most s, and

$$\tilde\lambda' = (r-\lambda'_s,\ldots,r-\lambda'_1),$$

i.e., the diagram of $\tilde\lambda$, when rotated 180°, fits together with the diagram of λ to form an $r\times s$ rectangle. It follows that with $m=2r$ and $n=2s$ we have

$$\begin{aligned}\prod_{i=1}^{m/2}\prod_{j=1}^{n/2}\left(4u\cos^2\frac{i\pi}{m+1}+4v\cos^2\frac{j\pi}{n+1}\right) &= \sum_\lambda s^{(-1)}_\lambda(\underbrace{1,\ldots,1}_{m-1})s^{(-1)}_{\tilde\lambda'}(\underbrace{1,\ldots,1}_{n-1})u^{|\lambda|}v^{|\tilde\lambda'|}\\ &= \sum_\lambda s^{(-1)}_\lambda(\underbrace{1,\ldots,1}_{m-1})s^{(2)}_{\tilde\lambda}(\underbrace{1,\ldots,1}_{n-1})u^{|\lambda|}v^{|\tilde\lambda'|}\end{aligned} \tag{4.2}$$

The sum on the right side of (4.2) has a simple combinatorial interpretation: Let us define a *dimer tableau* to be an array (p_{ij}) with entries chosen from the alphabet $A\cup A'$, where $A=\{1,\ldots,m-1\}$ and $A'=\{1',\ldots,(n-1)'\}$ such that if $p_{i,j}=\alpha$ and $p_{i+1,j}=\beta$, one of the following holds:

(1) $\alpha\in A$ and $\beta\in A'$,

(2) $\alpha,\beta\in A$ and $\alpha\le\beta+1$,

(3) $\alpha,\beta\in A'$ and $\alpha<\beta-1'$,

and if $p_{ij}=\gamma$ and $p_{i,j+1}=\delta$, one of the following holds:

(1) $\gamma\in A$ and $\delta\in A'$,

(2) $\gamma,\delta\in A$ and $\gamma<\delta-1$,

(3) $\gamma,\delta\in A'$ and $\gamma\le\delta+1'$.

(The total order, addition, and subtraction in A' have their obvious meaning.) A dimer tableau with i entries in A and j entries in A' is assigned the weight u^iv^j. Then we have

Theorem 8. *Let m and n be even. Then the number of coverings of an $m \times n$ rectangle by $2i$ vertical dominoes and $2j$ horizontal dominoes is equal to the number of $(m/2) \times (n/2)$ dimer tableaux with i entries in $\{1, \ldots, m-1\}$ and j entries in $\{1', \ldots, (n-1)'\}$.*

A simple bijection between these two sets would be interesting. Such a bijection is easy to construct for $m = 2$. Here the dimer tableau has only one row, which consists of some number of 1's followed by a sequence of elements of $\{1', 2', \ldots, (n-1)'\}$, each at least $2'$ more than its predecessor. Given such a tableau, we construct a covering of a $2 \times n$ rectangle by putting the left end of a pair of horizontal dominoes at the positions corresponding to the primed numbers, and fill up the remaining space with vertical dominoes. Figure 3 illustrates the correspondence for 2×4 rectangles.

Figure 3

5. Trace generating functions

Let us now return to Theorem 3. We would like to count tableaux in which an entry l in position (i, j) is assigned the weight x_{i-j}^l. The *trace* of a plane partition (see Stanley [64]) is the sum of the elements on the main diagonal, and generating functions with these weights are called *trace generating functions* because they generalize enumeration by trace.

To apply Theorem 3 to this situation, we use the relabeling function $\mathbf{f}$ for which $f_i(l)$ is the ordered pair (i, l), which is assigned the weight x_i^l. If we take $d = \infty$ in (3.3) then $h(a, b, c) = H_{\mathbf{f}}(a, b, c, d)$ is easy to evaluate:

$$h(a,b,c) = \frac{(x_a x_{a+1} \cdots x_{c-1})^b}{(1 - x_a x_{a+1} \cdots x_{c-1})(1 - x_{a+1} x_{a+2} \cdots x_{c-1}) \cdots (1 - x_{c-1})}, \tag{5.1}$$

for $a < c$, with $h(a, b, a) = 1$ and $h(a, b, c) = 0$ for $a > c$. Then Theorem 3 yields

Theorem 9. *Let λ and μ be partitions with k parts, and let the integers b_i satisfy $b_{i+1} \geq b_i$. Then the trace generating function for tableaux T of shape $\lambda - \mu$ with every part in row i at least b_i is*

$$\left| h(\mu_i - i + 1, b_i, \lambda_j - j + 1) \right|_1^k,$$

where $h(a, b, c)$ is defined by (5.1).

If μ is empty and the b_i are all zero (or equivalently by column-strictness, $b_i = i - 1$) then there is an explicit product expression for the trace generating function, due to Gansner [22], using the Hillman-Grassl correspondence [31]. To describe Gansner's result, we define the *hook lengths* and *contents* of a diagram. The hook length of a square in a diagram is the number of squares to its right plus the number of squares below it plus one. Thus the hook length of square (i, j) is $\lambda_i + \lambda'_j - i - j + 1$. The *content* of square (i, j) is $j - i$. The hook lengths and contents of the diagram for the partition (431) are as follows:

6	4	3	1
4	2	1	
1			

Hook lengths

0	1	2	3
−1	0	1	
−2			

Contents

Since every entry of row i of a tableau of shape λ with nonnegative parts is at least $i - 1$, we can factor out

$$\prod_{i=1}^{k} (x_{-i} x_{-i+1} \cdots x_{-i+\lambda_i - 1})^{i-1}$$

from the trace generating function for tableaux to leave the trace generating function for reverse plane partitions (with no strictness condition). Gansner [22, p. 132, Theorem 4.1] showed that the trace generating function for reverse plane partitions of shape λ with nonnegative entries may be described as follows: Let us define the *hook* of a square s in the diagram of a partition to be the set of all squares to the right of s or below s, together with s. The *content* $c(s)$ of square (i,j) is $j-i$. Let $X(s)$ be the product $\prod x_{c(t)}$ over all squares t in the hook of s. Then Gansner's result is that the trace generating function for reverse plane partitions of shape λ with nonnegative entries is

$$\prod_s \frac{1}{1-X(s)}$$

where the product is over all squares s in the diagram of λ.

Another proof of Gansner's theorem has been given by Eğecioğlu and Remmel [19] by evaluating a determinant related to those we shall discuss in Section **?**

6. Hook Schur functions.

Let us consider fillings of the diagram of $\lambda-\mu$ using two copies of the integers, 1, 2, 3, $\ldots$, and $1', 2', 3', \ldots$. We order them by $1 < 1' < 2 < 2' \cdots$. We consider tableaux with these entries such that the unprimed numbers are column-strict and the primed numbers are row-strict. More precisely, for each i we allow at most one i in each column and at most one i' in each row. For simplicity we do not restrict the parts in each row. Let us weight each i by x_i and each i' by y_i, $i = 1, 2, \ldots$, and weight a tableau by the product of the weights of its entries. Let us write $s^*_{\lambda/\mu}$ for the sum of the weights of these tableaux.

Now let h_n be the coefficient of u^n in $\prod_{i=1}^{\infty}(1+y_iu)/\prod_{i=1}^{\infty}(1-x_iu)$, so

$$h^*_n = \sum_{k=0}^{n} h_k(x)e_{n-k}(y).$$

Theorem 10.

$$s^*_{\lambda/\mu} = \left| h^*_{\lambda_i-\mu_j+j-i} \right|_1^k.$$

We give here only a sketch of a proof. In the next section we shall consider a generalization (which is less clear geometrically).

We work with paths with vertical and horizontal steps as before, but we also allow diagonal steps, and we label them as in Figure 4. One can check that applying Corollary 2 to this digraph yields Theorem 10.

Figure 4

A result equivalent to Theorem 10 was first proved by Stanley [63] using the Littlewood-Richardson rule. The proof sketched above was given by Remmel [52] in a less general form. A related result was given by Thomas [68,Theorem 3].

The symmetric functions s^*_λ are sometimes called *hook Schur functions* because a tableau counted by $s^*_\lambda(x_1,\ldots,x_m;y_1,\ldots,y_n)$ lies inside the "hook" $\{\,(i,j) \mid 1 \le i \le m \text{ or } 1 \le j \le n\,\}$. They have also been studied by Berele and Regev [7] and Worley [74].

7. Generalized Schur functions

The t-tableaux of Section 3 and the hook tableaux of Section 4 are both examples of arrays in which one relation holds between elements which are adjacent in a row and another holds between elements which are adjacent in a column. We may ask if there are similar determinant formulas corresponding to other relations.

Let R be an arbitrary relation on a set X. For each $i \in X$, let x_i be an indeterminate. Let

$$h^R_n = \sum x_{i_1}x_{i_2}\cdots x_{i_n},$$

where the sum is over all $i_1 \; R \; i_2 \; R \cdots R \; i_n$.

We define the *R-Schur function* $s^R_{\lambda/\mu}$ by

$$s^R_{\lambda/\mu} = \left| h^R_{\lambda_i - \mu_j + j - i} \right|.$$

We now ask, under what conditions is it true that $s^R_{\lambda/\mu}$ counts arrays (p_{ij}) of shape $\lambda - \mu$ with entries in X which satisfy

$$p_{ij} \; R \; p_{i,j+1} \tag{7.1}$$

and

$$p_{ij} \; S \; p_{i+1,j}. \tag{7.2}$$

for some relation S? By looking at $s^R_{(11)}$ and $s^R_{(21)}$ we infer that the only reasonable choice for S is the relation $\bar{R} = \{\, (a,b) \mid b \not R \; a \,\}$, where $\not R$ is the negation of R. (The examples mentioned above are of this form.) So we define an R-tableau to be an array (p_{ij}) which satisfies (7.1) and (7.2) with $S = \bar{R}$.

It is not hard to show that if $\lambda - \mu$ is a skew-hook then the $s^R_{\lambda/\mu}$ counts R-tableaux for any relation R. This is because counting R-tableaux of a skew-hook shape is the same as counting sequences $i_1 \dots i_n$ in which $\{\, j \mid i_j \not R \; i_{j+1} \,\}$ is a specified subset of $\{1, \dots, n-1\}$. The desired formula is easily obtained by inclusion-exclusion. (See MacMahon [44, Vol. 1, pp. 197–202] for the special case in which R is "$\le$" and Stanley [66] for some other special cases. The general case can be found in Gessel [23] and Goulden and Jackson [27, p. 254].)

However, for other shapes we need additional conditions on R. Take, for example, $X = \{1, 2, 3\}$ and $R = \{(1,2), (2,3)\}$. Then we have (writing h_i for h^R_i) $h_0 = 1$, $h_1 = x_1 + x_2 + x_3$, $h_2 = x_1x_2 + x_2x_3$, $h_3 = x_1x_2x_3$, and

$$s^R_{22} = \begin{vmatrix} h_2 & h_3 \\ h_1 & h_2 \end{vmatrix} = x_1^2x_2^2 + x_2^2x_3^2 + x_1x_2^2x_3 - x_1^2x_2x_3 - x_1x_2x_3^2,$$

which has negative terms, although the positive terms do correspond to the three R-tableaux

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 2 & 3 \\ \hline \end{array}$$

There is a simple characterization of relations with this property. We prove the generalization of Corollary for these relations by restating the proof of Theorem 1 in terms of tableaux.

A relation R on a set X is called *semitransitive* (see, e.g., [12] or [21]) if it satisfies the following condition:

($*$) For all a, b, c, d in X, if $a \; R \; b \; R \; c$ then $a \; R \; d$ or $d \; R \; c$.

It is easily verified that the following property is equivalent to ($*$), and although less symmetrical, is more convenient for our proof:

For all a, b, c, d in X, if $a \; \bar{R} \; b \; R \; c \; R \; d$ then $a \; R \; d$.

It is easy to show that R is semitransitive if and only if $\bar{R}$ is.

Theorem 11. *If R is semitransitive then $s^R_{\lambda/\mu}$ counts R-tableaux of shape $\lambda - \mu$.*

Proof. We work with arrays (a_{ij}) defined for $i = 1, \dots, k$; $\mu_i < j \le m_i$ for some numbers m_i, and satisfying $a_{ij} \; R \; a_{i,j+1}$. A *failure of the first kind* of such an array is a position (i,j) such that $a_{ij} \; \not{\bar{R}} \; a_{i+1,j}$, i.e., $a_{i+1,j} \; R \; a_{ij}$. A *failure of the second kind* is a position (i,j) with a_{ij} undefined (but $i \ge 1$) and $a_{i+1,j}$

defined. It is clear that an array with no failures is an R-tableau. As in the proof of Theorem 1, we shall define an involution ϵ on the set of arrays with failures that has the right properties.

There is one technical point that we should mention first. In the involution used in the proof of Theorem 1, two nonadjacent paths are sometimes switched. It is more convenient to use a different choice of intersecting paths as the model for this proof; otherwise we would have to consider failures between nonadjacent rows, which correspond to intersections of nonadjacent paths. As a first attempt, we might try choosing the least i such that paths i and $i+1$ intersect; however, this does not give an involution. For example, in Figure 5a

Figure 5

paths 2 and 3 would switch at x, giving Figure 5b, but the "involution" applied to Figure 5b would try to switch the new paths 1 and 2 at y. Thus a slightly more complicated choice is necessary.

We can modify the involution as follows: call an intersection point x of paths i and j *early* if x is the first intersection point on both paths. Then it is easy to see that for the rectangular grid digraphs we are using, the set of early intersections does not change when a switch is made at an early intersection, every early intersection involves two consecutive paths (as long as the paths are numbered correctly), and every nondisjoint k-path has at least one early intersection. (The last statement does not hold in general for arbitrary acyclic digraphs.) Thus we can choose our switching point to be the 'least' early intersection.

Returning to the proof of Theorem 11, we define an *early failure* to be a failure (i,j) such that there are no failures $(i-1,j')$ with $j'<j$, no failures (i,j') with $j'<j$, and no failures $(i+1,j')$ with $j'\le j$. It is clear that every array with a failure has an early failure. We define the *earliest* failure of an array to be the early failure (i,j) with i as small as possible. For example, if R is "$\le$" then in the array

1	4
2	3
3	3
4	4
4	

$(4,1)$ is an early failure, $(2,2)$ is the earliest failure, and $(1,2)$ is a failure which is not early.

To construct the involution ϵ on arrays with failures, we perform a switching operation on rows i and $i+1$, where i is the row of the earliest failure. The switching is described most easily by (noncommutative!) diagrams for several cases: First suppose we have a failure of the first kind between b and e where C and G are the rest of the rows:

$$\begin{tikzcd}
\cdots & a \arrow[r, "R"] \arrow[d, "\bar{R}"] & b \arrow[r] & C & \\
\cdots & d \arrow[r, "R"] & e \arrow[r, "R"] \arrow[u, "R"] & f \arrow[r] & G
\end{tikzcd}$$

We change this to

$$\begin{tikzcd}
\cdots & a \arrow[r, "R"] \arrow[d, "\bar{R}"] & f \arrow[r] & G & \\
\cdots & d \arrow[r, "R"] & e \arrow[r, "R"] \arrow[u, "R"] & b \arrow[r] & C
\end{tikzcd}$$

which also has a failure of the first kind in the same place. (Here we have $a\ R\ f$ since $a\ \bar{R}\ d\ R\ e\ R\ f$.) This case also applies if a or a and d are absent. If f is absent, we switch the same way, and we obtain a failure of the second kind.

Similarly, if the earliest failure is of the second kind, above e in

$$\begin{tikzcd}
\cdots & a \arrow[d, "\bar{R}"] & & & \\
\cdots & d \arrow[r, "R"] & e \arrow[r, "R"] & f \arrow[r] & G
\end{tikzcd}$$

we change it to

$$\begin{tikzcd} \cdots & a \arrow[r, "R"] \arrow[d, "\bar{R}"] & f \arrow[r] & G \\ \cdots & d \arrow[r, "R"] & e \arrow[u, "R"'] & \end{tikzcd}$$

which has a failure of the first kind. As before, we have $a\ R\ f$ since $a\ \bar{R}\ d\ R\ e\ R\ f$. Also a or a and d may be absent with no change. If f is absent, we get a failure of the second kind.

It is not hard to verify that ϵ preserves the position of the earliest failure and thus is an involution. We leave it to the reader to verify that this involution cancels the unwanted terms in $s^R_{\lambda/\mu}$.

We can give a strong converse of Theorem 11: although semitransitivity is sufficient for $s^R_{\lambda/\mu}$ to count R-tableaux of any shape, it is necessary for $s^R_{(22)}$ to count R-tableaux of shape (22). The following theorem shows that $s^R_{(22)}$ expresses exactly the degree to which R fails to be semitransitive. We omit the proof, which is straightforward.

Theorem 12. *For any relation R the sum of the weights of the R-tableaux of shape* (22) *is*

$$s_{(22)} + \sum x_a x_b x_c x_d,$$

where the sum is over all (a, b, c, d) satisfying $a\ R\ b\ R\ c$, $a \not R\ d$, and $d \not R\ c$.

Theorem 12 could easily be generalized to include part restrictions on the rows, and in fact we could have a different set of elements on each diagonal and a different relation between each pair of consecutive diagonals, as long as they satisfy the appropriate generalizations of $(*)$.

What can we say about the structure of semitransitive relations? Probably the most surprising fact is that they can all be built up in a simple way from semitransitive relations which are either reflexive or irreflexive. (A relation R on X is *reflexive* if $a\ R\ a$ for all a in X and R is *irreflexive* if $a \not R\ a$ for all a in X.)

If R is a semitransitive relation on X, it is clear that R is reflexive if and only if $\bar{R}$ is irreflexive. We note also that since $a\ R\ b\ R\ c$ implies $a\ R\ a$ or $a\ R\ c$, if R is irreflexive, then it is transitive, and is thus a strict partial order. Such a relation is called a *partial semiorder* [21]. (Thus the condition of being a partial semiorder is stronger than the condition of being a partial order!)

If (R_1, X_1) and (R_2, X_2) are two semitransitive relations, where $X_1 \cap X_2 = \emptyset$, we define the *sum* $R_1 \oplus R_2$ of R_1 and R_2 to be the relation on $X_1 \cup X_2$ given by $R_1 \cup R_2 \cup (X_1 \times X_2)$. It is easily checked that if R_1 and R_2 are semitransitive, then $R_1 \oplus R_2$ is semitransitive. For example, a total order is a sum of reflexive points and a strict (i.e., irreflexive) total order is a sum of irreflexive points.

A semitransitive relation on X is *irreducible* if it is not the sum of semitransitive relations on nonempty subsets of X.

Theorem 13. *An irreducible semitransitive relation is either reflexive or irreflexive.*

Proof. Suppose that (R, X) is neither reflexive nor irreflexive. Then there exist a, b in X with $a\ R\ a$ and $b \not R\ b$. Since $a\ R\ a\ R\ a$, either $a\ R\ b$ or $b\ R\ a$. Without loss of generality, we may assume $a\ R\ b$. Let

$$X_2 = \{\, x \in X \mid x \not R\ x \text{ and } a\ R\ x \,\} \cup \{\, x \in X \mid \text{ for some } y \in X, y \not R\ y \text{ and } a\ R\ y\ R\ x \,\}$$

Let $X_1 = X - X_2$. We shall show that R is the sum of its restrictions to X_1 and X_2. We first note the following easily proved facts about semitransitive relations

(i) If $u\ R\ u$ then for all v in X, $u\ R\ v$ or $v\ R\ u$.

(ii) If $u\ R\ v$ and $v\ R\ u$ then $u\ R\ u$.

(iii) If $u\ R\ v\ R\ w$ and either $u \not R\ u$ or $w \not R\ w$ then $u\ R\ w$.

It is clear that $a \in X_1$ and $b \in X_2$, so X_1 and X_2 are nonempty. Now suppose that $p \in X_1$ and $q \in X_2$. To prove the theorem we need only show that $p\ R\ q$ and $q \not R\ p$.

First we prove that $p\ R\ q$. We consider two cases. In the first case, we assume that $q \not R\ q$ and $a\ R\ q$. We know that either $a\ R\ p$ or $p\ R\ a$, by (i). If $p\ R\ a$ then $p\ R\ a\ R\ q$, so by (iii), $p\ R\ q$. If $a\ R\ p$ then we have $p\ R\ p$, since otherwise p would be in X_2. Thus by (i), if $p \not R\ q$ then $q\ R\ p$, so $a\ R\ q\ R\ p$, and thus $p \in X_2$, a contradiction.

In the second case, there exists y with $y \not R\ y$ and $a\ R\ y\ R\ q$. Then by the first case, we have $p\ R\ y$, so $p\ R\ y\ R\ q$. Then by (iii), $p\ R\ q$ unless $p\ R\ p$. Thus by (i), if $p \not R\ q$ then $q\ R\ p$ and we have $q\ R\ p\ R\ y$, so $q\ R\ y$ by (iii). Thus by (ii), $y\ R\ y$, a contradiction.

It remains to prove that $q \not R\ p$. But if $q\ R\ p$ then since we have just shown that $p\ R\ q$, it follows from (ii) that $q\ R\ q$. Thus there must exist some y with $y \not R\ y$ and $a\ R\ y\ R\ q$. Therefore we have $y\ R\ q\ R\ p$ so by (iii), $y\ R\ p$. But since $y \in X_2$, it follows from what we have already proved that $p\ R\ y$, so by (ii) we have $y\ R\ y$, a contradiction.

It follows that all semitransitive relations can be constructed from partial semiorders.

The following construction shows that partial semiorders are easy to find: Start with a strict total order $<$ on X. Let f be function from X to X such that for all $a, b \in X$, $f(a) \geq a$ and $a \leq b$ implies $f(a) \leq f(b)$. Then let

$$R = \{\, (a, b) \mid f(a) < b \,\}. \tag{7.3}$$

To see that R is semitransitive, suppose $a\ R\ b\ R\ c$ but $a \not R\ d$. Then we have $f(a) < b$, $f(b) < c$, and $f(a) \geq d$. Then $d \leq f(a) < b$, so $f(d) \leq f(b) < c$, and thus $d\ R\ c$. The condition $f(a) \geq a$ impies that R is irreflexive. It is interesting to note that the number of such functions on an n-element set is the Catalan number C_n. The relation on the integers given by $a\ R\ b$ iff $a \leq b - t$, where $t \geq 1$, comes from the function $f(x) = x + t - 1$.

Not every partial semiorder is of this form; a counterexample is a disjoint union of two 2-element chains. To see this, note that if R is defined by (7.3), then the sets $R_x = \{\, y \mid x\ R\ y \,\}$ over all $x \in X$ are totally ordered by inclusion.

Edrei [18] (see also Karlin [34, p.412]) proved the following theorem, which was conjectured by Schoenberg (see [61]): Let r_n be real numbers, with $r_n = 0$ for $n < 0$. Then all determinants $s^{\mathbf{r}}_{\lambda/\mu} = \left| r_{\lambda_i - \mu_j + j - i} \right|_1^k$ for all k are nonnegative if and only if

$$\sum_{n=0}^{\infty} r_n u^n = C u^\lambda e^{\gamma u} \prod_{i=1}^{M} (1 + \alpha_i u) \Big/ \prod_{i=1}^{N} (1 - \beta_i u),$$

where α_i, β_i, and γ are positive real numbers, $C \geq 0$, $\sum(\alpha_i + \beta_i) < \infty$, and M and N may be infinite. It follows that for any finite partial semiorder R, if we set $P_R(u) = \sum_i h_i^R u^i$, then for any nonnegative values of the x_i, $P_R(u)$ is a polynomial in u with negative roots.

The hook Schur functions give a strengthening of the easy half of Edrei's theorem: if the α_i, β_i, and γ are indeterminates, then $s^{\mathbf{r}}_{\lambda/\mu}$ has nonnegative coefficients as a polynomial in these variables. (The parameter γ can be accounted for via the homomorphism θ of Section 3, by introducing a third set of labels, each of which can appear at most once.)

There is another way of looking at R-Schur functions: we can interpret them as evaluations of ordinary Schur functions. For the moment, let us think of h_n, $n \geq 1$, as indeterminates, with $h_0 = 1$, and define the skew Schur functions $s_{\lambda/\mu}$ as polynomials in the h_n. Then $s^R_{\lambda/\mu}$ is the image of $s_{\lambda/\mu}$ under the substitution $h_n \mapsto h_n^R$. It follows that any polynomial identity among the $s_{\lambda/\mu}$ remains true when $s^R_{\lambda/\mu}$ is substituted for $s_{\lambda/\mu}$. For example, we have the formulas

$$(h_1)^n = \sum_\lambda f^\lambda s_\lambda,$$

over all partitions λ of n, where f^λ is the number of standard Young tableaux of shape λ, and

$$s_\mu s_\nu = \sum_\nu c^\lambda_{\mu\nu} s_\lambda,$$

where the integers $c^\lambda_{\mu\nu}$ are given by the Littlewood-Richardson rule [43, p. 68]. It is reasonable to expect that these formulas can be explained combinatorially by analogs of the Robinson-Schensted correspondence [56, 59] and the *jeu de taquin* of Schützenberger [62]. This has been done in the special case of the hook Schur functions by Remmel [52] and Worley [74].

8. Stanley's formula

Much of the theory of plane partitions is devoted to counting plane partitions or tableaux by the sum of their entries. To accomplish this we set $x_i = q^i$ in the formulas of Section 3. First we evaluate $h_n(x_a, x_{a+1}, \ldots, x_b)$ with $x_i = q^i$. Using the well-known expansions of $(1-z)(1-zq)\cdots(1-zq^n)$ and its reciprocal [1, p. 19] we have

$$\sum_{n=0}^{\infty} h_n(q^a, q^{a+1}, \ldots, q^b) z^n = \frac{1}{(1-zq^a)\cdots(1-zq^b)}$$

$$= \frac{1}{(zq^a)_{b-a+1}} = \sum_{j=0}^{\infty} z^j q^{aj} \begin{bmatrix} b-a+j \\ j \end{bmatrix}$$

and

$$\sum_{n=0}^{\infty} e_n(q^a, q^{a+1}, , \ldots, q^b) z^n = (1+zq^a)\cdots(1+zq^b)$$

$$= \prod_{i=0}^{b-a} (1+q^i z q^a) = \sum_j z^j q^{\binom{j}{2}+aj} \begin{bmatrix} b-a+1 \\ j \end{bmatrix}.$$

So

$$h_n(q^a, \ldots, q^b) = q^{an} \begin{bmatrix} b-a+n \\ n \end{bmatrix}$$

and

$$e_n(q^a, \ldots, q^b) = q^{an+\binom{n}{2}} \begin{bmatrix} b-a+1 \\ n \end{bmatrix}.$$

[Kreweras [39, p. 64] gave the determinant $|\binom{y_i - y'_j + r}{i-j+r}|$ for counting r-tuples of paths between two given paths of heights y_i and y'_j.]

Applying these formulas to Corollary 4, we obtain

Theorem 14. *The generating function for tableaux of shape $\lambda - \mu$ in which parts in row i are at least a_i and at most b_i, where $a_i \le a_{i+1}$ and $b_i \le b_{i+1}$, is*

$$\left| h_{\lambda_i - \mu_j + j - i}(q^{a_j}, \ldots, q^{b_i}) \right|_1^k = \left| q^{a_j(\lambda_i - \mu_j + j - i)} \begin{bmatrix} b_i - a_j + \lambda_i - \mu_j + j - i \\ \lambda_i - \mu_j + j - i \end{bmatrix} \right|_1^k \tag{8.1}$$

and the corresponding generating function for row-strict tableaux, where $a_{i+1} - a_i \ge \mu_{i+1} - \mu_i - 1$ and $b_{i+1} - b_i \ge \lambda_{i+1} - \lambda_i - 1$, is

$$\left| q^{E_{ij}} \begin{bmatrix} b_i - a_j + 1 \\ \lambda_i - \mu_j + j - i \end{bmatrix} \right|_1^k,$$

where

$$E_{ij} = \binom{\lambda_i - \mu_j + j - i}{2} + a_j(\lambda_i - \mu_j + j - i). \tag{8.2}$$

Analogous formulas for column-strict plane partitions are easily obtained from Theorem 14, since replacing each entry with its negative changes a tableau to a column-strict plane partition. Then replacing a_i and b_i with their negatives and q by q^{-1} in Theorem 14, and using the identity $\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q$, we obtain:

Theorem 15. *The generating function for column-strict plane partitions of shape $\lambda-\mu$ in which parts in row i are at most a_i and at least b_i, where $a_i \geq a_{i+1}$ and $b_i \geq b_{i+1}$ is*

$$\left| q^{b_i(\lambda_i-\mu_j+j-i)} \begin{bmatrix} a_j - b_i + \lambda_i - \mu_j + j - i \\ \lambda_i - \mu_j + j - i \end{bmatrix} \right|_1^k . \tag{8.3}$$

We can easily modify formula (8.3) to count (not necessarily column-strict) plane partitions: if we add i to every part in row i of a column-strict plane partition counted by (8.3), we increase the sum by $\sum_i(\lambda_i-\mu_i)$ and we obtain a plane partition of shape $\lambda-\mu$ in which parts in row i are at most a_i+i and at least b_i+i. Setting $A_i = a_i + i$ and $B_i = b_i + i$, we obtain:

Theorem 16. *The generating function for plane partitions of shape $\lambda-\mu$ in which parts in row i are at most A_i and at least B_i, where $A_i \geq A_{i+1}-1$ and $B_i \geq B_{i+1}-1$ is*

$$q^{\Sigma_i i(\lambda_i-\mu_i)} \left| q^{(B_i-i)(\lambda_i-\mu_j+j-i)} \begin{bmatrix} A_j - B_i + \lambda_i - \mu_j \\ \lambda_i - \mu_j + j - i \end{bmatrix} \right|_1^k .$$

There are many determinants of q-binomial coefficients which have explicit formulas as quotients. Some of these are very difficult to evaluate [2–4, 6, 46, 47, 49**?**]. In this section we give an easy evaluation of a determinant by induction, which yields a result of Stanley counting tableaux of a given shape with a given maximum part size. (For other work on evaluation of determinants of matrices of binomial and q-binomial coefficients, see [10,29, 45, 48, and 69]. In the next section we use a simple summation formula to evaluate a determinant and give two (**?**) applications which seem to be new.

Let us consider the problem of counting tableaux of shape λ, with parts $0, 1, \cdots, b-1$. By formula (8.1) the generating function for these tableaux is

$$\left| \begin{bmatrix} b + \lambda_i + j - i \\ \lambda_i + j - i \end{bmatrix} \right|_1^k .$$

However, in such a tableau the smallest part in row i must be at least $i-1$ (since $\mu = 0$) so this determinant is also equal to

$$\left| q^{(j-1)(\lambda_i+j-i)} \begin{bmatrix} b - 1 - (j-1) + \lambda_i + j - i \\ \lambda_i + j - i \end{bmatrix} \right|_1^k = \left| q^{(j-1)(\lambda_i+j-i)} \begin{bmatrix} b + \lambda_i - i \\ \lambda_i + j - i \end{bmatrix} \right|_1^k , \tag{8.4}$$

and this determinant turns out to be easier to evaluate.

We now define the *hook lengths* and *contents* of a diagram. The hook length of a square in a diagram is the number of squares to its right plus the number of squares below it plus one. Thus the hook length of square (i,j) is $\lambda_i + \lambda'_j - i - j + 1$. The *content* of square (i,j) is $j-i$. The hook lengths and contents of the diagram for the partition (431) are as follows:

6	4	3	1
4	2	1	
1			

Hook lengths

0	1	2	3
-1	0	1	
-2			

Contents

We write $h(x)$ and $c(x)$ for the hook length and content of x, and we set $H(\lambda) = \prod_x (1 - q^{h(x)})$ and $C_a(x) = \prod_x (1 - q^{a+c(x)})$, where the product is over all squares x of the diagram of λ.

Theorem 17. (Stanley [65]) *The generating function for tableaux of shape* λ *with parts* $0, 1, \cdots, b-1$ *is* $q^e C_b(\lambda)/H(\lambda)$, *where* $e = \sum_{i=1}^{k}(i-1)\lambda_i$.

Proof. We evaluate the determinant (8.4) by induction. First note that if $\lambda_k = 0$ (8.4) reduces to the formula for $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$.

Now first suppose that b is greater than k, the number of parts of λ. Note that

$$C_b(\lambda) = C_{b-1}(\lambda)\frac{(1-q^{b+\lambda_1-1})(1-q^{b+\lambda_2-2})\cdots(1-q^{b+\lambda_k-k})}{(1-q^{b-1})(1-q^{b-2})\cdots(1-q^{b-k})}.$$

Then by induction the determinant is

$$\begin{aligned}\left|q^{(j-1)(\lambda_i+j-i)}\begin{bmatrix} b+\lambda_i-i \\ b-j\end{bmatrix}\right| &= \left|q^{(j-1)(\lambda_i+j-i)}\frac{1-q^{b+\lambda_i-i}}{1-q^{b-j}}\begin{bmatrix} b+\lambda_i-i-1 \\ b-j-1\end{bmatrix}\right| \\ &= \prod_{i=1}^{k}\frac{(1-q^{b+\lambda_i-i})}{(1-q^{b-i})}\cdot q^e\frac{C_{b-1}(\lambda)}{H(\lambda)} = q^e\frac{C_b(\lambda)}{H(\lambda)}.\end{aligned}$$

If $b = k$ and $\lambda_k \neq 0$ then the first column must be 0, 1, 2, …, $b-1$. So the tableau is determined by the entries in columns 2, 3, …, λ_1, which may constitute an arbitrary tableau of shape $\tilde{\lambda} = (\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_k - 1)$. Thus by induction the generating function is

$$q^{0+1+\cdots+(k-1)}q^{(\lambda_2-1)+2(\lambda_3-1)+\cdots+(k-1)(\lambda_k-1)}\frac{C_b(\tilde{\lambda})}{H(\tilde{\lambda})} = q^e\frac{C_b(\tilde{\lambda})}{H(\tilde{\lambda})}.$$

Now $H(\lambda) = H(\tilde{\lambda})\cdot(1-q^{\lambda_1+k-1})(1-q^{\lambda_2+k-2})\cdots(1-q^{\lambda_k})$ and $C_b(\lambda) = C_b(\tilde{\lambda})\cdot(1-q^{b+\lambda_1-1})\cdots(1-q^{b+\lambda_k-k})$ and thus when $b = k$,

$$\frac{C_b(\tilde{\lambda})}{H(\tilde{\lambda})} = \frac{C_k(\tilde{\lambda})}{H(\tilde{\lambda})} = \frac{C_k(\lambda)}{H(\lambda)} = \frac{C_b(\lambda)}{H(\lambda)}.$$

9. Some quotient formulas

Next we consider some determinants of binomial coefficients which can be evaluated explicitly. For simplicity, we consider here only the case $q = 1$.

We use the notation $(a)_n$ to denote $a(a+1)\cdots(a+n-1)$.

Lemma 18.

$$\left|\frac{1}{(\alpha_i)_j}\right|_0^n = \frac{\displaystyle\prod_{0\le i<j\le n}(\alpha_i-\alpha_j)}{\displaystyle\prod_{i=0}^{n}(\alpha_i)_n}$$

Proof. The formula is equivalent to

$$\left|\frac{(\alpha_i)_n}{(a_i)_j}\right|_0^n = \prod_{0\le i<j\le n}(\alpha_i-\alpha_j)$$

The left side is

$$\left|(\alpha_i+j)(\alpha_i+j+1)\cdots(\alpha_i+n-1)\right|_0^n.$$

Since the entries in column j of this determinant are the values of a polynomial of degree $n-j$ evaluated at the points α_i, elementary column operations reduce this to a Vandermonde determinant.

Lemma 19.

$$\left|\frac{(\alpha_i-\beta_j)_j}{(\alpha_i)_j}\right|_0^n = \frac{\prod_{i=0}^{n}(\beta_i)_i \prod_{0\le i<j\le n}(\alpha_j-\alpha_i)}{\prod_{i=0}^{n}(\alpha_i)_n}.$$

Proof. Let A be the matrix $\left(\frac{1}{(\alpha_i)_j}\right)$ and let B be the matrix $\left(\frac{(\beta_j)_i(-j)_i}{i!}\right)$. Then the (i,j) entry of $C=AB$ is

$$\sum_{l=0}^{n}\frac{1}{(\alpha_i)_l}\frac{(\beta_j)_l(-j)_l}{l!} = \frac{(\alpha_i-\beta_j)_j}{(\alpha_i)_j}$$

by Vandermonde's theorem [5, p. 3]. Thus $|C|=|A||B|$. Since B is upper triangular,

$$|B| = \prod_{j=0}^{n}\frac{(\beta_j)_j(-j)_j}{j!} = (-1)^{\binom{n}{2}}\prod_{j=0}^{n}(\beta_j)_j$$

and the result follows from 18

We now give two applications of Lemma 19. First let us evaluate the determinant $\left|\binom{\beta-\alpha_i+j}{\alpha_i+j-1}\right|_0^n$. Since $(\alpha_i-\beta-j)_j = (-1)^j(\beta-\alpha_i+1)_j$, we have

$$\begin{aligned}\binom{\beta-\alpha_i+j}{\alpha_i+j-1} &= \binom{\beta-\alpha_i}{\alpha_i-1}\frac{(\beta-\alpha_i+1)_j}{(\alpha_i)_j}\\ &= (-1)^j\frac{(\beta-\alpha_i)!}{(\beta-2\alpha_i+1)!\,(\alpha_i-1)!}\frac{(\alpha_i-\beta-j)_j}{(\alpha_i)_j}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\binom{\beta-\alpha_i+j}{\alpha_i+j-1}\right|_0^n &= (-1)^{\binom{n+1}{2}}\prod_{i=0}^{n}\frac{(\beta-\alpha_i)!\,(\beta+i)_i}{(\beta-2\alpha_i+1)!\,(\alpha_i-1)!\,(\alpha_i)_n}\prod_{0\le i<j\le n}(\alpha_j-\alpha_i)\\ &= \prod_{i=0}^{n}\frac{(\beta-\alpha_i)!\,(\beta+i)_i}{(\beta-2\alpha_i+1)!\,(\alpha_i-1)!\,(\alpha_i)_n}\prod_{0\le i<j\le n}(\alpha_i-\alpha_j)\\ &= \prod_{i=0}^{n}\frac{(\beta-\alpha_i)!\,(\beta+i)_i}{(\beta-2\alpha_i+1)!\,(\alpha_i+n-1)!}\prod_{0\le i<j\le n}(\alpha_i-\alpha_j). \qquad (9.1)\end{aligned}$$

This determinant can clearly be interpreted as counting certain tableaux. The value of the determinant can be simplified and expressed in a form very similar to Stanley's hook length-content formula. The hook lengths are the same, but the contents are different. If (i,j) is a square of the diagram of λ we define $d(i,j)$ as follows:

$$d(i,j) = \begin{cases} -(\lambda_i+j-2i+1) & \text{if } i\le j;\\ \lambda'_j+i-2j & \text{if } i>j.\end{cases}$$

These numbers can be described as follows: if x is a diagonal square, $-d(x)$ is one more than the number of squares to the right of x, and as we move right from a diagonal square d decreases by 1 for each square. If x is a subdiagonal square, $d(x)$ is 2 more than the number of squares below x, and as we move down from a subdiagonal square, d increases by 1 for each square. Here are the values of $d(i,j)$ for the partition (44331):

-5	-6	-7	-8	-9
5	-4	-5	-6	-7
6	4	-1		
7	5	2		
8	6			

In this section we let $H(\lambda)$ be the product of the hook lengths of λ. We will need the following lemma (see Macdonald [43, p. 9]):

Lemma 20. *For any partition λ,*

$$\frac{\displaystyle\prod_{1\le i<j\le k}(\lambda_i-\lambda_j-i+j)}{\displaystyle\prod_{i=1}^{k}(\lambda_i-i+k)!}=\frac{1}{H(\lambda)}.$$

Theorem 21. *The number of tableaux of shape λ with positive integer parts, in which the largest part in row i is at most $r-2\lambda_i+2i-1$ is*

$$\frac{1}{H(\lambda)}\prod_{x\in\lambda}\big(r+d(x)\big).$$

Proof. In (9.1) we set $\alpha_i=\lambda_{i+1}-i+1$, $k=n+1$, and $\beta=r+1$. When i and j are replaced by $i-1$ and $j-1$, the determinant becomes

$$\left|\binom{r-\lambda_i+i+j-2}{\lambda_i-i+j}\right|_1^k,$$

and the combinatorial interpretation follows from Theorem 14. The value of the determinant is

$$\prod_{i=1}^{k}\frac{(r-\lambda_i+i-1)!\,(r+i)_{i-1}}{(r-2\lambda_i+2i-2)!\,(\lambda_i-i+k)!}\prod_{1\le i<j\le k}(\lambda_i-\lambda_j+j-i)$$

$$\begin{aligned}&=\frac{1}{H(\lambda)}\prod_{i=1}^{k}\frac{(r+2i-2)(r+2i-3)\cdots(r+2i-2\lambda_i-1)}{(r+i-1)(r+i-2)\cdots(r+i-\lambda_i)}\\&=\frac{1}{H(\lambda)}\prod_{i=1}^{k}\prod_{j=1}^{\lambda_i}\frac{(r+2i-2j)(r+2i-2j-1)}{(r+i-j)}\\&=\frac{1}{H(\lambda)}\prod_{(i,j)\in\lambda}\frac{(r+2i-2j)(r+2i-2j-1)}{(r+i-j)}\end{aligned}$$

We break the product into two factors.

First we have

$$\prod_{\substack{(i,j)\in\lambda\\ i\le j}} \frac{(r+2i-2j)(r+2i-2j-1)}{(r+i-j)} = \prod_{i=1}^{k}\prod_{l=0}^{\lambda_i-i} \frac{(r-2l)(r-2l-1)}{(r-l)}$$

$$= \prod_{i=1}^{k} \frac{\prod_{m=0}^{2\lambda_i-2i+1}(r-m)}{\prod_{n=0}^{\lambda_i-i}(r-n)}$$

$$= \prod_{i=1}^{k}\prod_{m=\lambda_i-i+1}^{2\lambda_i-2i+1}(r-m).$$

With the substitution $m=j+\lambda_i-2i+1$, this becomes

$$\prod_{i=1}^{k}\prod_{j=i}^{\lambda_i}(r-j-\lambda_i+2i-1) = \prod_{x\in\lambda+}\bigl(r+d(x)\bigr).$$

Next we have, with $k'=\lambda_1$

$$\prod_{\substack{(i,j)\in\lambda\\ i> j}} \frac{(r+2i-2j)(r+2i-2j-1)}{(r+i-j)} = \prod_{j=1}^{k'}\prod_{l=1}^{\lambda'_j-j} \frac{(r+2l-1)(r+2l)}{(r+l)}$$

$$= \prod_{j=1}^{k'} \frac{\prod_{m=1}^{2\lambda'_j-2j}(r+m)}{\prod_{n=1}^{\lambda'_j-j}(r+n)}$$

$$= \prod_{j=1}^{k'}\prod_{m=\lambda'_j-j+1}^{2\lambda'_j-2j}(r+m).$$

With the substitution $m=i+\lambda'_j-2j$,

$$\prod_{j=1}^{k'}\prod_{i=j+1}^{\lambda'_j}(r+i+\lambda'_j-2j) = \prod_{x\in\lambda-}\bigl(r+d(x)\bigr),$$

completing the proof.

Another determinant we can evaluate using Lemma 19 is $\left|C_{i+\alpha_j}\right|_0^n$, where C_n is the Catalan number defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = 2^{2n}\frac{(1/2)_n}{(2)_n}.$$

Since

$$C_{i+\alpha_j} = 2^{2i+2\alpha_j}\frac{(1/2)_{i+\alpha_j}}{(2)_{i+\alpha_j}} = 2^{2i+2\alpha_j}\frac{(1/2)_{\alpha_j}(1/2+\alpha_j)_i}{(2)_{\alpha_j}(2+\alpha_j)_i},$$

the determinant is equal to

$$2^{n(n+1)+2\Sigma_j \alpha_j} \frac{\prod_j (1/2)_{\alpha_j}}{\prod_j (2)_{\alpha_j}} \left| \frac{(1/2+\alpha_j)_i}{(2+\alpha_j)_i} \right|_0^n$$

Now by the lemma

$$\left| \frac{(1/2+\alpha_j)_i}{(2+\alpha_j)_i} \right|_0^n = \frac{\prod_{i=0}^n (3/2)_i}{\prod_{j=0}^n (\alpha_j+2)_n} \prod_{0 \le i < j \le n} (\alpha_j - \alpha_i).$$

Simplifying, we find that the determinant is

$$\prod_{0 \le i < j \le n} (\alpha_j - \alpha_i) \prod_{j=0}^{n} \frac{(2\alpha_j)!}{\alpha_j!\,(\alpha_j+n+1)!} \prod_{i=0}^{n} \frac{(2i+1)!}{i!}.$$

See Viennot [71] for an evaluation of a special case of this determinant, using the qd-algorithm of Padé approximant theory. See also de Sainte-Catherine and Viennot [14].

We recall that C_i is the number of paths from $(0,0)$ to $(2i,2i)$ which never go above (but may touch) the line $x=y$.

Now assume that $0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_n$ and let the points P_i and Q_i be defined by $P_i(n-i,n-i)$ and $Q_i = (n+\alpha_i, n+\alpha_i)$ for $i = 0, \ldots, n$. If we consider the digraph of lattice points with only those steps that lie below the line $x=y$, then $(\mathbf{P},\mathbf{Q})$ is nonpermutable. Thus the determinant $|C_{i+\alpha_j}|_0^n$ is the number of disjoint $(n+1)$-paths from $\mathbf{P}$ to $\mathbf{Q}$.

These paths may be expressed as tableaux: we consider separately the cases $\alpha_0 = 0$ and $\alpha_0 \ne 0$.

First we consider the case $\alpha_0 = 0$. In Figure 6 an example with $\alpha_0 = 0$, $\alpha_1 = 2$, and $\alpha_2 = 5$ is given.

Figure 6

It is clear that path i begins with $2i$ horizontal steps, and thus we may remove these steps, and associate what remains with a tableau in the usual way. Thus Figure 6 corresponds to the tableau

1	1	6
2		

In general, the tableau will have shape $(\alpha_n - n, \alpha_{n-1} - n + 1, \ldots, \alpha_1 - 1)$. The requirement that all steps fall below the main diagonal is equivalent to the condition that $p_{ij} \le 2n + j - i$. This condition is implied by the inequalities for the last elements in each column.

If we make the substitution $\lambda_i = \alpha_{n+1-i} - (n+1-i)$, (so that $\alpha_i = \lambda_{n+1-i} + i$) for $i = 1, \ldots, n$ and set $k = n$, then we obtain the following:

Theorem 22. *The number of tableaux (p_{ij}) of shape λ with nonnegative entries satisfying $p_{ij} \le 2k + j - i$, or equivalently, $p_{\lambda'_j,j} \le 2k + j - \lambda'_j$, is*

$$\frac{1}{(k+1)!} \prod_{i=1}^{k} (\lambda_i + k - i + 1)! \prod_{1 \le i < j \le k} (\lambda_i - \lambda_j - i + j)$$
$$\prod_{i=1}^{k} \frac{(2\lambda_i + 2k - 2i + 2)!}{(\lambda_i + k - i + 1)!\,(\lambda_i - i + 2k + 2)!} \prod_{i=0}^{k} \frac{(2i+1)!}{i!}.$$

Next we consider the case $\alpha_0 \neq 0$. Figure 7 shows an example with $\alpha_0 = 2$, $\alpha_1 = 4$, $\alpha_2 = 5$. Here path

Figure 7

i begins with $2i+1$ horizontal steps, which may be removed as before. Figure 7 corresponds to the tableau

0	1
2	2
3	

In general, the tableau will have shape $(\alpha_n - n - 1, \alpha_{n-1} - n, \ldots, \alpha_0 - 1)$ and the restriction on the parts is $p_{ij} \le 2n + 1 + j - i$. Now we make the substitution $\lambda_i = \alpha_{n-i+1} - n + i - 2$ for $i = 1$ to $n+1$ and we set $k = n + 1$. Then we have the following:

Theorem 23. *The number of tableaux (p_{ij}) of shape λ, with nonnegative parts satisfying $p_{ij} \le 2k - 1 + j - i$, or equivalently, $p_{\lambda'_j, j} \le 2k - 1 + j - \lambda'_j$ is*

$$\prod_{1 \le i < j \le k} (\lambda_i - \lambda_j - i + j) \prod_{i=1}^{k} \frac{(2\lambda_i + 2k - 2i + 2)!}{(\lambda_i + k - i + 1)!\,(\lambda_i + 2k - i + 1)!} \prod_{i=0}^{k-1} \frac{(2i+1)!}{i!}.$$

10. Fibonomial coefficients

We now consider a matrix of binomial coefficients whose characteristic polynomial can be explicitly evaluated in terms of polynomials closely related to Gaussian polynomials. The coefficients of the characteristic polynomial are sums of minors which we may interpret using the theory we have developed. In particular, we obtain the first known combinatorial interpretation for the Fibonomial coefficients.

Carlitz [9] showed that the characteristic polynomial of the matrix

$$\left(\binom{i}{n-j}\right)_{i,j=0,\ldots,n}$$

is

$$x^{n+1} + \sum_{r=1}^{n+1} (-1)^{\binom{r+1}{2}} \left\{ {n+1 \atop r} \right\}_F x^{n+1-r},$$

where $\left\{ {m \atop j} \right\}_F$ is the *Fibonomial coefficient*

$$\frac{F_m F_{m-1} \cdots F_{m-j+1}}{F_1 F_2 \cdots F_j}.$$

Here F_j is the Fibonacci number ($F_0 = 0$, $F_1 = 1$, and $F_j = F_{j-1} + F_{j-2}$ for $j \ge 2$).

First we generalize Carlitz's result. Let Ψ be the linear transformation on the vector space of polynomials in x of degree at most n defined by

$$\Psi(A(x)) = x^n A\big(s(1 + \frac{1}{x})\big)$$

for any such polynomial $A(x)$, where s is arbitrary.

Since $\Psi(x^i) = x^n s^i (1 + x^{-1})^i = \sum_{j=n-i}^{n} x^j \binom{i}{n-j} s^i$, the matrix of Ψ with respect to the basis $\{x^i\}_{i=0,\ldots,n}$ is

$$\left(\binom{i}{n-j} s^i\right)_{i,j=0,\ldots,n}.$$

Now we have

$$\Psi\big((1+ax)^k(1+bx)^{n-k}\big) = (as)^k(bs)^{n-k}\left(1+\frac{1+as}{as}x\right)^k\left(1+\frac{1+bs}{bs}x\right)^{n-k}.$$

Thus if $(1+as)/as = a$ and $(1+bs)/bs = b$, i.e.,

$$a, b = \frac{s \pm \sqrt{s^2+4s}}{2s}$$

then $(1+ax)^k(1+bx)^{n-k}$ will be an eigenvector for Ψ with eigenvalue $(as)^k(bs)^{n-k}$. As long as $s^2+4s \neq 0$ these will be distinct for $k = 0, \ldots, n$, and thus give all the eigenvalues of Ψ.

By continuity, we may remove the restriction on s and we obtain:

Theorem 24. *The eigenvalues of the matrix* $M = \left(\binom{i}{n-j}s^i\right)_{i,j=0,\ldots,n}$ *are* $\alpha^k\beta^{n-k}$, $k = 0, \ldots, n$, *where*

$$\alpha, \beta = \frac{s \pm \sqrt{s^2+4s}}{2},$$

and thus the characteristic polynomial of M *is*

$$|(xI - M)| = \prod_{k=0}^{n}(x - \alpha^k\beta^{n-k}) \tag{10.1}$$

where I *is the identity matrix.*

To express the product simply, it is convenient to introduce the *homogeneous Gaussian polynomials*, defined by

$$\begin{bmatrix} m \\ j \end{bmatrix}_{p,q} = \frac{(p^m - q^m)(p^{m-1} - q^{m-1})\cdots(p^{m-j+1} - q^{m-j+1})}{(p-q)(p^2-q^2)\cdots(p^j - q^j)}.$$

These are related to the ordinary Gaussian polynomials by

$$\begin{bmatrix} m \\ j \end{bmatrix}_{p,q} = q^{mj-j^2}\begin{bmatrix} m \\ j \end{bmatrix}_{p/q}.$$

Then the following formula is equivalent to a form of the q-binomial theorem:

$$\prod_{k=0}^{n}(x + p^k q^{n-k}) = \sum_{k=0}^{n+1} x^{n+1-k}(pq)^{\binom{k}{2}}\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q}.$$

Thus by (10.1), we have

$$|(xI - M)| = \sum_{k=0}^{n+1} x^{n+1-k}(-1)^{\binom{k+1}{2}}s^{\binom{k}{2}}\begin{bmatrix} n+1 \\ k \end{bmatrix}_{\alpha,\beta}, \tag{10.2}$$

since $\alpha\beta = -s$.

Now let $G_n = (\alpha^n - \beta^n)/(\alpha - \beta)$. The following facts about G_n are easily verified:

$$G_0 = 0, \quad G_1 = 1, \quad \text{and } G_n = s(G_{n-1} + G_{n-2})$$

$$\sum_{n=0}^{\infty} G_n u^n = \frac{u}{1 - s(u+u^2)}$$

$$G_n = \sum_{i=\lceil n/2 \rceil}^{n} s^i \binom{i}{n-i}.$$

Note that for $s=1$, G_n reduces to F_n. Let us define $\left\{{m \atop j}\right\}$ to be $\left[{m \atop j}\right]_{\alpha,\beta}$, so that

$$\left\{{m \atop j}\right\} = \frac{G_m G_{m-1} \cdots G_{m-j+1}}{G_1 G_2 \cdots G_j}.$$

We note that from the easily proved formula $G_{m+n} = G_m G_{n+1} + sG_{m-1}G_n$ we obtain the recurrence

$$\left\{{m \atop j}\right\} = G_{m-j+1}\left\{{m-1 \atop j-1}\right\} + sG_{j-1}\left\{{m-1 \atop j}\right\} \tag{10.3}$$

We now turn to the combinatorial interpretation. We know that the coefficient of x^{n+1-k} in $|(xI-M)|$ is $(-1)^k$ times the sum of all the $k \times k$ principal minors of M, i.e., the minors obtained by choosing k rows and the same k columns from M. Such a minor is a determinant

$$\left| \binom{r_i}{n-r_j} s^{r_i} \right|_1^k . \tag{10.4}$$

To get a determinant in the form we can interpret, we reverse the order of the columns in (10.4); then (10.4) is equal to $(-1)^{\binom{k}{2}} s^{\Sigma r_i}$ times the determinant

$$\left| \binom{r_i}{n-r_{k+1-j}} \right|_1^k \tag{10.5}$$

The determinant (10.5) is easily interpreted by our theory. It is in fact a *binomial determinant*, as studied in Gessel and Viennot [25], and it has the following interpretation: define points P_i and Q_i by $P_i = (0,-i)$ and $Q_i = (-n+i, -n+i)$. For any subset $R = \{r_1 < r_2 < \cdots < r_k\}$ of $\{0,1,\ldots,n\}$, let $N(R)$ be the number of nonintersecting k-paths from $(P_{r_1}, \cdots, P_{r_k})$ to $(Q_{r_k}, \cdots, Q_{r_1})$. Then $\left|\binom{r_i}{n-r_{k+1-j}}\right| = N(R)$.

Theorem 25. *For any subset R of $\{0,1,\ldots,n\}$ let $\|R\| = \sum_{i \in R} i$ and let $N(R)$ be defined as above. Then*

$$\sum_R s^{\|R\|} N(R) = s^{\binom{k}{2}} \left\{{n+1 \atop k}\right\},$$

where the sum is over all k-subsets R of $\{0,1,\ldots,n\}$.

Several questions related to Theorem 15 present themselves. First it would be nice to have a more natural interpretation than the one we have given. Second, is there a combinatorial interpretation to the recurrence (10.3)? R. Stanley has asked if there is a binomial poset associated with the Fibonomial coefficients; i.e., a ranked poset such that for all x and y in the poset if $x < y$ and $r(y) - r(x) = n$ then there are exactly $\left\{{n \atop k}\right\}_F$ points z with $x < z < y$ and $r(z) - r(x) = k$. Our work constructs the right number of objects, but it is not clear how to partially order them to obtain such a binomial poset.

11. Jacobi's theorem

There is a theorem of Jacobi [32; 41, p. 153–156] which relates the minors of a matrix to minors of the inverse matrix. It may be stated in the following form: Let M be an invertible matrix with rows and columns indexed by some set $I = \{a, a+1, \cdots, a+n\}$. (In our applications, a is 0 or 1.) Let $L = M^{-1}$. We shall call the matrix $M^* = \big((-1)^{i+j} L_{ij}\big)$ the *sign-inverse* of M.

If A and B are subsets of I of the same size, let $M[A|B]$ denote the minor of M corresponding to the rows in A and the columns in B. Jacobi's theorem asserts that if N is the sign-inverse of M then

$$M[A|B] = |M| \cdot N[I-B|I-A] \tag{11.1}.$$

In many cases, especially when M is a triangular matrix with 1's on the diagonal, (11.1) can be interpreted combinatorially. See, for example, Gessel and Viennot [25, Section 4].

One of the simplest examples is the case in which M is the matrix (h_{i-j}). It is easily verified that $M^* = (e_{i-j})$. Then Jacobi's theorem gives the expression of a skew Schur function in terms of the e_i.

The following theorem gives a pair of sign-inverse matrices to which we can apply Jacobi's theorem.

Lemma 26. *Let $a_0, a_1, \dots$ and $b_0, b_1, \dots$ be arbitrary, and define p_i and q_i by*

$$\sum_{i=0}^{\infty} p_i u^i = \left(\sum_{i=0}^{\infty} (-1)^i a_i u^i\right)^{-1}$$

and

$$\sum_{i=0}^{\infty} q_i u^i = \left(\sum_{i=0}^{\infty} (-1)^i b_i u^i\right) \Big/ \left(\sum_{i=0}^{\infty} (-1)^i a_i u^i\right)$$

Then

(i) *The matrices (a_{j-i}) and (p_{j-i}) are sign-inverse (where $a_k = p_k = 0$ for $k < 0$),*

(ii) *The matrices*

$$U = \begin{pmatrix} 1 & b_0 & b_1 & \cdots & b_n \\ 0 & a_0 & a_1 & \cdots & a_n \\ 0 & 0 & a_0 & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 1 & q_0 & q_1 & \cdots & q_n \\ 0 & p_0 & p_1 & \cdots & p_n \\ 0 & 0 & p_0 & \cdots & p_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & p_0 \end{pmatrix}$$

are sign-inverse.

The proof is straightforward.

Now let us take $a_i = h_{im+r}$ for fixed m and r, where $0 \le r < m$, and $b_i = h_{im+s}$. Then

$$\sum_{i=0}^{\infty} p_i u^i = \left(\sum_{i=0}^{\infty} h_{im+r} u^i\right)^{-1}$$

and

$$\sum_{i=0}^{\infty} q_i u^i = \frac{\sum_{i=0}^{\infty} (-1)^i h_{im+s} u^i}{\sum_{i=0}^{\infty} (-1)^i h_{im+r} u^i}.$$

(The variable u is actually redundant in these formulas.) The significance of the restriction $0 \le r < m$ is that it implies that the sum could as well start at $-\infty$ instead of at 0.[why**?**]

Before we apply Jacobi's theorem we need to relate complements of sets to conjugate partitions. By reasoning as in Macdonald [43, p. 15; see also (1.7), p. 3] it follows from Jacobi's theorem that if λ and μ are two partitions with at most s parts and largest part at most t, where $s + t = w$, and if M and N are sign-inverse matrices with rows and columns indexed 0, 1, ..., w, then we have

$$M[\{\lambda_i + s - i\}_{1\le i\le s} | \{\mu_j + s - j\}_{1\le j\le s}] = |M| \cdot N[\{s - 1 + i - \mu_i'\}_{1\le i\le t} | \{s - 1 + j - \lambda_j'\}_{1\le j\le t}]$$

Now since $s_{\lambda/\mu} = \left|h_{\lambda_i - \mu_j + j - i}\right|$ we have for the symmetric functions p_i

$$|p_{\lambda_i - \mu_j - i + j}| = h_r^{-(s+t)} |h_{(\lambda_i' - \mu_j' - i + j)m + r}|_{1\le i,j\le t} = h_r^{-(s+t)} s_{\tilde{\lambda}/\tilde{\mu}},$$

where

$$\tilde{\lambda}_i = m\lambda'_i - (m-1)i + r + C$$
$$\tilde{\mu}_i = m\mu'_i - (m-1)i + C$$

where C is large enough to make $\tilde{\lambda}$ and $\tilde{\mu}$ nonnegative.

The simplest case is that in which $\lambda = (n), \mu = \emptyset$. So $\lambda'_i = 1$ for $i = 1, \ldots, n$. We may take $s = 1$, $t = n$. We get

$$\begin{aligned} \tilde{\lambda}_i &= m - (m-1)i + r + C, \quad i = 1, \ldots, n \\ \tilde{\mu}_i &= -(m-1)i + C, \quad i = 1, \ldots, n \end{aligned}.$$

We may take $C = (m-1)n$, and we obtain

$$\begin{aligned} \tilde{\lambda}_i &= (m-1)(n-i) + m + r \\ \tilde{\mu}_i &= (m-1)(n-i). \end{aligned} \tag{11.2}$$

Thus $\tilde{\lambda}_i - \tilde{\mu}_i = m + r$ and $\tilde{\lambda}_{i+1} - \tilde{\mu}_i = r + 1$.

Figure 8

Applying the homomorphism θ of Section 2 and setting $u = 1$ we get

$$\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{mn+r}}{(mn+r)!\,(x^r/r!)^{n-1}} \right)^{-1} \tag{11.3}$$

as the exponential generating function for Young tableaux of shape $\tilde{\lambda} - \tilde{\mu}$ given by (11.2). For example, if $m = 2$ and $r = 1$ (11.3) reduces to $x^{3/2}/\sin(x^{3/2})$. Since

$$\frac{x}{\sin x} = \sum_{n=0}^{\infty} d_n \frac{x^{2n}}{(2n)!},$$

where d_n is related to the Bernoulli number by $d_n = (-1)^{n+1}(2^{2n} - 2)B_{2n}$, it follows that

$$\frac{x^{3/2}}{\sin(x^{3/2})} = \sum_{n=0}^{\infty} d_n \frac{(3n)!}{(2n)!} \frac{x^{2n}}{(2n)!}$$

and thus $\dfrac{(3n)!}{(2n)!} d_n$ is the number of Young tableaux of shape $(n+2, n+1, \ldots, 3) - (n-1, n-2, \ldots, 0)$. This result may be compared with a combinatorial interpretation of $1 \cdot 3 \cdots (2n+1) d_{2n}$ given in Gessel and Viennot [25, p. 315].

12. Salié numbers and Faulhaber numbers

In view of Jacobi's theorem, it is natural to ask when matrices of binomial coefficients to which our theory applies have inverses which can be expressed in some kind of explicit form. In this section we discuss a closely related pair of such matrices. The entries of their inverses are numbers which have arisen in other contexts, but are not well known. One of them, according to Edwards [17], was studied by Faulhaber [20] in the seventeenth century in connection with formulas for sums of powers. The other was apparently first considered by Salié [58] in 1963 in connection with the number-theoretic properties of the coefficients of $\cosh z/\cos z$. (See also Hammersley [ ] and Dumont and Zeng [16].)

Both arrays of numbers can be defined in three ways: as entries of the inverse of a matrix of binomial coefficients, by generating functions, and by formulas for sums of powers. We shall start with the generating functions.

We define the Salié numbers $s(n,k)$ by

$$\sum_{n,k=0}^{\infty} s(n,k)t^k \frac{x^{2n}}{(2n)!} = \frac{\cosh \sqrt{1+4t}\,\frac{x}{2}}{\cosh \frac{x}{2}} \tag{12.1}$$

and we define the Faulhaber number $f(n,k)$ by

$$\sum_{n,k=0}^{\infty} f(n,k)t^k \frac{x^{2n+1}}{(2n+1)!} = \frac{\cosh \sqrt{1+4t}\,\frac{x}{2} - \cosh \frac{x}{2}}{t \sinh \frac{x}{2}}. \tag{12.2}$$

It is easily seen that $s(n,k) = f(n,k) = 0$ for $k > n$. As we shall see later, $|s(n,k)| = (-1)^{n-k}s(n,k)$ and $|f(n,k)| = (-1)^{n-k}f(n,k)$. The first few values of these numbers are as follows (zeros are omitted):

$s(n,k)$:

$n\backslash k$	0	1	2	3	4			
0	1							
1		1						
2		−1	1					
3		3	−3	1				
4		−17	17	−6	1			
5		155	−155	55	−10	1		
6		−2073	2073	−736	135	−15	1	
7		38227	−38227	13573	−2492	280	−21	1

$f(n,k)$:

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1		1/2						
2		−1/6	1/3					
3		1/6	−1/3	1/4				
4		−3/10	3/5	−1/2	1/5			
5		5/6	−5/3	17/12	−2/3	1/6		
6		−691/210	691/105	−118/21	41/15	−5/6	1/7	
7		35/2	−35	359/12	−44/3	14/3	−1	1/8

Next we prove the sums of powers formulas. Let $S_r(m) = \sum_{i=0}^{m} i^r$, with $0^0 = 1$, so that $S_0(m) = m+1$.

Theorem 27.

$$S_{2n+1}(m) = \frac{1}{2}\sum_{k=0}^{n} f(n,k)\big(m(m+1)\big)^{k+1}. \tag{12.3}$$

Proof. We have

$$\begin{aligned}
\sum_{r=0}^{\infty} S_r(m)\frac{x^r}{r!} &= 1 + e^x + \cdots + e^{mx} = \frac{e^{(m+1)x}-1}{e^x-1} \\
&= \frac{e^{(m+\frac12)x} - e^{-x/2}}{e^{x/2}-e^{-x/2}} = \frac{\sinh(m+\frac12)x + \cosh(m+\frac12)x + \sinh\frac{x}{2} - \cosh\frac{x}{2}}{2\sinh\frac{x}{2}}.
\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} S_{2n+1}(m)\frac{x^{2n+1}}{(2n+1)!} = \frac{\cosh(m+\frac12)x - \cosh\frac{x}{2}}{2\sinh\frac{x}{2}}.$$

Now set $t = m(m+1)$, so $\sqrt{1+4t} = 2m+1$. Then

$$
\begin{aligned}
\sum_{n=0}^{\infty} S_{2n+1}(m)\frac{x^{2n+1}}{(2n+1)!} &= \frac{\cosh\sqrt{1+4t}\,\frac{x}{2} - \cosh\frac{x}{2}}{2\sinh\frac{x}{2}} \\
&= \frac{t}{2}\sum_{n,k=0}^{\infty} f(n,k)t^k\frac{x^{2n+1}}{(2n+1)!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\sum_{k=0}^{n} f(n,k)\big(m(m+1)\big)^{k+1},
\end{aligned}
$$

and the result follows by equating coefficients of $\dfrac{x^{2n+1}}{(2n+1)!}$.

Formula (12.3) was first studied by Faulhaber [20] in the seventeenth century. Faulhaber's work is described in Edwards [17] Schneider [60], and Knuth . Formula (12.3) was rediscovered by Jacobi [33], who gave the recurrence (in our notation)

$$
2n(2n+1)f(n-1,k-2) = 2k(2k-1)f(n,k-1) + k(k+1)f(n,k).
$$

[Give analogous formula for Salié numbers, from Concrete Math. 7.52] As far as we know, the generating function (12.2) is new.

There is a companion formula which we state for completeness, although we will not use it (see Edwards [17]):

$$
S_{2n}(m) = (m+\frac{1}{2})\sum_{k=0}^{n}\bar{f}(n,k)\big(m(m+1)\big)^k,
$$

for $n > 0$, where

$$
1 + \sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\sum_{k=0}^{n}\bar{f}(n,k)t^k = \frac{\sinh\sqrt{1+4t}\,\frac{x}{2}}{\sqrt{1+4t}\sinh\frac{x}{2}},
$$

and moreover,

$$
\bar{f}(n,k) = \frac{k+1}{2n+1}f(n,k).
$$

Next we consider the analogous formula for Salié numbers. Let

$$
T_r(m) = \frac{1}{2}\sum_{i=-m}^{m}(-1)^{m-i}i^r,
$$

so if r is odd $T_r(m) = 0$, and for $n > 0$,

$$
T_{2n}(m) = \sum_{i=1}^{m}(-1)^{m-i}i^{2n}.
$$

Theorem 28.

$$
T_{2n}(m) = \frac{1}{2}\sum_{k=0}^{n}s(n,k)\big(m(m+1)\big)^k.
$$

Proof. We have

$$
\begin{aligned}
\frac{1}{2}\sum_{r=0}^{\infty}\left(\sum_{i=-m}^{m}(-1)^{m-i}i^r\right)\frac{x^r}{r!} &= \frac{1}{2}\sum_{i=-m}^{m}(-1)^{m-i}e^{ix} = \frac{e^{x(m+\frac{1}{2})}+e^{-x(m+\frac{1}{2})}}{2(e^{x/2}+e^{-x/2})} \\
&= \frac{\cosh(m+\frac{1}{2})x}{2\cosh\frac{x}{2}} = \frac{\cosh\sqrt{1+4t}\,\frac{x}{2}}{2\cosh\frac{x}{2}} = \frac{1}{2}\sum_{n,k=0}^{\infty}s(n,k)\frac{x^{2n}}{(2n)!}t^k
\end{aligned}
$$

and the theorem follows by equating coefficients of $\dfrac{x^{2n}}{(2n)!}$.

There is a companion formula for $\sum_{i=1}^{2n+1}(-1)^{m-i}i^{2n+1}$ that we leave to the reader.

Next we relate the Faulhaber and Salié numbers to inverses of matrices of binomial coefficients.

Theorem 29. *The inverse of the matrix* $\left(\binom{i+1}{2i-2j+1}\right)_{i,j=0,\cdots m}$ *is the matrix* $\big(f(i,j)\big)_{i,j=0,\cdots m}$, *and the inverse of the matrix* $\left(\binom{i}{2i-2j}\right)_{i,j=0,\cdots m}$ *is the matrix* $\big(s(i,j)\big)_{i,j=0,\cdots m}$.

Proof. We have

$$\big(i(i+1)\big)^{r+1} - \big((i-1)i\big)^{r+1} = 2\sum_{l \text{ odd}}\binom{r+1}{l}i^{2r+2-l}.$$

Summing on i from 0 to m we obtain

$$\big(m(m+1)\big)^{r+1} = 2\sum_{l \text{ odd}}\binom{r+1}{l}S_{2r+2-l}(m) = 2\sum_{j=0}^{r}\binom{r+1}{2r-2j+1}S_{2j+1}(m)$$

and the first assertion follows from Theorem 27.

Similarly, we have

$$\big(i(i+1)\big)^{r} + \big((i-1)i\big)^{r} = 2\sum_{l \text{ even}}\binom{r}{l}i^{2r-l}.$$

Multiplying by $(-1)^{m-i}$, summing from $i=-m$ to m, and dividing by 2, we obtain

$$\big(m(m+1)\big)^{r} = 2\sum_{l \text{ even}}\binom{r}{l}T_{2r-l}(m) = 2\sum_{j=0}^{r}\binom{r}{2r-2j}T_{2j}(m)$$

and the second assertion follows from Theorem 28.

The proof just given follows Edwards [17] (who considered only the Faulhaber numbers).

Note that since the matrix $\binom{i}{2i-2j}$ is lower triangular with 1's on the diagonal, its inverse has integer entries, so the Salié numbers are integers. We can now explain Salié's interest in these numbers [58]. (See also Comtet [13, pp. 86–87].) Salié was studying the numbers S_{2n} defined by

$$\frac{\cosh x}{\cos x} = \sum_{n=0}^{\infty} S_{2n}\frac{x^{2n}}{(2n)!}.$$

In equation (12.1) if we replace x by $2ix$, where $i=\sqrt{-1}$, and set $t=-\frac{1}{2}$ we obtain

$$\frac{\cosh x}{\cos x} = \sum_{n=0}^{\infty} 2^{2n}(-1)^n\frac{x^{2n}}{(2n)!}\sum_{k=0}^{n}s(n,k)(-\frac{1}{2})^k$$

and thus

$$S_{2n} = 2^{2n}(-1)^n\sum_{k=0}^{n}s(n,k)\left(-\frac{1}{2}\right)^k = \sum_{k=0}^{n}s(n,k)(-1)^{n-k}2^{2n-k}.$$

Thus S_{2n} is divisible by 2^n, as shown by Salié. (Salié used the notation $c_{i-j}^{(i)}$ for our $s(i,j)$.)

It follows from equation (12.4) below that $s(n,n) = 1$, $s(n,n-1) = -\binom{n}{2}$, and $s(n,n-2) = \binom{n-1}{2}\binom{n}{2} - \binom{n}{4}$, so that we have in fact,

$$2^{-n} S_{2n} \equiv 1 + 2\binom{n}{2} + 4\binom{n-1}{2}\binom{n}{2} - 4\binom{n}{4} \pmod 8.$$

Salié obtained a congruence (mod 16) for the numbers $2^{-n}S_{2n}$ by a different method. See also Carlitz [8].

Next we consider the combinatorial interpretation of the Salié numbers and Faulhaber numbers. First we note the following lemma, which follows easily from the formula for the inverse of a matrix.

Lemma 30. *Let $\left(A_{ij}\right)_{i,j=0,\ldots,m}$ be an invertible lower triangular matrix, and let $(B_{ij}) = (A_{ij})^{-1}$. Then for $0 \le k \le n \le m$, we have*

$$B_{n,k} = \frac{(-1)^{n-k}}{A_{k,k}A_{k+1,k+1}\cdots A_{n,n}} \left|A_{k+i+1,k+j}\right|_{i,j=0,\ldots,n-k-1}.$$

Since the Salié numbers are entries of $\left(\binom{i}{2i-2j}\right)^{-1}$, we have

$$s(n,k) = (-1)^{n-k} \left| \binom{k+i+1}{2i-2j+2} \right|_0^{n-k-1} \tag{12.4}$$

and since the Faulhaber numbers are entries of $\left(\binom{i+1}{2i-2j+1}\right)^{-1}$, we have

$$f(n,k) = (-1)^{n-k} \frac{k!}{(n+1)!} \left| \binom{k+i+2}{2i-2j+3} \right|_0^{n-k-1}. \tag{12.5}$$

Although we can give a combinatorial interpretation of these determinants using Theorem 14, the simplest interpretations are most easily derived directly from the paths.

For the Salié numbers, we define the points $P_m = (2m, -2m)$ and $Q_m = (2m, -m)$. Then it follows from (12.4) that $(-1)^{n-k}s(n,k)$ is the number of disjoint $(n-k)$-paths from $(P_k, P_{k+1}, \ldots, P_{n-1})$ to $(Q_{k+1}, Q_{k+2}, \ldots, Q_n)$. Figure 9 illustrates a 3-path counted by $|s(5,2)|$. We note that it is immediate from

Figure 9

this combinatorial interpretation that $|s(n,1)| = |s(n,2)|$. To represent these $(n-k)$-paths most simply as tableaux, we assign to the horizontal segment from $(i,-j)$ to $(i+1,-j)$ the label $i-j+1$ and make the labels on each path into a row of the tableau, shifting as usual. Thus the 3-path of Figure 9 becomes the tableau

$$\begin{array}{cccc} & & 3 & 5 \\ & 2 & 4 & \\ 1 & 2 & & \end{array} \tag{12.6}$$

Converting the conditions on the paths into conditions on the tableau, we have

Theorem 31. *$|s(n,k)| = (-1)^{n-k}s(n,k)$ is the number of row-strict tableaux of shape $(n-k+1, n-k, \ldots, 2) - (n-k-1, n-k-2, \ldots, 0)$ with positive integer entries in which the largest entry in row i is at most $n+1-i$.*

We may convert the tableau into a sequence of integers by reading each row from left to right, starting with the last row, so that (12.6) becomes 1 2 2 4 3 5. Checking the conditions on this sequence, we find that we are counting sequences $a_1 a_2 \cdots a_{2n-2k}$ of positive integers satisfying $a_{2i-1} < a_{2i}$, $a_{2i} \geq a_{2i+1}$, and $a_{2i} \leq k+i$ for each i. This combinatorial interpretation is closely related to the combinatorial interpretation of the Genocchi numbers given by Dumont and Viennot [15]. See also Viennot [70].

Next we move on the Faulhaber numbers. Let $P_m = (2m, -2m)$ as before, and let $R_m = (2m+1, -m)$. Then $(-1)^{n-k}\dfrac{(n+1)!}{k!}f(n,k)$ is the number of nonintersecting $(n-k)$-paths from $(P_k, P_{k+1}, \ldots, P_{n-1})$ to $(R_{k+1}, R_{k+2}, \ldots, R_n)$. As in the case of the Salié numbers, we represent these $(n-k)$-paths as tableaux,

Figure 10

so the 4-path of Figure 10 becomes

$$\begin{array}{cccccc} & & & \boxed{1} & \boxed{3} & \boxed{5} \\ & & \boxed{3} & \boxed{4} & \boxed{5} & \\ & \boxed{1} & \boxed{3} & \boxed{4} & & \\ \boxed{1} & \boxed{2} & \boxed{3} & & & \end{array} \tag{12.7}$$

and we have

Theorem 32. $(-1)^{n-k}\dfrac{(n+1)!}{k!}f(n,k)$ *is the number of row-strict tableaux of shape* $(n-k+2, n-k+1, \ldots, 2) - (n-k-1, n-k-2, \ldots, 0)$ *with positive integer entries in which the largest entry in row* i *is at most* $n+2-i$.

We can also represent these tableaux as sequences, so for example, (12.7) becomes 1 2 3 1 3 4 3 4 5 1 3 5, and we have

Theorem 33. $(-1)^{n-k}\dfrac{(n+1)!}{k!}f(n,k)$ *is the number of sequences* $a_1 a_2 \cdots a_{3n-3k}$ *of positive integers satisfying* $a_{3i-2} < a_{3i-1} < a_{3i}$, $a_{3i-1} \geq a_{3i+1}$, $a_{3i} \geq a_{3i+2}$, *and* $a_{3i} \leq k+i+1$ *for all* i.

We now derive some explicit formulas for the Salié numbers in terms of the Genocchi numbers, and for the Faulhaber numbers in terms of the Bernoulli numbers.

We first consider the Salié numbers. We shall need the formula [55, Ex. 2, pp. 153–154]

$$\left(\frac{1-\sqrt{1+4t}}{2}\right)^i = \sum_{k=i}^{\infty} \frac{i}{2k-i}\binom{2k-i}{k}(-t)^k. \tag{12.8}$$

Now

$$\begin{aligned} \cosh \sqrt{1+4t}\,\frac{x}{2} &= \cosh\left(\left(\frac{1-\sqrt{1+4t}}{2}\right)x - \frac{x}{2}\right) \\ &= \cosh\left(\frac{1-\sqrt{1+4t}}{2}\right)x \,\cosh\frac{x}{2} - \sinh\left(\frac{1-\sqrt{1+4t}}{2}\right)x \,\sinh\frac{x}{2} \end{aligned}$$

so

$$\begin{aligned} \frac{\cosh\sqrt{1+4t}\,\frac{x}{2}}{\cosh\frac{x}{2}} = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \sum_{k=2j}^{\infty} \frac{2j}{2k-2j}\binom{2k-2j}{k}(-t)^k \\ - \tanh\frac{x}{2}\sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!}\sum_{k=2j+1}^{\infty}\frac{2j+1}{2k-2j-1}\binom{2k-2j-1}{k}(-t)^k. \end{aligned}$$

The Genocchi numbers G_n are defined [13, p. 49] by

$$\frac{2x}{e^x+1} = x(1-\tanh\frac{x}{2}) = \sum_{n=1}^{\infty} G_n \frac{x^n}{n!}.$$

(Thus $G_{2n+1}=0$ for $n>0$ and $G_{2n}=2(1-2^{2n}B_{2n})$, where B_{2n} is the Bernoulli number.) Then we have

$$\sum_{n=0}^{\infty} s(n,k)\frac{x^{2n}}{(2n)!} = (-1)^k\left(\sum_{j=0}^{\lfloor k/2\rfloor}\frac{2j}{2k-2j}\binom{2k-2j}{k}\frac{x^{2j}}{(2j)!}\right.$$
$$\left.- x\tanh\frac{x}{2}\sum_{j=0}^{\infty}\frac{1}{2k-2j-1}\binom{2k-2j-1}{k}\frac{x^{2j}}{(2j)!}\right)$$

and thus

$$s(n,k) = (-1)^k\left(\frac{2n}{2k-2n}\binom{2k-2n}{k} + \sum_{j=0}^{\lfloor (k-1)/2\rfloor}\frac{1}{2k-2j-1}\binom{2k-2j-1}{k}\binom{2n}{2j}G_{2n-2j}\right).$$

Since $\binom{2k-2n}{k}=0$ for $n>k/2$, we have

$$s(n,k) = (-1)^k\sum_{j=0}^{\lfloor (k-1)/2\rfloor}\frac{1}{2k-2j-1}\binom{2k-2j-1}{k}\binom{2n}{2j}G_{2n-2j} \tag{12.9}$$

for $n>k/2$, and since $s(n,k)=0$ for $n<k$, we also obtain the identity for Genocchi numbers

$$\sum_{j=0}^{\lfloor (k-1)/2\rfloor}\frac{1}{2k-2j-1}\binom{2k-2j-1}{k}\binom{2n}{2j}G_{2n-2j} = -\frac{2n}{2k-2n}\binom{2k-2n}{k},\quad n<k.$$

The first few instances of (12.9) are

$$\begin{aligned}
s(n,1) &= -G_{2n},\quad n\ge 1\\
s(n,2) &= G_{2n},\quad n\ge 2\\
s(n,3) &= -2G_{2n} - \frac{1}{3}\binom{2n}{2}G_{2n-2},\quad n\ge 2\\
s(n,4) &= 5G_{2n} + \binom{2n}{2}G_{2n-2},\quad n\ge 3
\end{aligned}$$

Similarly, for the Faulhaber numbers we have

$$\begin{aligned}
\frac{\cosh\sqrt{1+4t}\,\frac{x}{2}-\cosh\frac{x}{2}}{t\sinh\frac{x}{2}} &= t^{-1}\coth\frac{x}{2}\left(\cosh\left(\frac{1-\sqrt{1+4t}}{2}\right)x-1\right)\\
&\quad - t^{-1}\sinh\left(\frac{1-\sqrt{1+4t}}{2}\right)x\\
&= x\coth\frac{x}{2}\sum_{j=1}^{\infty}\frac{x^{2j-1}}{(2j)!}\sum_{k=2j}^{\infty}\frac{2j}{2k-2j}\binom{2k-2j}{k}(-1)^k t^{k-1}\\
&\quad - \sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!}\sum_{k=2j+1}^{\infty}\frac{2j+1}{2k-2j-1}\binom{2k-2j-1}{k}(-1)^k t^{k-1}\\
&= x\coth\frac{x}{2}\sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!}\sum_{k=2j+1}^{\infty}\frac{1}{2k-2j}\binom{2k-2j}{k+1}(-1)^{k+1}t^k\\
&\quad + \sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!}\sum_{k=2j}^{\infty}\frac{2j+1}{2k-2j+1}\binom{2k-2j+1}{k+1}(-1)^k t^k.
\end{aligned}$$

Now since

$$x \coth \frac{x}{2} = 2 \sum_{n=0}^{\infty} B_{2n} \frac{x^{2n}}{(2n)!},$$

where B_m is the Bernoulli number, we have

$$\begin{aligned} f(n,k) = (-1)^{k+1} &\sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \frac{1}{k-j} \binom{2k-2j}{k+1} \binom{2n+1}{2j+1} B_{2n-2j} \\ &+ (-1)^k \frac{2n+1}{2k-2n+1} \binom{2k-2n+1}{k+1}. \end{aligned}$$

As before, this yields the formula

$$f(n,k) = (-1)^{k+1} \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \frac{1}{k-j} \binom{2k-2j}{k+1} \binom{2n+1}{2j+1} B_{2n-2j} \tag{12.10}$$

for $n > k/2$, and the identity for Bernoulli numbers

$$\sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \frac{1}{k-j} \binom{2k-2j}{k+1} \binom{2n+1}{2j+1} B_{2n-2j} = \frac{2n+1}{2k-2n+1} \binom{2k-2n+1}{k+1}, \quad n < k.$$

The first few instances of (12.10) are as follows:

$$\begin{aligned} f(n,1) &= (2n+1)B_{2n}, \quad n \geq 1 \\ f(n,2) &= -2(2n+1)B_{2n}, \quad n \geq 2 \\ f(n,3) &= 5(2n+1)B_{2n} + \frac{1}{2}\binom{2n+1}{3} B_{2n-2}, \quad n \geq 2 \end{aligned}.$$

It follows in particular that we have given a combinatorial interpretation to the product

$$(-1)^{n-1}(n+1)!\,(2n+1)B_{2n}.$$

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