I-graphs and the corresponding configurations

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Abstract

We consider the class of I-graphs $I(n, j, k)$, which is a generalization over the class of the generalized Petersen graphs. We study different properties of I-graphs such as connectedness, girth and whether they are bipartite or vertex-transitive. We give an efficient test for isomorphism of I-graphs and characterize the automorphism groups of I-graphs.

Regular bipartite graphs with girth at least 6 can be considered as Levi graphs of some symmetric combinatorial configurations. We consider configurations which arise from bipartite I-graphs. Some of them can be realized in the plane as cyclic astral configurations, i.e. as geometric configurations with maximal isometric symmetry.

1 Introduction

Trivalent or cubic graphs form an extensively studied class of graphs. Since they are sparse, trivalent graphs can be readily drawn and visualized. Many graph theoretical problems can be reduced to the trivalent case. The purpose of this paper is the study of I-graphs, a special class of trivalent graphs. I-graphs were introduced in [6] and form a natural generalization of generalized Petersen graphs [18]. An I-graph is described by three integer parameters. We determine the necessary and sufficient conditions for testing whether two I-graphs are isomorphic or not. We also classify I-graphs in terms of girth, bipartiteness, and automorphism group.

Bipartite cubic graphs with girth at least 6 can be considered as incidence graphs (or Levi graphs) of combinatorial configurations. Although configurations are mathematical objects known for more than 150 years, the connection between them and certain classes of graphs has not been widely investigated. Here we contribute some results concerning configurations arising from I-graphs.

From the combinatorial point of view, some results follow from the properties of their Levi graphs, for example about symmetry and about being triangle- or quadrangle-free, etc.

From the geometric point of view, there is an interesting connection to (cyclic) astral configurations introduced in [14]. These configurations can be realized in the Euclidean plane with maximal possible cyclic symmetry. In [5] the authors proved among other things that the Levi graph of the smallest astral triangle-free configuration is the generalized Petersen graph $G(18, 5)$, which is, in turn, an I-graph.

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Only bipartite I-graphs can serve as Levi graphs of astral configurations, but Levi graphs of astral configurations are not necessarily I-graphs. This fact gives us a motivation to study a new class of bipartite cubic graphs, which we call C-graphs. The C-graphs generalize the class of Levi graphs of all astral configurations and the class of bipartite I-graphs. We define them by means of covering graphs. Using this definition we are not only able to easily give answers about combinatorial properties of the specific configuration (e.g. existence of triangles, quadrangles, etc.) but also to "encode" the information about its astral realization, if it exists.

1.1 Generalized Petersen graphs and I-graphs

The generalized Petersen graph $G(n,k)$ is a graph with vertex set

$$V(G(n,k)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$$

and edge set

$$E(G(n,k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, \ldots, n-1\}.$$

Throughout the paper subscripts are to be read modulo n. Note that $G(n,k)$ is isomorphic to $G(n, n-k)$ and $G(n, n/2)$ is not simple. Therefore, for $n \geq 3$, we consider only graphs $G(n,k)$ where $k < n/2$. For $n \leq 2$ we allow two exceptions, $G(1,1)$ and $G(2,1)$, compare Figures 2 and 3.

Generalized Petersen graphs constitute a standard family of graphs which represents a generalization of the renowned Petersen graph $G(5,2)$. This important and well known family of graphs that was introduced in 1969 by Mark Watkins [18] possesses a number of interesting properties. For example, $G(n,r)$ is vertex transitive if and only if $n = 10$, $r = 2$ or $r^2 \equiv \pm 1 \pmod{n}$. It is a Cayley graph if and only if $r^2 \equiv 1 \pmod{n}$. It is arc-transitive only in the following seven cases: $(n,r) = (4,1), (5,2), (8,3), (10,2), (10,3), (12,5), (24,5)$. The family contains some very important graphs. Among others the n-prism $G(n,1)$, the Dürer graph $G(6,2)$, the Möbius-Kantor graph $G(8,3)$, the dodecahedron $G(10,2)$, the Desargues graph $G(10,3)$, etc.

The generalized Petersen graphs form a special case of the so-called I-graphs, see [6]. The *I-graph* $I(n,j,k)$ is a graph with vertex set

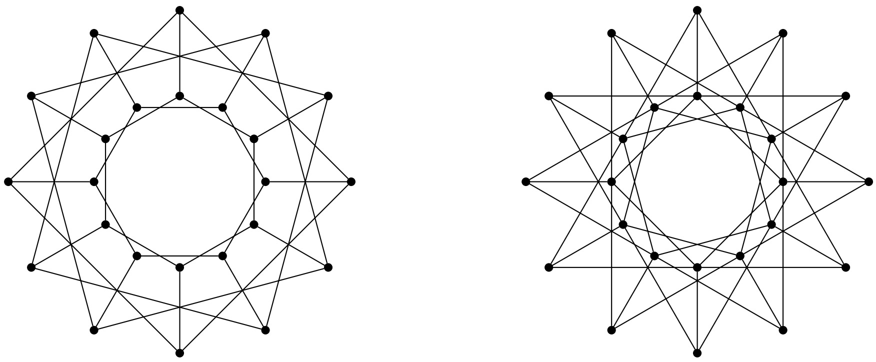
$$V(I(n,j,k)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$$

and edge set

$$E(I(n,j,k)) = \{u_i u_{i+j}, u_i v_i, v_i v_{i+k} : i = 0, \ldots, n-1\}.$$

Since $I(n,j,k) = I(n,k,j)$ we will usually assume that $j \leq k$. Clearly $G(n,k) = I(n,1,k)$. Following the usual representation of these graphs where we draw vertices u_i on one circle and vertices v_i on another circle (with smaller radius), we call the vertices on these two concentric circles *the vertices on the outer rim* and *the vertices on the inner rim.* Edges between the two rims are called *spokes.* The class of graphs $I(n,j,k)$ contains the class $G(n,k)$. We call an I-graph that is connected and not isomorphic to a generalized Petersen graph a *proper I-graph.* The smallest proper I-graphs are $I(12,2,3)$ and $I(12,3,4)$ and are depicted in Figure 1.

If we restrict to vertex transitive graphs, the two classes coincide (compare Theorem 7 and Corollary 8). Nevertheless, I-graphs that are not generalized Petersen graphs bring some new features that are worth studying. For example, in [10] Frucht, Graver, and Watkins characterize the automorphism groups of the generalized Petersen graphs. In particular, they show that the automorphism groups of non-vertex transitive generalized Petersen graphs are all dihedral. We show that this is not necessarily the case for proper I-graphs.



(a) (b)

Figure 1: I-graphs $I(12, 2, 3)$ (a) and $I(12, 3, 4)$ (b) are the smallest proper I-graphs, i.e. connected I-graphs that are not generalized Petersen graphs.

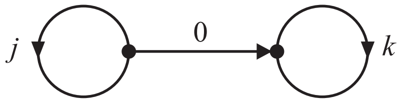


Figure 2: I-graph $I(n, j, k)$ is a $\mathbb{Z}_n$ covering graph over the handcuff graph $G(1, 1)$ with voltages and edge directions shown above.

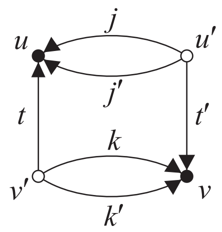


Figure 3: Graph $G(j, j', t', k, k', t)$ is voltage graph over $G(2, 1)$. A $\mathbb{Z}_n$ covering graph over this graph is denoted by $C(n, j, j', t', k, k', t)$.

1.2 Covering graphs and C-graphs

To simplify the description of large graphs, the concept of *voltage graphs* and *covering graphs* is generally used, see for example [13] or [19]. Using this method of description, $I(n, j, k)$ is a $\mathbb{Z}_n$ covering graph over the *handcuff graph* $G(1, 1)$ shown in Figure 2.

Now, let us define another family which generalizes bipartite I-graphs. Let $C(n, j, j', t', k, k', t)$ denote a $\mathbb{Z}_n$ covering graph over the voltage graph $G(j, j', t', k, k', t)$ shown in Figure 3. Since the voltages on any spanning tree can be set to 0, we may assume that $j' = t' = k' = 0$. Therefore we denote the graph $C(n, j, 0, 0, k, 0, t)$ by $C(n, j, k, t)$. We will refer to this graph by *C-graph*. As we will see in the next section, bipartite I-graphs form a subset of C-graphs. However, these two sets are not equal, an example of a C-graph which is not a bipartite I-graph is $C(3, 1, 1, 0)$.

Regular bipartite graphs with girth at least 6 can be considered as Levi graphs of symmetric configurations. In the last two sections we consider combinatorial configurations which arise from C-graphs.

2 Properties of I-graphs

Proposition 1. *The graph $I(n, j, k)$ is connected if and only if $\gcd(n, j, k) = 1$. If $\gcd(n, j, k) = d > 1$, then the graph $I(n, j, k)$ consists of d copies of $I(n/d, j/d, k/d)$.*

Proof. We use the fact that the covering graph is connected if and only if the local group of any vertex is equal to the whole voltage group, see [13] or [19]. In the case when the voltage group is cyclic and voltages on some spanning tree of the base graph are all 0, the covering graph is connected if and only if the greatest common divisor of non-zero voltages is relatively prime to the order of the group. In our case this means that $I(n, j, k)$ is connected if and only if $\gcd(n, j, k) = 1$. □

Theorem 2. *A connected graph $I(n, j, k)$ is bipartite if and only if n is even and j and k are odd.*

Proof. We know that a graph is bipartite if and only if it does not contain odd cycles. There are three types of cycles in $I(n, j, k)$, cycles with edges only in the outer rim, cycles with the edges only in the inner rim, and cycles with edges in both rims. Cycles of the first type have length $n/\gcd(n, j)$ while cycles of the second type have length $n/\gcd(n, k)$. Throughout the proof, i^a denotes a sequence of a edges on the inner rim, s denotes a spoke, and o^b denotes a sequence of b edges on the outer rim.

Let us assume that $G = I(n, j, k)$ is bipartite. Since all the cycles are even, $n/\gcd(n, j)$ must be even. Thus, n must be even. Both j and k cannot be even, otherwise $I(n, j, k)$ is not connected by Proposition 1. Without loss of generality we may assume that j is odd. We have to prove, that also k is odd. We prove this by contradiction. Suppose that k is even. Then we can find an odd cycle C in G which has the form $C = si^{k'}so^{j'}$, where $k' = \operatorname{lcm}(j, k)/k$ and $j' = \operatorname{lcm}(j, k)/j$. Since j is odd and k is even, j' is even and k' is odd. Therefore, $2 + k' + j'$, the length of C, is odd. Note that in the case when $\operatorname{lcm}(j, k)/j$ is greater than the length of the cycles in the outer rim, $l = n/\gcd(j, n)$, then j' should be taken modulo l. Since we assume that l is even, the parity of j' remains unchanged. The same should be done for k'.

Conversely, suppose n is even, j and k are odd. Then all the cycles on the outer rim and all the cycles on the inner rim are even. Any other cycle can only be of the form $C = i^{k_1}so^{j_1}si^{k_2}so^{j_2}s\ldots$. In order for C to be a cycle, the equality $k_1 \cdot k + j_1 \cdot j + k_2 \cdot k + j_2 \cdot j + \cdots \equiv 0 \pmod{n}$ must hold. The equality can be simplified to $k' \cdot k + j' \cdot j \equiv 0 \pmod{n}$. Since j and k are both odd and n is even, this equation can be fulfilled only if k' and j' are of the same parity. But then C is an even cycle and $I(n, j, k)$ is bipartite. □

Proposition 3. *A bipartite I-graph $I(n, j, k)$ is isomorphic to the C-graph $C(\frac{n}{2}, j, k, \frac{j+k}{2})$.*

Proof. Since $I(n, j, k)$ is bipartite we may assume that n is even and j and k are odd. An isomorphism between $C(\frac{n}{2}, j, k, \frac{j+k}{2})$ and $I(n, j, k)$ is then given by the following rules:

$$u_i \mapsto u_{2i}, \quad u'_i \mapsto u_{2i+j}, \quad v_i \mapsto v_{2i+j}, \quad v'_i \mapsto v_{2i+j+k},$$

for $i = 0, 1, \ldots, \frac{n}{2} - 1$. Here u_i, u'_i, v_i, v'_i on the left sides of the rules denote the vertices of C-graph, which we get from the fibers over u, u', v, v', respectively; see Figure 3. □

Proposition 4. *If $j \neq \pm k$ then $I(n, j, k)$ has a cycle of length 8. If $j = \pm k$ then in $I(n, j, k)$ there exists a cycle of length 4.*

Proof. If $j \neq \pm k$, the walk $u_0u_jv_jv_{j+k}u_{j+k}u_kv_kv_0u_0$ is a cycle since $0, j, j + k$ and k are all different. If $j = \pm k$, then the walk $u_0u_jv_jv_0u_0$ clearly forms a 4-cycle. □

Theorem 5. *Let $j, k < n/2$. The graph $I(n, j, k)$ has girth*

- 3 *if and only if* $n = 3j$ *or* $n = 3k$.
- 4 *if and only if it has girth greater than* 3 *and* $n = 4j$ *or* $n = 4k$ *or* $j = k$.
- 5 *if and only if it has girth greater than* 4 *and* $n = 5j$ *or* $n = 5k$ *or* $2n = 5j$ *or* $2n = 5k$ *or* $2k = j$ *or* $2j = k$ *or* $2k = -j$ *or* $2j = -k$.
- 6 *if and only if it has girth greater than* 5 *and* $n = 6j$ *or* $n = 6k$ *or* $2j = 2k$ *or* $2j = -2k$ *or* $3k = j$ *or* $3k = -j$ *or* $3j = k$ *or* $3j = -k$.
- 7 *if and only if it has girth greater than* 6 *and* $n = 7j$ *or* $n = 7k$ *or* $2n = 7j$ *or* $2n = 7k$ *or* $3n = 7j$ *or* $3n = 7k$ *or* $4k = j$ *or* $4k = -j$ *or* $4j = k$ *or* $4j = -k$ *or* $3j = 2k$ *or* $3j = -2k$ *or* $3k = 2j$ *or* $3k = -2j$.
- 8 *if and only if it has girth greater than* 7.

All the numbers are to be read modulo n.

Proof. Note that the last point follows from Proposition 4. We will denote by i, o and s an arbitrary edge from the inner rim, outer rim and a spoke, respectively. In a graph $I(n, j, k)$ we have 3 different classes of cycles with length at most 7: $ii \dots i$, $oo \dots o$ and $ii \dots isoo \dots os$ which has length at least 4. We will call the latter cycle of type 3. We only give an outline of the proof, which is basically the consideration of all possible cases. If the cycle of type $oo \dots o$ has length say l, the following must hold: $n = la$ for some a and $\gcd(n, j) = a$. Similarly, for the cycle of type $ii \dots i$ to have length l, $n = la$ for some a and $\gcd(n, k) = a$ must hold. We also bear in mind that $j, k < n/2$. The only cycles of type 3 which have length 4 are the ones of the form $isos$, which are possible only if $k = j$. The cycles of type 3 which have length 5 are $iisos$ and $isoos$. They are possible if $2k = j$ or $2k = -j$ in the first case and $2j = k$ or $2j = -k$ in the second case. The proof for girth 6 and 7 is then completed by considering conditions for the cycles of type $iisoos$, $isooos$, $iiisos$, $iiiisos$, $iiisoos$, $iisooos$ and $isoooos$ to exist. □

With the following theorem we give only a partial answer to the problem of distinguishing between two I-graphs. The complete algorithm for checking whether two I-graphs are isomorphic is given at the end of this section.

Theorem 6. *Let n, j, k, and a be positive integers such that $\gcd(n, j, k) = 1$ and $\gcd(n, a) = 1$. Then the graph $I(n, aj, ak)$ is isomorphic to $I(n, j, k)$.*

Proof. Since a and n are relatively prime, the numbers at, $t = 0, \dots, n-1$ are all different modulo n and can be used to label the vertices of $I(n, aj, ak)$; the labels are now to be read modulo an.

Let u_{at}, v_{at}, $t = 0, \dots, n-1$, denote the vertices of the outer rim and the inner rim of $I(n, aj, ak)$, respectively, and x_i, y_i, $i = 0, \dots, n-1$, denote the vertices of the outer rim and the inner rim of $I(n, j, k)$. We define a mapping $\varphi : V(I(n, aj, ak)) \to V(I(n, j, k))$ with

$$\varphi(u_{ta}) = x_t, \quad \varphi(v_{ta}) = y_t, \qquad t = 0, 1, \dots, n-1$$

This is clearly a bijection. It is easy to see that $\varphi(u_{ta}) = u_a$ and $\varphi(v_{ta}) = y_t$ also for $t \geq n$. We must show that φ is also a homomorphism. The edges in $I(n, aj, ak)$ are of the form $u_{ta}u_{ta+aj}$, $u_{ta}v_{ta}$, and $v_{ta}v_{ta+ak}$ and the edges in $I(n, j, k)$ are of the form $x_t x_{t+j}$, $x_t y_t$, and $y_t y_{t+k}$, $t = 0, \dots, n-1$. Since $\varphi(u_{ta}) = x_t$, $\varphi(u_{ta+aj}) = \varphi(u_{(t+j)a}) = x_{t+j}$, $\varphi(v_{ta}) = y_t$, and $\varphi(v_{ta+ak}) = y_{t+k}$, the edges from $I(n, aj, ak)$ map to the edges in $I(n, j, k)$. This finishes the proof. □

Theorem 7. *Let n, j, and k be positive integers such that* $\gcd(n,j,k) = 1$*,* $\gcd(n,j) \neq 1$ *and* $\gcd(n,k) \neq 1$*. Then the graph $I(n,j,k)$ is neither vertex transitive nor edge transitive.*

Proof. Since $\gcd(n,k) \neq \gcd(n,j)$, the edges from the inner rim form cycles of different length than the edges from the outer rim. Therefore there is no automorphism of $I(n,j,k)$ which interchanges the vertices from the inner and the outer rim setwise. Thus, if $I(n,j,k)$ is vertex transitive, there exists an automorphism which maps a cycle from the outer rim to a cycle consisting of the edges of the inner rim, outer rim and spokes. That implies that $I(n,j,k)$ is also edge-transitive.

The proof that no I-graph satisfying the conditions of the theorem is edge-transitive is very similar to the proof from [10] that only seven sporadic examples of the generalized Petersen graphs can be edge transitive. Therefore we will only give the outline of the proof.

We find all possible types of 8-cycles in $I(n,j,k)$ as we did in the proof of Theorem 5 for smaller cycles, and the conditions which must be fulfilled by n, j, and k such that particular types of cycles in $I(n,j,k)$ exist. For each type of cycles we count the number of outer edges, inner edges and spokes lying on such a cycle; given a cycle z in $I(n,j,k)$, we denote by $r(z)$ the number of outer edges, by $s(z)$ the number of spokes and by $t(z)$ the number of inner edges in z. Let Z denote the set of all 8-cycles in $I(n,j,k)$. We define $R = \sum_{z \in Z} r(z)$, $S = \sum_{z \in Z} s(z)$, and $T = \sum_{z \in Z} t(z)$. We then find the numbers R, S, and T for all different choices of n, j, k depending on the types of 8-cycles that $I(n,j,k)$ contains and prove that either $R \neq S$ or $R \neq T$ or $S \neq T$. $\square$

Corollary 8. *A graph $I(n,j,k)$ is a generalized Petersen graph if and only if* $\gcd(n,j) = 1$ *or* $\gcd(n,k) = 1$*. If* $\gcd(n,j) = 1$ *then $I(n,j,k) = G(n,r)$, where r is the solution of the equation* $k \equiv r \cdot j \pmod{n}$*.*

Remark. If $\gcd(j,n) = 1$, the equation $k \equiv r \cdot j \pmod{n}$ is always solvable and the solution r can be obtained by the extended Euclidean algorithm. If $r \geq n/2$ we can take the graph $G(n, n-r)$.

Proof. If $\gcd(j,n) = 1$, the vertices on the outer rim form the cycle $u_0 u_j u_{2j} \ldots\ u_{(n-1)j} u_0$ of length n. We place the vertices of the inner rim in that same order. To get from u_0 to u_k using the edges of the outer rim, we have to take r edges, where r is the solution of the equation $k \equiv r \cdot j \pmod{n}$. Vertex v_0 is then connected to v_k, which is r places away in the reordered graph. The same holds for the other vertices of the inner rim, since all the arithmetic is modulo n. A similar proof can be used in the case $\gcd(n,k) = 1$.

Now suppose that $\gcd(n,j) > 1$ and $\gcd(n,k) > 1$. If for some r the graph $I(n,j,k)$ is isomorphic to the generalized Petersen graph $G(n,r)$, there is an isomorphism of $I(n,j,k)$ to $G(n,r)$ which maps a cycle of length n, consisting of outer edges, inner edges and spokes, to the outer edges of $G(n,r)$. That implies that $I(n,j,k)$ is edge-transitive, a contradiction to Theorem 7. $\square$

The following proposition is partly proved in [18].

Proposition 9. *Let n, r, and s be positive integers such that* $s \not\equiv \pm r \pmod{n}$*. Then $G(n,r)$ is isomorphic to $G(n,s)$ if and only if* $r \cdot s \equiv \pm 1 \pmod{n}$*.*

Remark. If $r \cdot s \equiv \pm 1 \pmod{n}$ then $\gcd(n,r) = \gcd(n,s) = 1$.

Proof. Let u_i, v_i, $i = 0, \ldots, n-1$, denote the vertices of the outer rim and the inner rim of $G(n,r)$, respectively, and x_i, y_i, $i = 0, \ldots, n-1$, denote the vertices of the outer rim and the inner rim of $G(n,s)$.

Let $r \cdot s \equiv \pm 1 \pmod{n}$. Then $\gcd(n, r) = \gcd(n, s) = 1$. The mapping $\varphi : V(G(n, r)) \to V(G(n, s))$ defined by

$$\varphi(u_i) = y_{si}, \quad \varphi(v_i) = x_{si}, \qquad i = 0, 1, \ldots, n-1$$

is an isomorphism from $G(n, r)$ to $G(n, s)$ since it is a bijection and the edges $u_i v_i$, $u_i u_{i+1}$ and $v_i v_{i+r}$, $i = 0, \ldots n-1$, from $G(n, r)$ map to the edges $x_{si} y_{si}$, $y_{si} y_{si+s}$ and $x_{si} x_{si+rs} = x_{si} x_{si\pm 1}$, $i = 0, \ldots n-1$, from $G(n, s)$.

Conversely, suppose $G(n, r)$ is isomorphic to $G(n, s)$. There are only seven different edge transitive generalized Petersen graphs $G(n, r)$ with unique parameters if $r < n/2$, see [10]. All other generalized Petersen graphs are not edge transitive and since $s \not\equiv \pm r \pmod{n}$, the only possible isomorphism between $G(n, r)$ and $G(n, s)$ maps the vertices of the outer rim of $G(n, r)$ to the vertices of the inner rim of $G(n, s)$ and vice versa. This is only possible if the vertices of the inner rim of both $G(n, r)$ and $G(n, s)$ form a cycle. It follows that $\gcd(n, r) = \gcd(n, s) = 1$ and $r \cdot s \equiv \pm 1 \pmod{n}$. $\square$

I-graphs with different parameters can be isomorphic as can be seen from Theorem 6 and Proposition 9. The following theorem, together with Theorem 6 and Proposition 9 provides a simple means to distinguish between different I-graphs.

Theorem 10. *Let n, j, k, j' and k' be positive integers such that $\gcd(j, k) = \gcd(j', k') = 1$, $\gcd(n, j) = \gcd(n, j') \neq 1$ and $\gcd(n, k) = \gcd(n, k') \neq 1$. Then the graph $I(n, j, k)$ is isomorphic to $I(n, j', k')$ if and only if $k \cdot j' \equiv \pm k' \cdot j \pmod{n}$.*

Proof. Let u_i, v_i, $i = 0, \ldots, n-1$, denote the vertices of the outer rim and the inner rim of $I(n, j, k)$, respectively, and let x_i, y_i, $i = 0, \ldots, n-1$, denote the vertices of the outer rim and the inner rim of $I(n, j', k')$.

Suppose $I(n, j, k)$ is isomorphic to $I(n, j', k')$ and let φ be the isomorphism of these two graphs. Let $j_1 = \gcd(n, j) = \gcd(n, j')$ and $k_1 = \gcd(n, k) = \gcd(n, k')$. Then the edges of the outer rim in both graphs form cycles of length n/j_1 and the edges of the inner rim form cycles of length n/k_1. No proper I-graph can be vertex transitive by Theorem 7 and since $n/j_1 \neq n/k_1$, the vertices of the outer rim in $I(n, j, k)$ map to the vertices of the outer rim in $I(n, j', k')$ and the vertices of the inner rim in $I(n, j, k)$ map to the vertices of the inner rim in $I(n, j', k')$. We may assume that $\varphi(u_0) = x_0$ and $\varphi(u_j) = x_{j'}$, otherwise we relabel the vertices of $I(n, j', k')$. The cycle $C_1 = u_0, u_j, u_{2j}, \ldots, u_{(n/j_1 - 1)\cdot j}$ maps to the cycle $C_1' = x_0, x_{j'}, x_{2j'}, \ldots, x_{(n/j_1-1)\cdot j'}$ and to preserve adjacency, the cycle $C_2 = v_0, v_k, v_{2k}, \ldots, v_{(n/k_1-1)\cdot k}$ maps to the cycle $C_2' = y_0, y_{k'}, y_{2k'}, \ldots, y_{(n/k_1-1)\cdot k'}$ (possibly in the opposite direction). Taking k steps along the cycle C_1 we reach u_{kj}, and taking k steps along the cycle C_1' we reach $x_{kj'}$. Therefore $\varphi(u_{kj}) = x_{kj'}$. Taking j steps along the cycle C_2 we reach v_{jk} and taking j steps along the cycle C_2' we reach $y_{\pm jk'}$. Therefore $\varphi(v_{jk}) = y_{\pm jk'}$. Since $\varphi(u_{jk})$ is adjacent to $\varphi(v_{jk})$, they have the same index and $k \cdot j' \equiv \pm j \cdot k' \pmod{n}$.

Conversely, suppose $k \cdot j' \equiv k' \cdot j \pmod{n}$. Given an integer a, the equation $ij + pk \equiv a \pmod{n}$ is solvable since j and k are relatively prime. The solutions are of the form $i = i_0 + tk$ and $p = p_0 - tj$ where $i_0 j + p_0 k = a$. We will show that the mapping $\varphi : I(n, j, k) \to I(n, j', k')$ with $\varphi(u_{ij+pk}) = x_{ij'+pk'}$ and $\varphi(v_{ij+pk}) = y_{ij'+pk'}$ is the desired isomorphism between $I(n, j, k)$ and $I(n, j', k')$.

First, let us show that φ is well-defined. Suppose $a \equiv ij + pk \equiv i_1 j + p_1 k \pmod{n}$ where $i = i_0 + tk$, $p = p_0 - tj$, $i_1 = i_0 + t_1 k$, $p_1 = p_0 - t_1 j$ and $i_0 j + p_0 k = a$. Then $\varphi(u_{ij+pk}) = x_{ij'+pk'} = x_{i_0 j' + p_0 k' + tkj' - tjk'} = x_{i_0 j' + p_0 k'}$ since $k \cdot j' \equiv k' \cdot j \pmod{n}$. Similarly, $\varphi(u_{i_1 j + p_1 k}) = x_{i_0 j' + p_0 k'}$. The proof that $\varphi(v_{ij+pk}) = \varphi(v_{i_1 j + p_1 k})$ is the same.

Clearly, φ is a surjection, because j' and k' are relatively prime and the equation $ij' + pk' \equiv a \pmod{n}$ is then solvable for any a. A surjective mapping between finite sets with equal number of elements is also a bijection.

We must prove that φ is also a homomorphism. The edges of type $u_a v_a$ obviously map to edges. Now let us consider the edges of type $u_a u_{a+j}$, where $a = ij + pk$. Then $\varphi(u_{a+j}) = \varphi(u_{(i+1)j+pk}) = x_{(i+1)j'+pk'} = x_{ij'+pk'+j'}$ is adjacent to $\varphi(u_a) = x_{ij'+pk'}$. The remaining edges are of type $v_a v_{a+k}$, where $a = ij+pk$: $\varphi(v_{a+k}) = \varphi(u_{ij+(p+1)k}) = y_{ij'+pk'+k'}$ is adjacent to $\varphi(v_a) = y_{ij'+pk'}$.

If $k \cdot j' \equiv -k' \cdot j \pmod{n}$, the mapping $\varphi : I(n, j, k) \to I(n, j', k')$ with $\varphi(u_{ij+pk}) = x_{ij'-pk'}$ and $\varphi(v_{ij+pk}) = y_{ij'-pk'}$ is an isomorphism. □

In the case where $n = jk$ or $n = 2jk$, j, k being relatively prime, the following corollary provides us with a simple method for choosing representatives of different isomorphism classes of I-graphs.

Corollary 11. *Let n, j, k be positive integers such that $\gcd(n, j) = j_1 \neq 1$, $\gcd(n, k) = k_1 \neq 1$, $\gcd(j, k) = 1$, and $n = j_1 k_1$ or $n = 2 j_1 k_1$. Then the graph $I(n, j, k)$ is isomorphic to the graph $I(n, j_1, k_1)$*

Algorithm. Algorithm for checking whether the graphs $I(n, j, k)$ and $I(n, j', k')$ are isomorphic.

Input: n, j, k, j', k'
Output: `True` if and only if $I(n, j, k) \cong I(n, j', k')$

1. $d \leftarrow \gcd(n, j, k)$; $d' \leftarrow \gcd(n, j', k')$ (number of connected components)
2. **if** $d \neq d'$ **then return** `False`
3. $n_0 \leftarrow n/d$; $j_0 \leftarrow j/d$; $k_0 \leftarrow k/d$; $j'_0 \leftarrow j'/d$; $k'_0 \leftarrow k'/d$ (reduction to a single connected component)
4. $a \leftarrow \gcd(j_0, k_0)$; $a' \leftarrow \gcd(j'_0, k'_0)$
5. $j_1 \leftarrow j_0/a$; $k_1 \leftarrow k_0/a$; $j'_1 \leftarrow j'_0/a'$; $k'_1 \leftarrow k'_0/a'$ (Theorem 6)
6. $P \leftarrow (\gcd(n_0, j_1) = 1$ **or** $\gcd(n_0, k_1) = 1)$; $P' \leftarrow (\gcd(n_0, j'_1) = 1$ **or** $\gcd(n_0, k'_1) = 1)$
7. **if** $P \not\Leftrightarrow P'$ **then return** `False`
8. **if** P **then** (Generalized Petersen graphs, use Proposition 9)
   - 8.1 let r be the smallest solution of the equation $k_1 \equiv r \cdot j_1 \pmod{n}$; let r' be the smallest solution of the equation $k'_1 \equiv r' \cdot j_1 \pmod{n}$
   - 8.2 **if** $r \equiv \pm r' \pmod{n}$ **or** $r \cdot r' \equiv \pm 1 \pmod{n}$ **then return** `True`
9. **if** $k \cdot j' \equiv k' \cdot j \pmod{n}$ **then return** `True` **else return** `False` (Theorem 10)

3 The automorphism groups of I-graphs

In [10], Frucht, Graver, and Watkins characterized the automorphism groups of the generalized Petersen graphs. In this section, we characterize the automorphism groups of proper I-graphs.

Define mappings ρ, τ with

$$\rho(u_a) = u_{a+1},\ \rho(v_a) = v_{a+1} \quad \text{and} \quad \tau(u_a) = u_{-a},\ \tau(v_a) = v_{-a},$$

which are clearly automorphisms of $I(n, j, k)$; ρ can be viewed as a rotation and τ as a reflection of the vertex set. The automorphism group of an $I(n, j, k)$ therefore contains as a subgroup the dihedral group D_n with $2n$ elements, generated by ρ and τ.

Throughout this section, the numbers n, j, k, j_1, and k_1 will be positive integers such that $\gcd(n, j) = j_1 \neq 1$, $\gcd(n, k) = k_1 \neq 1$, $\gcd(j, k) = 1$. According to Theorem 6, Proposition 1, and Corollary 8, with such parameters all different connected proper I-graphs $I(n, j, k)$ will be considered. Since j and k are relatively prime, the equation $ij + pk \equiv a \pmod n$ can be solved for any integer a and we will use the numbers $ij + pk$ to label the vertices of $I(n, j, k)$. By x_{ij+pk} we will denote a vertex of the graph $I(n, j, k)$, which can be either u_{ij+pk} or v_{ij+pk}.

The following Proposition shows that apart from the usual rotations and reflections of the whole vertex set, some I-graphs also admit automorphisms, which rotate the cycles on the outer rim and reflect the cycles on the inner rim or rotate the cycles on the inner rim and reflect the cycles on the outer rim.

Proposition 12. *Let $n = j_1k_1$ or $n = 2j_1k_1$. Then the graph $I(n, j, k)$ has automorphisms φ and ψ, defined by*

$$\varphi(x_{ij+pk}) = x_{-ij+pk} \quad \textit{and} \quad \psi(x_{ij+pk}) = x_{ij-pk}$$

for $i, p \in \mathbb{Z}$.

Proof. The proof that φ and ψ are well-defined and bijective is the same as in the proof of Theorem 10, the only difference is that we use the fact that $2kj \equiv 0 \pmod n$. Now it is easy to verify that φ and ψ are also homomorphisms. ☐

Theorem 13. *If $n/(j_1k_1) > 2$, then the automorphism group of the graph $I(n, j, k)$ is isomorphic to the dihedral group D_n otherwise it is isomorphic to the group Γ, defined by*

$$\Gamma = \langle \rho, \varphi, \psi \mid \rho^n = \varphi^2 = \psi^2 = 1, \varphi\psi = \psi\varphi, \rho\varphi = \varphi\rho^a, \rho\psi = \psi\rho^b \rangle, \tag{1}$$

where $b \equiv -a \pmod n$ and a is the inverse in $\mathbb{Z}_n$ of the element $sk - tj$ such that $sk + tj \equiv 1 \pmod n$.

Proof. By Theorem 7, $I(n, j, k)$ is not vertex transitive and therefore any automorphism of $I(n, j, k)$ sends cycles containing only vertices of the outer (inner) rim to cycles containing only vertices of the outer (inner) rim.

If a vertex u_a on the cycle $C_1 = u_a, u_{a+j}, \ldots, u_{a+(n/j_1-1)j}$ is fixed, the cycle is either fixed or it is reflected to $u_a, u_{a+(n/j_1-1)j}, u_{a+(n/j_1-2)j}, \ldots, u_{a+j}$. Any automorphism that fixes u_a also fixes v_a, since $I(n, j, k)$ is not vertex-transitive.

Therefore the image of two cycles, one consisting of outer edges and the other consisting of inner edges, determines any automorphism of $I(n, j, k)$. This can be seen as follows. Without loss of generality we may assume that the images of cycles $u_0, u_j, \ldots, u_{(n/j_1-1)j}$ and $v_0, v_k, \ldots, v_{(n/k_1-1)k}$ are known and that u_0 maps to u_0. Since the image of the eight-cycle $u_0v_0v_ku_ku_{k+j}v_{k+j}v_ju_j$ is then uniquely determined, also the image of the cycle $u_k, u_{k+j}, \ldots, u_{k+(n/j_1-1)j}$ is determined. Now we have two cycles consisting of the outer-edges, each of them having at least one vertex adjacent to any cycle consisting of inner-edges, and these two vertices are not $n/2$ places apart. That means that the images of all inner cycles are determined, which determines the whole automorphism.

Let $C_2 = v_a, v_{a+k}, \ldots, v_{a+(n/k_1-1)k}$. Then there are exactly $n/(j_1k_1)$ vertices of C_1 adjacent to the vertices of C_2.

When $n/(j_1k_1) > 2$, C_1 and C_2 have at least three pairs of adjacent vertices and they are either both fixed or both reflected by the same automorphism of $I(n,j,k)$. Therefore the stabilizer of u_a has only 2 elements; namely, the identity and the reflection, which fixes u_a and v_a. Therefore the automorphism group of $I(n,j,k)$ has $2n$ elements and must be the group D_n itself.

If $n/(j_1k_1) \leq 2$, then C_1 and C_2 have at most two pairs of adjacent vertices and one of them can be fixed while the other one is reflected by the same automorphism of $I(n,j,k)$. Therefore the stabilizer of the vertex u_a has at most 4 elements and the automorphism group of $I(n,j,k)$ has at most $4n$ elements.

Let φ, ψ be as in Proposition 12 and ρ, τ be defined as in the beginning of this section. It is easy to verify the relations $\rho^n = \varphi^2 = \psi^2 = 1$ and $\tau = \varphi\psi = \psi\varphi$.

Now we will show that $\rho\varphi = \varphi\rho^a$. Since j and k are relatively prime, there exist s and t such that $sk + tj \equiv 1 \pmod{n}$. If $sk+tj$ is relatively prime to n and $n = j_1k_1$ or $n = 2j_1k_1$, then also $sk - tj$ is relatively prime to n and we can define a such that $a(sk - tj) \equiv 1 \pmod{n}$. Let x_{ij+pk} be a vertex of $I(n,j,k)$. Then

$$\rho(\varphi(x_{ij+pk})) = x_{-ij+pk+1}$$

and

$$\varphi(\rho^a(x_{ij+pk})) = \varphi(x_{ij+pk+a(sk+tj)}) = x_{-(i+as)j+(p+as)k} = x_{-ij+pk+a(sk-tj)}.$$

The desired equality $\rho\varphi = \varphi\rho^a$ holds, because $a(sk - tj) \equiv 1 \pmod{n}$. The proof that $\rho\psi = \psi\rho^b$, where $b \equiv -a \pmod{n}$ is similar.

Since Γ contains $\tau = \varphi\psi$, the dihedral group D_n is contained in Γ as a proper subgroup, and Γ must have at least $4n$ elements. Therefore it is the whole automorphism group of $I(n,j,k)$. $\square$

Remark. For the generators of Γ we can also take $\rho, \tau = \varphi\psi$ (reflection), and φ; now

$$\Gamma = \langle \rho, \tau, \varphi \mid \rho^n = \tau^2 = \varphi^2 = 1, \rho\tau\rho = \tau, \varphi\tau\varphi = \tau, \varphi\rho\varphi = \rho^a \rangle .$$

From this presentation it is evident that D_n, generated by ρ and τ, is a subgroup of Γ and that Γ is actually a semidirect product of D_n and C_2.

4 Configurations

An *incidence structure* is a triple $(P, \mathcal{B}, I)$ where P denotes the set of *points*, $\mathcal{B}$ the set of *blocks*, and $I \subseteq P \times \mathcal{B}$ is the incidence relation. If $(p, B) \in I$ we say that the point p and the block B are *incident*. A *symmetric combinatorial configuration* (v_r) is an incidence structure of v points and v blocks, called *lines*, such that r lines are incident with each point, r points are incident with each line, and two lines meet in at most one point. In the case of configurations we use geometric expressions and call blocks *lines*, say that the point $p \in P$ *lies* on the line $B \in \mathcal{B}$ if $(p, B) \in I$, etc.

Incidence structures and hence combinatorial configurations are closely related to graphs. Let $G(\mathcal{C})$ be a bipartite graph with one set of the bipartition representing points of the incidence structure $\mathcal{C}$, the other set of the bipartition representing lines of $\mathcal{C}$, and with an edge joining two vertices if and only if the corresponding point and line are incident in $\mathcal{C}$. The graph $G(\mathcal{C})$ is called *incidence graph* or *Levi graph* of the incidence structure $\mathcal{C}$. The following proposition characterizes symmetric configurations in terms of their Levi graphs.

Proposition 14. *An incidence structure is a* (v_r) *configuration if and only if its Levi graph is* r*-regular with girth at least* 6*.*

For the proof and more about configurations and graphs see [8, 11, 12]. For enumeration results about (v_3) configurations the reader is referred to [1].

Example 1. The generalized Petersen graph $G(10, 3)$ is bipartite graph with girth 6 (by Theorem 5). Thus, it is an incidence graph of a (10_3) configuration. This is the well-known Desargues configuration.

The use of geometric expressions shows the strong connection between combinatorial and geometric configurations.

A *geometric* (v_r) *configuration* is a set of v points and v lines in the Euclidean plane, such that precisely r of the lines pass through each of the points, and each of the lines passes through precisely r points. Clearly, each geometric configuration determines a combinatorial configuration, while the reverse is not always true. For example, it is well known that the only combinatorial (7_3) configuration (projective plane of order 2) cannot be realized with points and lines in the Euclidean plane.

The problem of realization of combinatorial (v_3) configurations in the Euclidean plane has long history and dates back to H. Schroeter [16] in 1888. The most intriguing result is due to E. Steinitz which (roughly) says that *every connected* (v_3) *configuration can be drawn in the plane with at most one curved line.* Later Steinitz conjectured [17] that every (v_3) with $v > 10$ can be realized as a geometric configuration. This has been contradicted in [9] by an example with $v = 16$. The counterexample from [9] can easily be extended to $v > 16$ and, moreover, to contradict the Steinitz theorem (see also [4]). The counterexamples base on the fact that we do not allow additional incidences between points and lines. By the term *additional incidence* we call a situation where a point and a line are incident in the realization in the plane, but not in the combinatorial configuration we attempt to realize geometrically.

Here we will be interested in the realizations which have a certain amount of symmetry. A geometric (v_3) configuration is said to be *astral* if both points and lines form two orbits under the group of (isometric) symmetries. This is the largest amount of symmetry any geometric (v_3) configuration can possess. In this paper we limit ourselves only to astral configurations with *cyclic* symmetries, although there exist astral configurations with dihedral symmetries. Figure 5 shows an example of a cyclic astral configuration.

We denote an astral configuration with cyclic symmetry by $\mathcal{A}(n, j, k, t)$, $1 \leq j, k < \frac{n}{2}$, $0 \leq t < n$, where n is the order of cyclic automorphism and the meaning of parameters j, k, t is evident from Figure 4; j can be understood as a "span" that a line from the first orbit makes between the points of the first orbit, k as a span that a line from the second orbit makes between the points of the second orbit, and t as a "shift" that a line from second orbit makes on the point of the first orbit.

Theorem 15. *The Levi graph of a cyclic astral* (v_3) *configuration is a C-graph. More precisely, the Levi graph of* $\mathcal{A}(n, j, k, t)$ *is* $C(n, j, k, t)$*.*

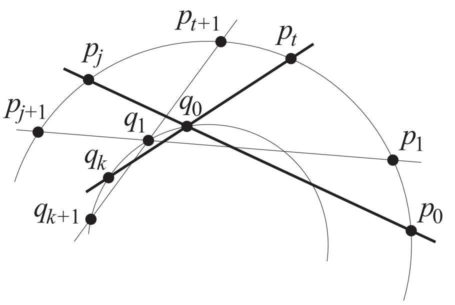


Figure 4: Construction of an astral configuration.

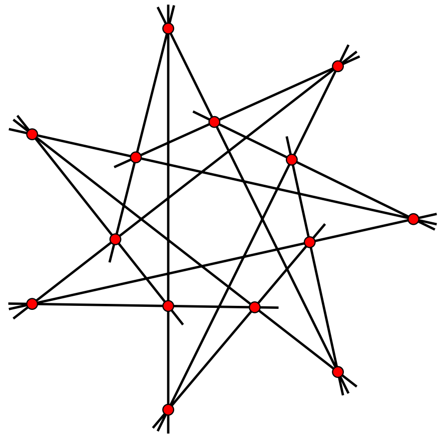


Figure 5: Astral (14_3) configuration $\mathcal{A}(7,1,3,2)$.

Proof. Let us denote points of $\mathcal{A}(n,j,k,t)$ by p_i (points in the first orbit) and q_i (points in the second orbit), see Figure 4, and lines by P_i and Q_i, $P_i = \{p_i, p_{i+j}, q_i\}$, $Q_i = \{q_i, q_{i+k}, q_{i+t}\}$. The isomorphism between Levi graph of $\mathcal{A}(n,j,k,t)$ and $C(n,j,k,t)$ is

$$p_i \mapsto u_i, \quad q_i \mapsto v_i, \quad P_i \mapsto u'_i, \quad Q_i \mapsto v'_i,$$

$i = 1, 2, \ldots, n$. The vertices of the C-graph $C(n,j,k,t)$ are denoted as in Figure 3. □

Now, we can reverse the question: Which C-graphs are Levi graphs of some astral configuration? By Proposition 14 such C-graphs must have girth at least 6.

Proposition 16. *C-graph $C(n,j,k,t)$ with $n > 2$, $0 < j,k < n/2$, $0 \le t < n$ has girth at least 6 if and only if the following inequalities hold:*

$$j + k \ne t, \quad j \ne t, \quad k \ne t, \quad t \ne 0.$$

Proof. If we consider $C(n,j,k,t)$ as a $\mathbb{Z}_n$ covering graph over the graph $G(j,0,0,k,0,t)$ shown in Figure 3, then we get a 4-cycle in $C(n,j,k,t)$ if and only if one of the four 4-cycles through u, u', v, v' lifts to a 4-cycle. This happens precisely when the voltages on the edges sum to 0; the four cases are: $t = 0$ or $j + k - t = 0$ or $j - t = 0$ or $k - t = 0$. The parallel edges cannot lift to a 2- or 4-cycle, since the voltages on them are 0 and j (or k) and we assume that $0 < j,k < n/2$. □

Hence, if integers n, j, k, t satisfy the conditions of Proposition 16, C-graph $C(n,j,k,t)$ determines a pair of combinatorially dual configurations. In fact, since there always exists an automorphism of C-graph which interchanges black and white vertices, it determines only one combinatorially self dual (self polar) configuration.

Proposition 17. *Each combinatorial astral configuration is (combinatorially) self polar.*

Proof. We need to prove that there exists an automorphism of order 2 of the C-graph $C(n,j,k,t)$ which interchanges black and white vertices. This automorphism is

$$u_i \mapsto u'_{-i}, \quad u'_i \mapsto u_{-i}, \quad v_i \mapsto v'_{-i-t}, \quad v'_i \mapsto v_{-i-t}$$

(according to the notation of vertices shown in Figure 3). □

We will call configurations arising from C-graphs *combinatorial astral configurations* and denote them by $C\mathcal{A}(n,j,k,t)$. Now, the question is whether a given combinatorial astral configuration can be realized as a (geometric) astral configuration. The necessary condition is given in the following Theorem.

Theorem 18. *If the combinatorial astral configuration $C\mathcal{A}(n,j,k,t)$ can be realized in the plane as an astral configuration $\mathcal{A}(n,j,k,t)$, then there exist real roots of the quadratic equation*

$$\begin{aligned} 0 = \cos\tfrac{k\pi}{n}(\cos\tfrac{2j\pi}{n}-1)\,x^2 + \\ \big(\cos\tfrac{k\pi}{n}(1-\cos\tfrac{2j\pi}{n}) - \sin\tfrac{(j+k-2t)\pi}{n}\sin\tfrac{j\pi}{n}\big)\,x + \sin\tfrac{(k-t)\pi}{n}\sin\tfrac{t\pi}{n}\,. \end{aligned} \tag{2}$$

Proof. We may assume that the cyclic automorphism acts as a rotation by $\frac{2\pi}{n}$ around $(0,0)$ and that the coordinates of p_0, see Figure 4, are $(1,0)$. Point q_0 must lie on a line through p_0 and p_j. Hence its coordinates are

$$\big(1 - x + \cos\tfrac{2j\pi}{n}, x\sin\tfrac{2j\pi}{n}\big)$$

with x being the distance between q_0 and p_0 relative to the length of the line segment between p_0 and p_j. The points q_0, q_k, and q_t must be collinear, which gives the equation (2). □

Remark. The quadratic equation (2) has a real solution if and only if its discriminant is not negative. This is true when

$$\sin\tfrac{(k-t)\pi}{n} \geq -\frac{\big(\sin\frac{(j+k-2t)\pi}{n} - 2\cos\frac{k\pi}{n}\sin\frac{j\pi}{n}\big)^2}{8\cos\frac{k\pi}{n}\sin\frac{t\pi}{n}}\,. \tag{3}$$

Note that the right side of the above inequality is always non-positive.

It turns out that it is possible that the solutions of (2) give a realization of the corresponding configuration with additional incidences. An example is shown in Figure 6. The following theorem tells when this happens.

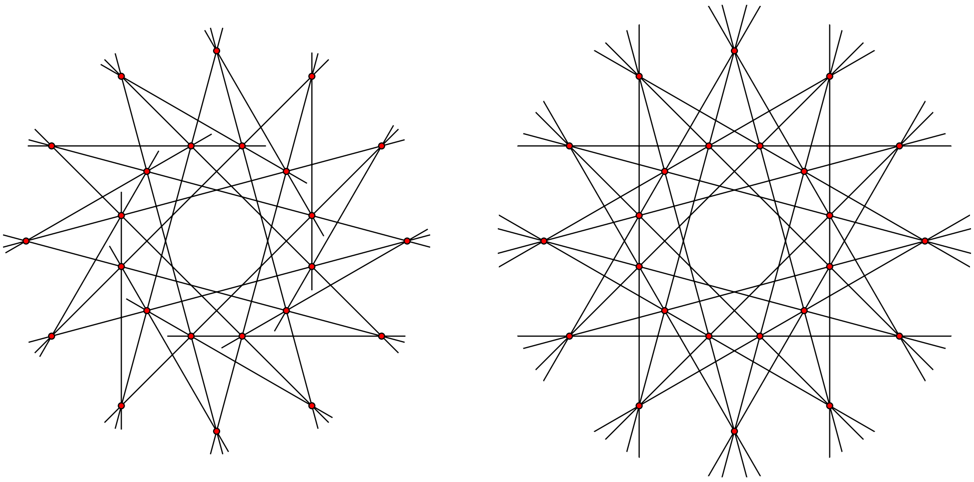
Theorem 19. *The combinatorial astral configuration $C\mathcal{A}(n,j,k,t)$ with parameters $n>4$, $0<j,k<n/2$, $0\leq t<n$ satisfying Equation* (2) *is realizable in the plane as astral configuration $\mathcal{A}(n,j,k,t)$ (i.e.* without *additional incidences), if there does not exist an integer l, $0\leq l<n/2$ such that*

$$\cos\tfrac{j\pi}{n}\cos\tfrac{k\pi}{n} = \cos\tfrac{l\pi}{n}\cos\tfrac{(k+j+l-2t)\pi}{n} \tag{4}$$

holds.

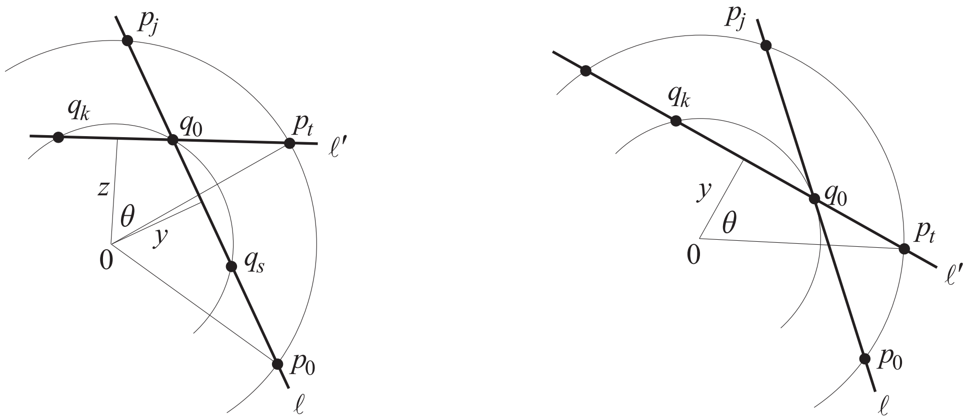
Proof. The two different situations where additional incidences occur are shown in Figures 7(a) and 7(b). In the first case, see Figure 7(a), we assume that line ℓ contains four configuration points, where p_0, p_j and q_0 are combinatorially incident with it and q_s is not. Next, let us assume that the radius of the outer orbit equals 1 and the radius of the inner orbit equals $R<1$. If y denotes the distance between ℓ and the origin, then we can express it either as $y=\cos\frac{j\pi}{n}$ or as $y=R\cos\frac{(n-s)\pi}{n}$. This gives $R=\big(\cos\frac{j\pi}{n}\big)/\big(\cos\frac{(n-s)\pi}{n}\big)$. Similarly, if z denotes the distance between ℓ' and the origin, then $z=R\cos\frac{k\pi}{n}$ and $z=\cos\theta$ where θ is the angle between p_t and midpoint of ℓ'. Hence, $R=\cos\theta/\cos\frac{k\pi}{n}$. Expressing θ from the equation $\theta+\frac{2t\pi}{n}=\frac{j\pi}{n}+\frac{(n-s)\pi}{n}+\frac{k\pi}{n}$ and taking $l=n-s$, $(l\geq 1)$, we obtain (4) from the two expressions for R.

The second case, see Figure 7(b), can be handled similarly: $R=\cos\frac{j\pi}{n}$ (radius of the second orbit), $R\cos\frac{k\pi}{n}=y=\cos\theta$, $\theta+\frac{2t\pi}{n}=\frac{j\pi}{n}+\frac{k\pi}{n}$. From this facts (4) follows for $l=0$. □



(a) (b)

Figure 6: "Astral" configuration $\mathcal{A}(12, 1, 5, 7)$ with additional incidences (a). If we extend the lines of the configuration, it is evident that $\mathcal{A}(12, 1, 5, 7)$ is a subconfiguration of some astral (24_4) configuration (b).



(a) (b)

Figure 7: Situations where additional incidences occur in $\mathcal{A}(n, j, k, t)$ (proof of Theorem 19).

Remarks.

1. When looking for a suitable value for $l > 0$ we consider integer solutions of the equation
$$\cos \tfrac{j\pi}{n} \cos \tfrac{k\pi}{n} = \cos \tfrac{l\pi}{n} \cos \tfrac{m\pi}{n}. \tag{5}$$
This kind of problems were studied, for example, in [7] and [15]. Integer solutions of this equation also imply (with some additional conditions) the existence of some astral (v_4) configuration, see [3]. See [2] for more on astral (v_4) configurations.

2. For $l = 0$ we consider integer solutions of
$$\cos \tfrac{j\pi}{n} \cos \tfrac{k\pi}{n} = \cos \tfrac{m\pi}{n}.$$
According to [7], the only solution that would come into question for a connected configuration is $n = 12$, $j = k = 3$, $m = 4$. See Example 4 below.

3. If $\mathcal{C} = \mathcal{A}(n, j, k, t)$ possesses additional incidences and l from Theorem 19 is greater than 0, then $\mathcal{C}$ is a subconfiguration of some astral (v_4) configuration. If $l = 0$, then this is not the case.

Example 2. The smallest example of a combinatorial astral configuration is, according to Proposition 16, $C\mathcal{A}(5, 2, 2, 1)$. It is also realizable as the astral configuration $\mathcal{A}(5, 2, 2, 1)$. See [3] or [9] for its picture.

Example 3. The configuration $\mathcal{A}(12, 1, 5, 7)$, see Figure 6, has additional incidences. This follows from Theorem 19. The corresponding value for l is 4 (other values are $n = 12$, $j = 5$, $k = 1$, $t = 7$). Here, as in all cases where $l > 0$, the considered configuration is a subconfiguration of some astral (v_4) configuration.

Example 4. The configuration $\mathcal{A}(12, 3, 3, 1)$ also contains additional incidences. But in contrast to the configuration from the previous example, it is not a subconfiguration of some astral (v_4) configuration; now we have $l = 0$.

Bipartite I-graphs are also C-graphs. Since we lose one parameter, Proposition 16 and Theorem 18 can be written in a simplified form for this family of graphs. Combinatorial configurations arising from $I(n, j, k)$ will be denoted by $I\mathcal{A}(n, j, k)$. Let us rewrite the two theorems for I-graphs.

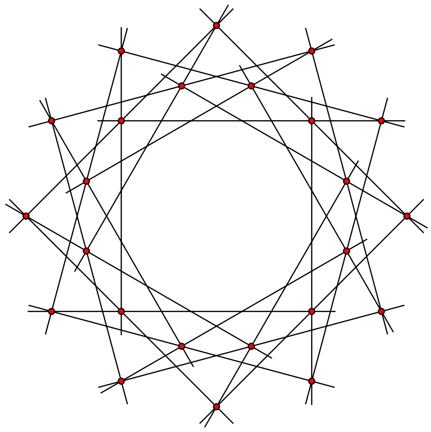
Proposition 20. *Let $n \geq 8$ be an even integer, $1 \leq j \leq k < \frac{n}{2}$ two odd integers, and $j \neq k$, $4j \neq n$, $4k \neq n$. Then there exists a combinatorial astral (n_3) configuration $I\mathcal{A}(n, j, k)$ which is isomorphic to $C\mathcal{A}\big(\frac{n}{2}, j, k, \frac{j+k}{2}\big)$.*

Proof. The result follows from Theorem 2, Theorem 5 and Proposition 14. ☐

Theorem 21. *If the combinatorial configuration $I\mathcal{A}(n, j, k)$ is realizable as astral configuration $\mathcal{A}\big(\frac{n}{2}, (j, k), \frac{j+k}{2}\big)$, then*
$$1 \leq j, k < \tfrac{n}{4} \quad \text{or} \quad \tfrac{n}{4} < j, k < \tfrac{n}{2}.$$

Proof. The result follows from Theorem 18. The quadratic equation (2) has in this case real solutions precisely when $1 \leq j, k < \frac{n}{4}$ or $\frac{n}{4} < j, k < \frac{n}{2}$ (the condition (3) reduces to $\cos \frac{2k\pi}{n} \cos \frac{2j\pi}{n} \geq 0$). ☐

Example 5. The smallest $I\mathcal{A}(n, j, k)$ configuration satisfying conditions of Theorem 21 (and those of Theorem 19) is a combinatorial (14_3) configuration $I\mathcal{A}(14, 1, 3)$ which can be realized as geometric configuration $\mathcal{A}(7, 1, 3, 2)$. See Figure 5.



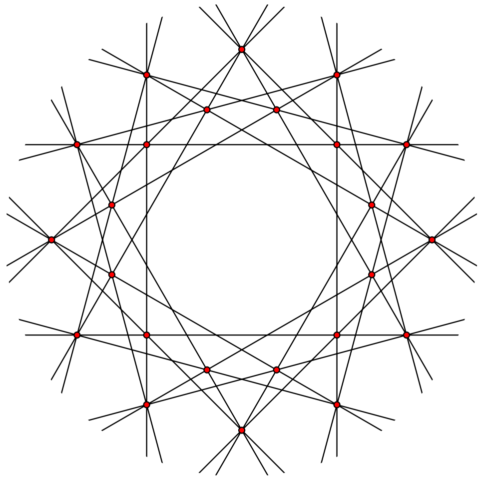


Figure 8: "Astral" configuration $\mathcal{A}(12, 3, 3, 1)$ with additional incidences (a). This is evident if we extend the lines (b).

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