Growth in Repeated Truncations of Maps

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Last revision: May 6, 2000

Dedicated to the memory of Mario Pezzana

Abstract

It is shown that the spherical growth function of repeated truncations of maps tend to a universal function

$$f(x) = (1+x)\prod_{n=0}^{\infty}(1+2x^{2^n}) = 1+3x+4x^2+6x^3+6x^4+6x^5+8x^6+12x^7+10x^8+6x^9+\dots$$

which is independent of the original map as well as from the initial vertex of the truncated map.

Let G be connected finite or locally finite graph, rooted as some vertex v. Define $\delta(G, v, n)$ to be the number of vertices at distance n from v. Furthermore, let f be the co-called *(spherical) growth function*. This means that $f(G, v; x)$ is the generating function for $\delta(G, v, n)$ of G at v.

$$f(G, v; x) = \sum_{n=0}^{\infty} \delta(G, v, n)x^n.$$

We may calculate the growth of iterated truncations of maps. Let $T(M)$ denote the *truncated* map M. By $T^n(M)$ we denote the $n-$th iterated truncation of M. Since $T(M)$ is cubic, $T^2(M)$ contains a 2-factor consisting of

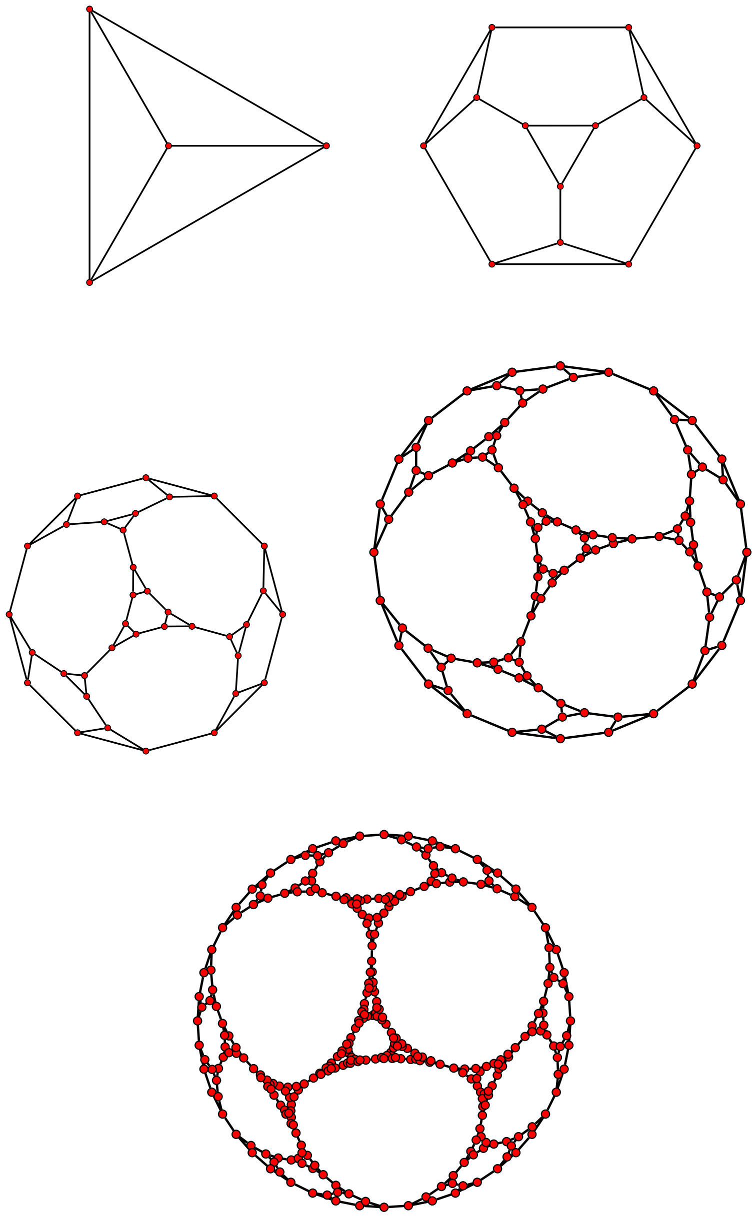

Figure 1: Tetrahedron and its first 4 repeated truncations.

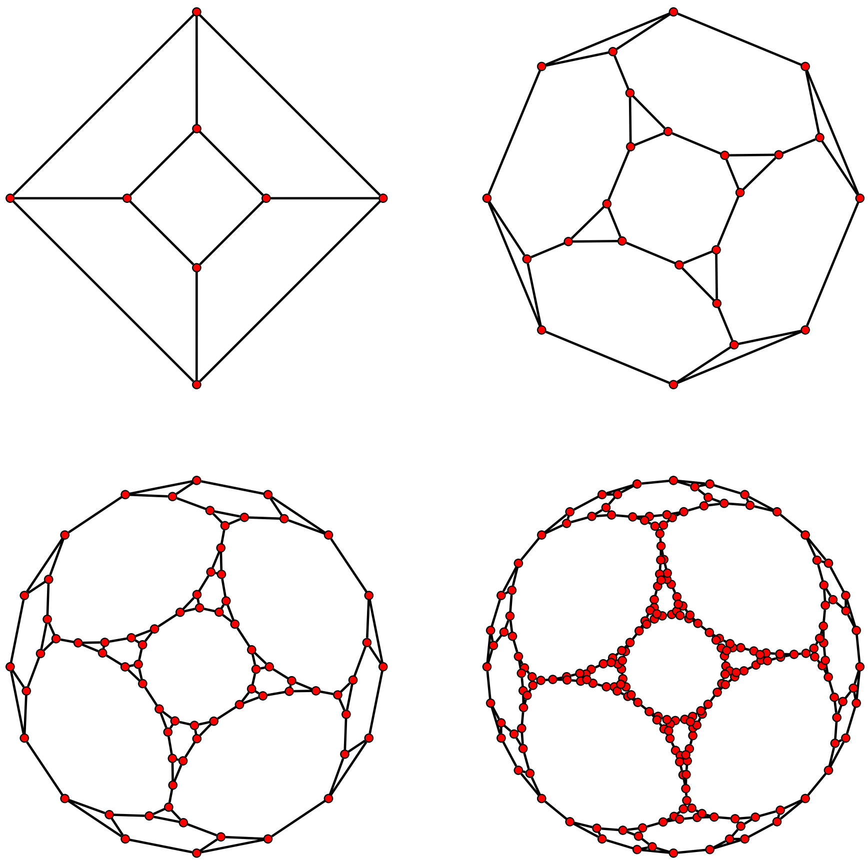

Figure 2: Cube and its first 3 repeated truncations.

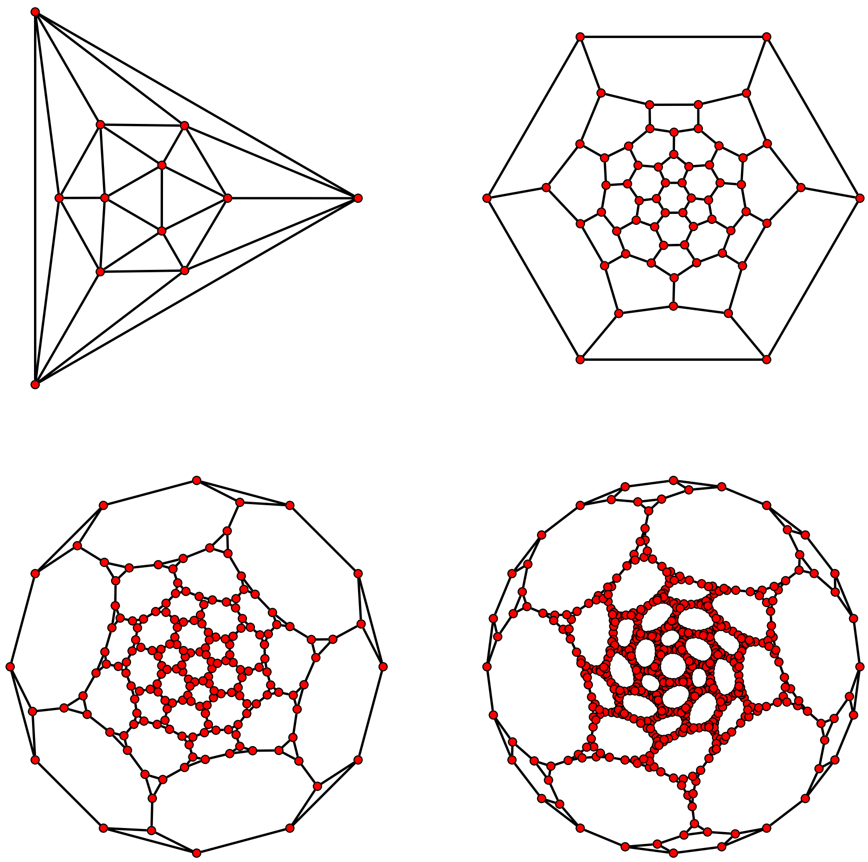

Figure 3: Icosahedron and its first 3 repeated truncations.

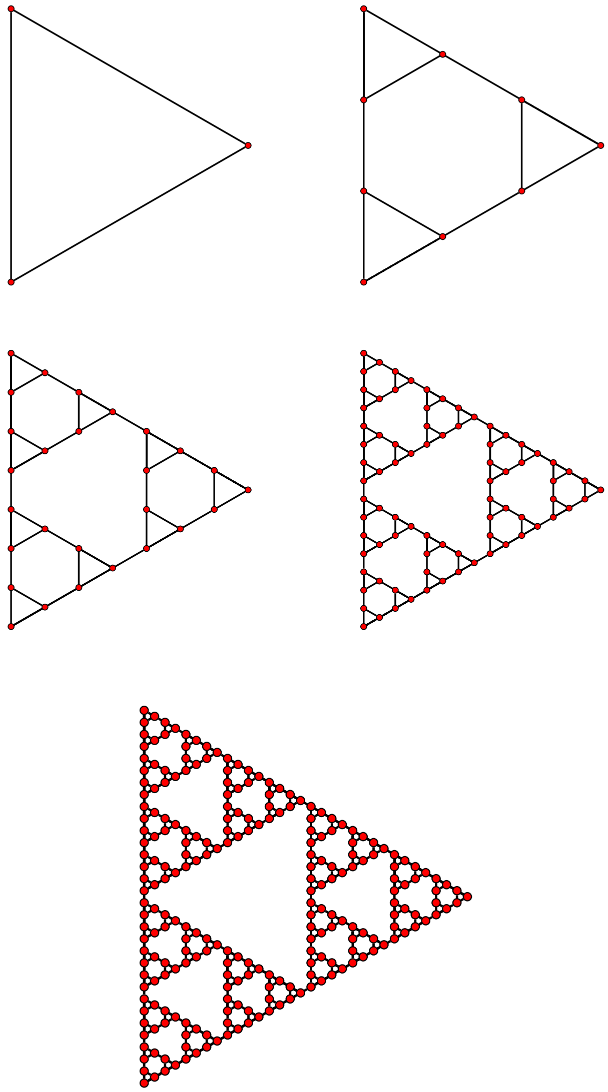

Figure 4: The first truncation produces a cubic map. The second one puts each vertex on a triangle. Each successive truncation subdivides the triangle in a fractal way as shown here. The resulting graph is called the *n-th extended Sierpiński triangle* and will be denote by S_n. Here we have $S_n, n = 0, 1, 2, 3, 4$.

triangles (a patchwork [4]) and a 1-factor. We will refer to the edges belonging to triangles as "blue" edges, and to other edges as the "red" edges. A walk with alternating blue-red edges is called an *alternating walk*. Label each red edge by x and blue edge by y.

Clearly we have: in $T^n(M), n \geq 2$ any shortest path is alternating. Also: $T^n(M)$ may be viewed as a $T^2(M)$ in which each triangle is subdivided in a *fractal way* (see Figure 4). In other words, each triangle S_0 is replaced be the (n-2)-nd extended Sierpiński triangle S_{n-2}. A word of caution here! Our notion of the (extended) Sierpiński triangle differs from the one that is used in chaos theory [1]. The first difference is that we consider graphs: (finite) abstract combinatorial structures while fractals arise from the study of iterating functions in some metric spaces. The second difference is more important. The classical Sierpiński triangle viewed as a graph would have triangles *touching*, i.e. it would involve vertices of valence four, while our approach separates the triangles by edges. In other words, the classical Sierpiński triangle is obtained from our extended Sierpiński triangle by contracting the "red" edges of the one-factor.

If the original map is cubic, like in the case of Figures 1 and 2, we may start with $T(M)$ instead of $T^2(M)$ in the argument that follows.

Let us choose a root vertex and label each vertex of $T^2(M)$ by $x^a y^b$, where $a + b$ is its distance from the root and there are a red edges and b blue edges on a shortest path. Clearly $|a - b| \leq 1$. For any subset S of vertices of $T^2(M)$ we may describe the labeling of the corresponding induced subgraph as the formal sum of vertex labels. In an even cycle we may reach the antipodal vertex in two ways $x^{a+1}y^a$ or $x^a y^{a+1}$. Later we will think of y being "longer" than x. That is why we will always select the first alternative and use the labeling $x^{a+1}y^a$. There are at most 6 types of triangles since each triangle is labeled in at most one of the six ways:

$$\begin{array}{lrcl}
P_5: & x^a y^b + x^{a+1} y^b + x^a y^{b+1} & = & x^a y^b (1 + x + y) \\
P_4: & x^a y^b + x^{a+1} y^b + x^a y^b & = & x^a y^b (2 + x) \\
P_3: & x^a y^b + x^{a+1} y^b + x^{a+1} y^b & = & x^a y^b (1 + 2x) \\
P_2: & x^a y^b + x^a y^{b+1} + x^a y^{b+1} & = & x^a y^b (1 + 2y) \\
P_1: & x^a y^b + x^a y^b + x^a y^{b+1} & = & x^a y^b (2 + y) \\
P_0: & x^a y^b + x^a y^b + x^a y^b & = & 3x^a y^b
\end{array}$$

The growth function $f_2(x)$ can be written as

$$\begin{aligned} f_2(x) &= P_5(x,x)(1+2x) + P_4(x,x)(2+x) \\ &\quad + P_3(x,x)(1+2x) + P_2(x,x)(1+2x) \\ &\quad + P_1(x,x)(2+x) + P_0(x,x)3 \end{aligned}$$

where

$$\begin{aligned} f_2(x,y) &= P_5(x,y)(1+x+y) + P_4(x,y)(2+x) \\ &\quad + P_3(x,y)(1+2x) + P_2(x,y)(1+2y) \\ &\quad + P_1(x,y)(2+y) + P_0(x,y)3 \end{aligned}$$

is the generating function for the number of vertices labeled $x^a y^b$ and P_i count the number of triangles of a given type. Note that the only nonzero coefficients are for $x^a y^b$ where $b = a$ or $b = a-1$ or $b = a+1$. All the P_is have an xy factor except P_2 where the two triangles nearest to the root provide an extra 1 term and an x term.

Every xy term becomes $x(yx)$ in moving from the nth truncation to the $(n+1)$st. Thus in going from the 2nd truncation to the nth one replaces y by $x^{2^{n-2}}$ for all y in the generating function for x alone. We can now write down the growth function of the $T^n(M)$ for an appropriate root:

$$\begin{aligned} f_n(x) = P_5(x,x^{2^{n-2}})t_n^{(5)}(x) + P_4(x,x^{2^{n-2}})t_n^{(4)}(x) + P_3(x,x^{2^{n-2}})t_n^{(3)}(x) \\ + P_2(x,x^{2^{n-2}})t_n^{(2)}(x) + P_1(x,x^{2^{n-2}})t_n^{(1)}(x) + P_0(x,x^{2^{n-2}})t_n^{(0)}(x) \end{aligned}$$

where $t_n^{(i)}(x)$ represent the growth in the n-th extended Sierpiński triangle S_n with a given boundary conditions and are given by the following recursion formulas:

$$\begin{aligned} t_n^{(5)}(x) &= (1+x)t_{n-1}^{(2)}(x) + x^{2^{n-2}} t_{n-1}^{(5)}(x) \\ t_n^{(4)}(x) &= (2+x)t_{n-1}^{(2)}(x) \\ t_n^{(3)}(x) &= (1+2x)t_{n-1}^{(2)}(x) \\ t_n^{(2)}(x) &= t_{n-1}^{(2)}(x)(1+2x^{2^{n-2}}) \\ t_n^{(1)}(x) &= 2t_{n-1}^{(2)}(x) + x^{2^{n-2}} t_{n-1}^{(1)}(x) \\ t_n^{(0)}(x) &= 3t_{n-1}^{(2)}(x) \end{aligned}$$

$$\begin{aligned} t_2^{(5)}(x) &= 1+2x \\ t_2^{(4)}(x) &= 2+x \\ t_2^{(3)}(x) &= 1+2x \\ t_2^{(2)}(x) &= 1+2x \\ t_2^{(1)}(x) &= 2+x \\ t_2^{(0)}(x) &= 3 \end{aligned}$$

It is not hard to see that

$$t_n^{(2)}(x) = \prod_{k=2}^{n}(1+2x^{2^{k-2}}) = 1+2x+2x^2+4x^3+2x^4+4x^5+4x^6+8x^7+...+2^n x^{2^{n-1}}$$

It makes sense to define the limit of $t_n^{(2)}(x)$ when n tends to infinity.

$$t^{(2)}(x) = \prod_{n=0}^{\infty}(1+2x^{2^n})$$

One can similarly get:

$$\begin{aligned} t^{(5)}(x) &= (1+x)t^{(2)}(x) \\ t^{(4)}(x) &= (2+x)t^{(2)}(x) \\ t^{(3)}(x) &= (1+2x)t^{(2)}(x) \\ t^{(1)}(x) &= 2t^{(2)}(x) \\ t^{(0)}(x) &= 3t^{(2)}(x) \end{aligned}$$

We have enough information for computing the exact growth function for $T^n(M)$ for a specific root. We may view $T^n(M)$ as a $T^2(M)$ in which each triangle is replaced by a copy of the (n-2)-fold extended Sierpiński triangle S_{n-2}. If we are given a root in $T^2(M)$ this defines a root in the corresponding $T^n(M)$.

For example, in the case of the cube C, we may label the vertices of $T(C)$ and obtain: $P_2(x,y) = 1+x, P_1(x,y) = 2x^3y^2$ and $P_i(x,y) = 0$, for $i = 5,4,3,0$. Hence the growth function of $T^n(M)$ can be computed by the above method. For the truncated tetrahedron we get: $P_2(x,y) = 1+x+2xy+2x^2y, P_5(x,y) = 2xy$ and $P_i(x,y) = 0$, for $i = 4,3,1,0$.

We may define the limit truncation $T^\infty(M)$ which is a planar graph that consists of infinitely many isomorphic components.

The same approach can be used to look at the growth of the "infinite" truncation of an arbitray map M:

$$\begin{aligned} f(x) &= (1+x)t^{(2)}(x) \\ &= 1+3x+4x^2+6x^3+6x^4+6x^5+8x^6+12x^7+\ldots \end{aligned}$$

It seems that in this case the growth function is *universal.* It is independent of the choice of the original map and vertex.

We may verify the result by an independent argument. Observe that the neighborhood of each vertex in a repeated truncation looks like two extended Sierpiński triangles joined by an edge. The first few extended Sierpiński triangles are depicted in Figure 4. The growth in the n-th extended Sierpiński triangle S_n from one of its corners is given by g_n, where:

$$\begin{aligned} g_0(x) &= (1+2x) \\ g_1(x) &= (1+2x^2)g_0(x) \\ g_2(x) &= (1+2x^4)g_1(x) \\ g_3(x) &= (1+2x^8)g_2(x) \\ &\cdots \end{aligned}$$

Let $g(x) = \lim_{n\to\infty} g_n(x)$. There is a very simple rule for generating the list a_n of coefficients of g_n from the list of coefficients a_{n-1} for g_{n-1}.

$$\begin{aligned} a_0 &= (1) \\ a_n &= a_{n-1} * 2a_{n-1} \end{aligned}$$

where $*$ denotes the concatenation of lists.

$$\begin{aligned} a_0 &= (1) \\ a_1 &= (1,2) \\ a_2 &= (1,2,2,4) \\ a_3 &= (1,2,2,4,2,4,4,8) \\ a_4 &= (1,2,2,4,2,4,4,8,2,4,4,8,4,8,8,16) \\ \cdots \end{aligned}$$

At each step we double the number of known coefficients of $g(x)$.

Hence we do get $f(x) = (1+x)t^{(2)}(x)$ as $g(x) = t^{(2)}(x)$.

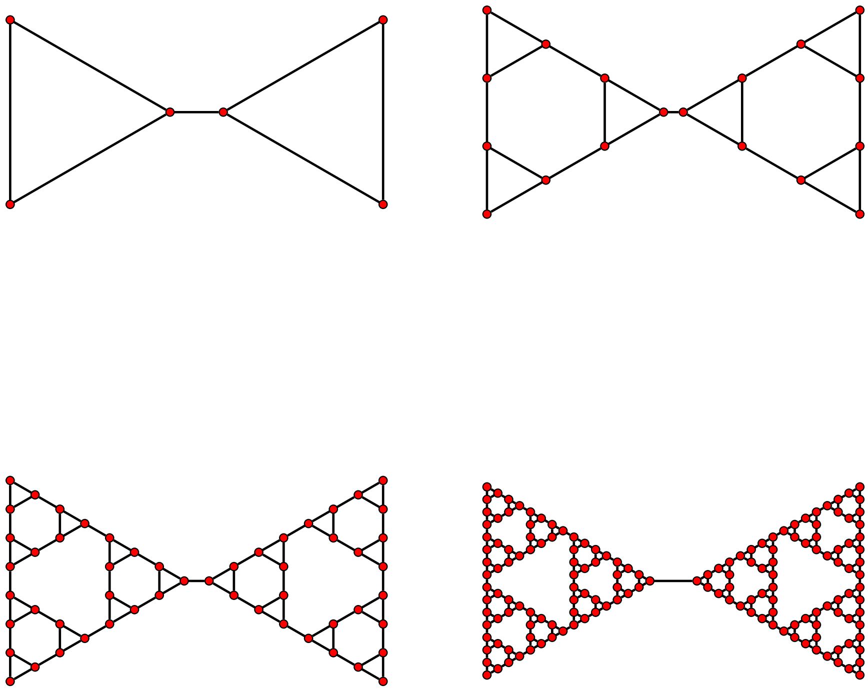

Figure 5: A connected component of the limit graph may be obtained by attaching two extended Sierpiński triangles on an edge.

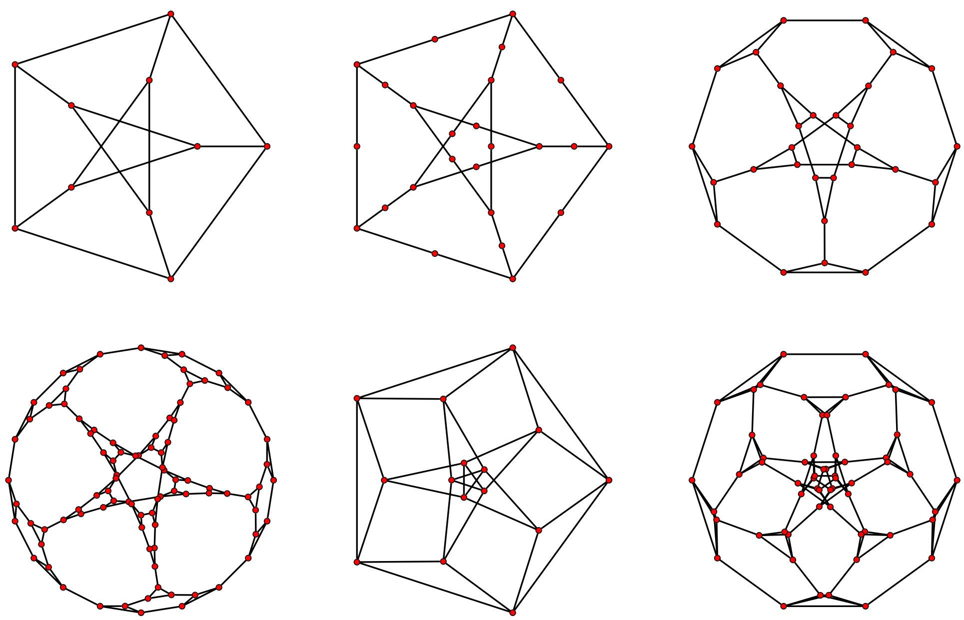

Figure 6: The Petersen graph P, and some if its derivative graphs: $S(P), T(P), T(T(P)), L(P), T(L(P))$.

There is a generalization of our approach. Namely, triangulation of a trivalent map is independent of the surface in which the graph is embedded. It can be described in a combinatorial way using two well-known concepts in graph theory: subdivision graphs $S(G)$ and line-graphs [5] $L(G)$. Define $T(G) = L(S(G))$. For trivalent graphs this operation coincides with map truncation. However, it can be studied for other graphs. For instance, if G is $k+1$-valent then $T(G)$ is also $k+1$-valent. The computation can be repeated for general k. Instead of extended Sierpiński triangles we get extended Sierpiński simplices: for $k = 3$ tetrahedra. The universal growth function becomes:

$$f(x) = (1+x)\prod_{j=0}^{\infty}(1+kx^{2^j}) = (1+x)(1+kx+kx^2+k^2x^3+kx^4+k^2x^5+k^2x^6+k^3x^7+...)$$

Note. We have not been very specific about the type of limiting process considered in this paper. A possible formalization is indicated in [2].

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