A solution to the Subbarao relation

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Abstract. In [1], Subbarao notes that the relation $n \cdot \sigma(n) = 2 \bmod \phi(n)$ holds for prime values of n, and also for $n = 4, 6, 22$. The existence of other composite solutions was not known. In a closely-related problem in [3], Carlos Rivera and Jud McCranie posed questions about the function $z(n) = \phi(n) + \sigma(n) - 2n$. A solution to the Rivera/McCranie problem is presented, then the Subbarao question is completely solved using properties of this function $z(n)$.

1 Definitions

Definition 1. *We define the function $z(n)$ on positive integers n by*

$$z(n) = \phi(n) + \sigma(n) - 2n \tag{1}$$

2 Solution of Carlos Rivera's Puzzle 76

Theorem 1. *$z(n) \geq 0$, with equality only holding if $n = 1$ or n is prime.*

Proof. We first note the following relations which follow directly from the definitions of $\phi(n)$ and $\sigma(n)$:

$$\sigma(n) = \sum_{d|n} d \tag{2}$$

$$n = \sum_{d|n} \phi(d) \tag{3}$$

By applying the Möbius inversion formula to (3) we have

$$\phi(n) = \sum_{d|n} \mu(n/d) \cdot d \tag{4}$$

We note both (2) and (4) include a term for $d = n$ that evaluates to n, and so we have the formula

$$\begin{aligned} z(n) &= (\phi(n) - n) + (\sigma(n) - n) \\ &= \sum_{d|n}^{d<n} \mu(n/d) \cdot d + \sum_{d|n}^{d<n} d \\ &= \sum_{d|n}^{d<n} (\mu(n/d) + 1) \cdot d \end{aligned} \tag{5}$$

Since each term in (5) is non-negative we conclude $z(n) \geq 0$. For equality to hold, either the sum must be empty, or each term must equal zero. An empty sum only occurs in the case $n = 1$, otherwise we require

$$\mu(n/d) = -1 \;\forall\; d|n,\; d < n \tag{6}$$

In particular, this means n cannot have two or more distinct prime factors. If n had distinct prime factors p and q, choose $d = n/pq$ and we have a contradiction with (6). Hence n must be a power of a prime.

Similarly, if $n = p^r$ for some $r > 1$, choose $d = 1$ and again we have a contradiction with (6).

Finally it is easy to demonstrate that, if n is prime, $\phi(n) = n - 1$ and $\sigma(n) = n + 1$.

Hence $z(n) = 0$ if and only if $n = 1$, or n is prime. □

3 The Subbarao relation

Theorem 2. *The Subbarao relation*

$$n \cdot \sigma(n) = 2 \; mod \; \phi(n) \tag{7}$$

has no composite solutions except for $n = 4, 6, 22$.

Proof. We have, from (1),

$$\sigma(n) = 2n + z(n) \bmod \phi(n)$$

and thus

$$n \cdot \sigma(n) = n(2n + z(n)) \bmod \phi(n)$$

Let us assume n is composite and (7) is true. Then

$$2n^2 + nz(n) - k\phi(n) = 2, \tag{8}$$

for some integer k. The proof now proceeds by cases.

Case I: If n has a repeated prime factor p^r for some $r > 1$, then p^{r-1} divides the left side of (8). Hence p^{r-1} must divide 2, so p must be 2, and r must be 2. $n = 4$ is a known solution. If $n = 4m$ for some odd integer $m > 1$, some prime factor of m will contribute an additional factor 2 to $\phi(n)$, so the left-hand side is divisible by 4 while the right-hand side is not, a contradiction.

Case II: Suppose now $n = 2p$ for some odd prime p. We note that $z(n) = 2$, and considering (8) modulo $(p - 1)$ gives

$$2.2^2 + 2.2 = 2 \bmod (p - 1).$$

Hence $(p - 1)$ divides 10, producing the solutions $p = 3$ and $p = 11$, corrsponding to the known composite solutions $n = 6$ and $n = 22$.

Case III: Suppose now $n = 2m$ for some odd composite number m with no repeated factor. We note that in (5), none of the μ terms can be zero, hence every term in the sum is even. Furthermore each prime factor of n contributes at least one factor 2 to $\phi(n)$. Hence 4 divides every term on the left-hand side of (8), a contradiction, and there are no solutions n of this form.

Case IV: Finally assume n is a product of distinct odd primes. As above, we see $z(n)$ is even, but now show that $z(n)$ is not divisible by 4. Let us define $f(x)$ as the number of prime factors of x. Then (5) becomes

$$z(n)/2 = \sum_{d|n, f(n/d) \ is \ even}^{d<n} d \tag{9}$$

We note every term in the sum in (9) is odd. We also note the number of terms in this sum is

$$\sum_{r=1}^{2r<=f(n)} \binom{f(n)}{2r} = 2^{f(n)-1} - 1,$$

each choice of $2r$ factors denoting all combinations of factors of n/d. Since $f(n) > 1$ in this case, we note the number of terms is therefore odd, hence $z(n)/2$ is an odd number. As before each prime factor of n contributes a factor of 2 to $\phi(n)$. Since n is also odd, we have

$$\begin{aligned} 2n^2 &= 2 \bmod 4, \\ nz(n) &= 2 \bmod 4, \\ \phi(n) &= 0 \bmod 4 \end{aligned}$$

and the left-hand side of (8) is again divisible by 4, a contradiction and thus there are no solutions n of this form.

It is immediately verified the Subbarao relation holds for all primes. Summarizing, the only composite numbers satisfying the Subbarao relation are $n = 4, 6, 22$. □

References

[1] Richard K. Guy, "Unsolved Problems In Number Theory", B37, p.92. (Springer-Verlag, New York, 1994).

[2] E.Bach; J.Shallit: "Algorithmic Number Theory". (Foundation of Computer Science Series, MIT Press, 1996).

[3] C. Rivera, J. McCranie: "Puzzle 76. z(n)=sigma(n)+phi(n)-2n", http://www.sci.net.mx/˜crivera/puzzles/puzz_076.htm