The Queens Separation Problem

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Abstract

We define a *legal placement* of Queens to be any placement in which any two attacking Queens can be separated by a Pawn. The *Queens separation number* is defined to be equal to the minimum number of Pawns which can separate some legal placement of m Queens on an order n chess board. We prove that $n+1$ Queens can be separated by 1 Pawn and conjecture that $n+k$ Queens can be separated by k Pawns for large enough n. We also provide some results on the separation number of other chess pieces.

Keywords: n-Queens problem, Queens separation

1 n-Queens and Queens separation

According to traditional chess rules, a Queen may move any number of squares horizontally, vertically, or diagonally so long as no other piece lies in its path. Questions regarding various placements of multiple Queens on chessboards were first posed in the mid 19th century. In 1848, Max Bezzel described the problem of placing eight Queens on an 8×8 board so that no two Queens attack each other [1]. Studies of this and related problems, such as the fewest number of Queens needed to attack or occupy every square of $n \times n$ rectangular and toroidal boards, can be found in [3, 5, 6].

The problem and its variants serve as models of backtracking programming techniques and are related to mathematical topics including domination in graphs, magic squares, and integer programming among others, while solutions involving neural nets, parallel processing, and genetic algorithms have appeared in the literature. A collection of some references to the n-Queens problem can be found in [7]. The question of the number of different solutions to the n-Queens problem had been considered as well [4]. A *fundamental solution* is a class of solutions such that all the members of the class are simply rotations or reflections of one another.

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In January 2004, the *Chess Variant Pages* [2] proposed a variation of the traditional 8-Queens problem. The new problem, posed as part of a contest on the site, was to place nine Queens on an 8×8 board by using the least number of Pawns possible in order to block all Queens that would otherwise attack each other. An example in which three Pawns were needed to separate the nine Queens was provided. Our solution for the problem that requires only one Pawn immediately suggests a generalization to boards of arbitrary order n with $n + k$ Queens, where $k \geq 1$ is an integer.

We define a *legal placement* of Queens to be any placement in which any two attacking Queens can be separated by a Pawn. The *Queens separation number* $s_Q(m, n)$ is defined to be the minimum number of Pawns which can separate some legal placement of m Queens on an $n \times n$ board. If $m > \beta(K_n)$, the independence number on the $n \times n$ Kings graph, then $s_Q(m, n)$ is undefined since any arrangement of m Queens will have at least two Queens on physically adjacent squares.

Notice that for $m \leq \beta(K_n)$, $0 \leq s_Q(m, n) \leq \mathtt{min}(8m, n^2 - \beta(K_n))$. Equality is obtained in the lower bound if $m \leq n$ and $n \geq 4$. To see the upper bound, either surround each Queen with eight Pawns or choose a set of vertices that would be maximally independent on the Kings graph, place m Queens in that set, and place Pawns on every vertex outside that set.

It can be seen that $s_Q(4, 3) = 5$ and $s_Q(6, 5) = 3$ but $s_Q(5, 4)$ is undefined since $5 > \beta(K_4)$. We prove that $s_Q(n + 1, n) = 1$ for $n > 5$ and conjecture that $s_Q(n + k, n) = k$ for large enough n.

2 An extra Queen

The basic notion for placing an extra Queen and a single Pawn on a board is to begin with a known n-Queens solution and add extra rows, columns, Queens, and a Pawn to get a solution to the Queens separation problem on a larger board.

Begin with an $(n + 2) \times (n + 2)$ board, where $n \geq 4$. Number the rows $-1, 0, \ldots, n$ and the columns $-2, -1, 0, \ldots, n-1$. The known n-Queens solution will be in the subsquare with rows and columns $0, \ldots, n - 1$.

For a board of order 5, a computer search shows that 3 Pawns are necessary in order to permit the placement of 6 Queens. However, for boards of order $n \geq 6$, it can be seen that 1 Pawn suffices to allow the placement of $n + 1$ Queens.

Theorem 1 *For* $n \geq 6$, $s_Q(n + 1, n) = 1$.

Proof Sketch. There are four patterns and two special cases to consider. The proof that each pattern holds involves elementary but tedious calculations which are shown in detail in the appendix. Consider an $(n+2) \times (n+2)$ board labeled as in Figure 1.

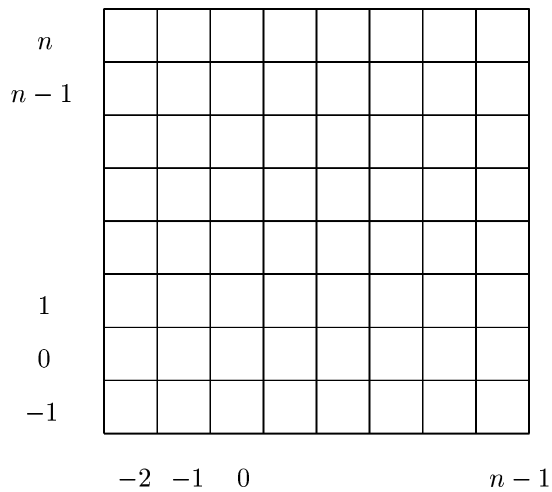

Figure 1: An $(n+2) \times (n+2)$ board.

Pattern I. Let n be even but $n \not\equiv 2 \mod 6$.
Solution: Place the Pawn at $(n/2-1,-1)$ and the Queens at $(n/2-1,-2)$, $(n,-1)$, $(-1,-1)$, $(2i+1,i)$ for $0 \leq i < n/2$, and $(2i-n,i)$ for $n/2 \leq i < n$.

Pattern II. Let n be even but $n \not\equiv 0 \mod 6$.
Solution: If $n = 8$, let $w = 4$. If $n = 10$, let $w = 7$. If $n > 10$, let $w = \lfloor (n+1)/4 \rfloor$. Place the Pawn at $(w,-1)$ and the Queens at $(w,-2)$, $(-1,-1)$, $(n,-1)$, $((n/2+2i-1) \mod n, i)$ for $0 \leq i < n/2$, and $((n/2+2i+2) \mod n, i)$ for $n/2 \leq i < n$.

Pattern II leaves an empty main diagonal, so it can be used to obtain solutions for some odd order chessboards.

Pattern III. Consider an $(n+3) \times (n+3)$ board with n even and $n \not\equiv 0 \mod 6$.
Solution: Number the rows $-1, 0, \ldots, n+1$ and the columns $-2, -1, 0, 1, \ldots, n$. Place Pawn and Queens as in Pattern II, with an additional Queen at $(n+1, n)$.

Pattern IV. Consider an $(n+3) \times (n+3)$ board with $n \equiv 0 \mod 6$ and $n \geq 12$.
Solution: Number the rows $-1, 0, \ldots, n+1$ and the columns $-3, -2, \ldots, n-1$. Place the Pawn at $(n/2, 2)$ and the Queens at $(n/2, -3)$, $(n+1, -2)$, $(-1,-1)$, $(n, 2)$, $(2i+1, i)$ for $0 \leq i < n/2$, and $(2i-n, i)$ for $n/2 \leq i < n$.

The cases of $n = 7$ and $n = 9$ must be considered separately, but Figures 2 and 3 show that $s_Q(8,7) = s_Q(10,9) = 1$. ■

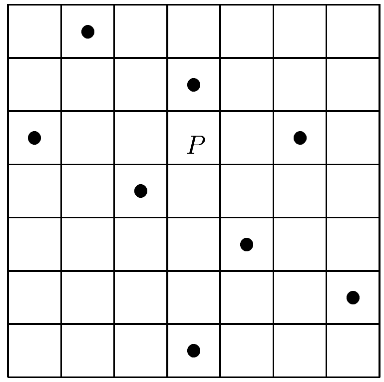

Figure 2: $s_Q(8,7) = 1$.

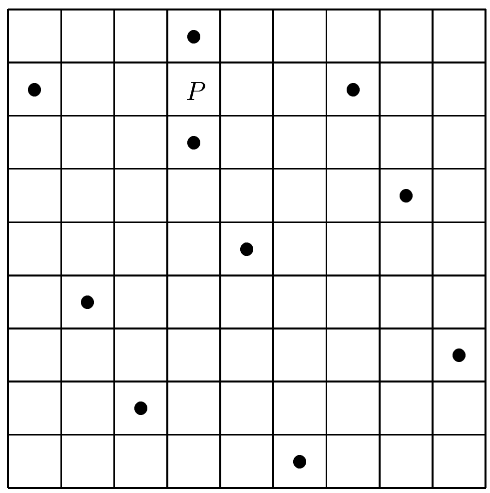

Figure 3: $s_Q(10,9) = 1$.

A computer search for small order boards provides the results in Table 1.

3 More Queens

Computer searches indicate that it may be possible to increase the number of Queens and the number of blocking Pawns in a one-to-one fashion as long as the order of the board is large enough.

n	Solutions	Fundamental solutions
5	0	0
6	16	2
7	20	3
8	128	16
9	396	52
10	2288	286
11	11152	1403
12	65172	8214

Table 1: $n+1$ Queens and 1 Pawn on $n \times n$ chessboard

n	Solutions	Fundamental solutions
6	0	0
7	4	1
8	18	5
9	160	32
10	698	147
11	6771	1428

Table 2: $n+2$ Queens and 2 Pawns on $n \times n$ chessboard

n	Solutions	Fundamental solutions
7	0	0
8	8	1
9	44	6
10	528	66

Table 3: $n+3$ Queens and 3 Pawns on $n \times n$ chessboard

Conjecture 2 *For each positive integer k and large enough n, $s_Q(n+k, n) = k$.*

4 Other chess pieces

The separation number can be considered for pieces other than the Queens as well. While separation is meaningless for the King, Knight, and Pawn, the Bishop and the Rook can be separated by placing Pawns between attacking pieces.

It can be easily seen that $s_R(m,n) \leq s_Q(m,n)$ when these values are defined. Furthermore, it is known [6] that n independent Rooks or $2n-2$ independent Bishops can be placed on an $n \times n$ board. From this, it follows that $s_R(m,n) = 0$ for $m \leq n$ and $s_B(m,n) = 0$ for $m \leq 2n-2$.

Proposition 3 *For $n \geq 3$, $s_B(2n-1,n) = 1$.*

Proof. Since the independence number $\beta_B = 2n-2$, $s_B(2n-1,n) \neq 0$. Place Bishops on the top row and bottom row, except for one corner. There are only two Bishops on opposite corners that attack each other, so place the Pawn between them. ■

Proposition 4 *For odd $n \geq 3$, $s_B(2n,n) = 1$.*

Proof. Place Bishops on the top row and the bottom row and the Pawn in the center square. ■

Proposition 5 *The Rooks separation number $s_R(n+k,n) = k$ for $n \geq k+2$.*

Proof. Clearly, $s_R(n+k,n) \geq k$. We show by induction a way to place $n+k$ Rooks on an $n \times n$ board. For $n = k+2$, place Pawns along the main diagonal from the upper left to the lower right except at the corners. Place Rooks along the two adjacent diagonals. This gives k Pawns and $n+k$ Rooks.

Given a solution for $n = i$, we can obtain a solution for $n = i+1$ by adding a column on the left or right and a row on top or bottom and placing the extra Rook in the newly added corner square. ■

Non-orthodox pieces are also interesting. An *Amazon* or *Maharaja* is a piece that can move as either a Queen or a Knight. A computer search has shown that when $n < 12$ there are no solutions for placing $n+1$ Amazons on an $n \times n$ board.

n	Solutions	Fundamental solutions
11	0	0
12	72	9
13	412	53

Table 4: $n+1$ Amazons and 1 Pawn on $n \times n$ chessboard

5 Open problems

1. Settle Conjecture 2. If the conjecture is true, what is "large enough"?

2. Describe the idea of using one chess piece to block another in terms of the effect that it has on the corresponding graph.

3. Consider rectangular or other shape boards. For example, on the torus one Pawn is insufficient and as many as eight Pawns may be necessary in order to safely place $n+1$ queens.

4. Consider the *upper Queens separation number* $S_Q(m,n)$, which is the maximum number of Pawns required to separate some legal placement of m Queens on an $n \times n$ board.

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Appendix: Proof of Theorem 1

Theorem 1 *For* $n \geq 6$, $s_Q(n+1, n) = 1$.

Proof. Clearly, $s_Q(n+1, n) > 0$. There are four primary patterns and some special cases to consider. If one Queen is placed at (x, y) and another is placed at (w, z) with $x \neq w$ and $y \neq z$, we need only check the Queens do not lie on the same diagonal. If the two Queens lie on the same negative slope diagonal, then $x + y = w + z$. If the two lie on the same positive slope diagonal, then $x - y = w - z$. So we need to compare all values of *row* + *column* and *row* − *column* for each pattern.

Pattern I. If $n \geq 4$ is even but $n \not\equiv 2 \mod 6$, then place the Pawn at $(n/2 - 1, -1)$ and the Queens at [*a*] $(n/2-1, -2)$, [*b*] $(n, -1)$, [*c*] $(-1, -1)$, [*d*] $(2i+1, i)$ for $0 \leq i < n/2$, and [*e*] $(2i - n, i)$ for $n/2 \leq i < n$.
proof:

	row + *column*		*row* − *column*
a	$n/2 - 3$	a'	$n/2 + 1$
b	$n - 1$	b'	$n + 1$
c	-2	c'	0
d	$3i + 1$	d'	$i + 1$
e	$3i - n$	e'	$i - n$

Case I: a versus b: $n/2 - 3 = n - 1 \Rightarrow -2 = n/2 \Rightarrow n = -4$. But $n \geq 4$ so these two Queens are not on the same diagonal.

Case II: a versus c: $n/2 - 3 = -2 \Rightarrow n/2 = 1 \Rightarrow n = 2$. But $n \geq 4$ so these two Queens are not on the same diagonal.

Case III: a versus d: $n/2 - 3 = 3i + 1$ for $0 \leq i < n/2$. This yields $3i = n/2 - 4 \Rightarrow i = n/2 - 4/3 = (n-8)/6$. But since $n \not\equiv 2 \mod 6$, $(n-8)/6$ is not an integer.

Case IV: a versus e: $n/2 - 3 = 3i - n$ for $n/2 \leq i < n$. Thus $3n/2 - 3 = 3i \Rightarrow i = n/2 - 1$. But $i \geq n/2$.

Case V: b versus c: $n - 1 = -2 \Rightarrow n = -1$. But $n \geq 4$.

Case VI: b versus d: $n - 1 = 3i + 1 \Rightarrow n - 2 = 3i \Rightarrow i = (n-2)/3$. Since $n \not\equiv 2 \mod 6$, $(n-2)/6$ is not an integer.

Case VII: b versus e: $n - 1 = 3i - n \Rightarrow 2n - 1 = 3i \Rightarrow i = (2n-1)/3$. If $n = 6k$, then $i = (12k - 1)/3$, which is not an integer. If $n = 6k + 4$, then $i = (12k + 8 - 1)/3 = (12k + 7)/3$, which is not an integer.

Case VIII: c versus d: $-2 = 3i + 1 \Rightarrow -3 = 3i \Rightarrow -1 = i$. But $i \geq 0$.

Case IX: c versus e: $-2 = 3i - n \Rightarrow 3i = n - 2 \Rightarrow i = (n-2)/3 < n/3 < n/2$. But $n/2 \leq i$.

Case X: d versus d: $3i + 1 = 3j + 1 \Rightarrow i = j$. There is no diagonal conflict.

Case XI: d versus e: $3i + 1 = 3j - n \Rightarrow j = (n+1)/3 + i$ which is not an integer since $n \not\equiv 2 \mod 6$.

Case XII: e versus e: $3i - n = 3j - n \Rightarrow i = j$. There is no diagonal conflict.

Case XIII: a' versus b': $n/2 + 1 = n + 1 \Rightarrow n = 0$. But $n \geq 4$.

Case XIV: a' versus c': $n/2 + 1 = 0 \Rightarrow n/2 = -1$.

Case XV: a' versus d': $n/2 + 1 = i + 1 \Rightarrow n/2 = i$. But $i < n/2$ for d'.

Case XVI: a' versus e': $n/2 + 1 = i - n \Rightarrow i = 3n/2 + 1 > n + 1 > n$. But $i < n$ for e'.

Case XVII: b' versus c': $n + 1 = 0 \Rightarrow n = -1$.

Case XVIII: b' versus d': $n + 1 = i + 1 \Rightarrow n = i$. But $i < n/2$ for d'.

Case XIX: b' versus e': $n+1 = i-n \Rightarrow 2n+1 = i$. But $i < n$ for e'.

Case XX: c' versus d': $0 = i+1 \Rightarrow -1 = i$. But $i \geq 0$ for d'.

Case XXI: c' versus d': $0 = i-n \Rightarrow i = n$. But $i < n$.

Case XXII: d' versus d': $i+1 = j+1 \Rightarrow i = j$. There is no diagonal conflict.

Case XXIII: d' versus e': $i+1 = j-n \Rightarrow j = n+i+1 > n$.

Case XXIV: e' versus e': $i-n = j-n \Rightarrow i = j$. There is no diagonal conflict.

Pattern II. Suppose n is even but $n \not\equiv 0 \mod 6$. If $n = 8$, let $w = 4$. If $n = 10$, let $w = 7$. If $n > 10$, let $w = \lfloor (n+1)/4 \rfloor$. Place the Pawn at $(w, -1)$ and the Queens at $(w, -2)$, $(-1, -1)$, $(n, -1)$, $((n/2 + 2i - 1) \mod n, i)$ for $0 \leq i < n/2$, and $((n/2 + 2i + 2) \mod n, i)$ for $n/2 \leq i < n$.
proof:

	row + column		*row − column*
a	$\lfloor (n+1)/4 \rfloor - 2$	a'	$\lfloor (n+1)/4 \rfloor + 2$
b	-2	b'	0
c	$n-1$	c'	$n+1$
d	$(n/2 + 2i - 1) \mod n + i$	d'	$(n/2 + 2i - 1) \mod n - i$
e	$(n/2 + 2i + 2) \mod n + i$	e'	$(n/2 + 2i + 2) \mod n - i$

Note that $\lfloor (n+1)/4 \rfloor = (n-2)/4$ for $n \equiv 2 \mod 4$ and $\lfloor (n+1)/4 \rfloor = n/4$ for $n \equiv 0 \mod 4$.

Case I: a versus b: For $n \equiv 2 \mod 4$, we have $(n-2)/4 - 2 = -2 \Rightarrow (n-2)/4 = 0 \Rightarrow n = 2$. But $n > 10$.

For $n \equiv 0 \mod 4$, we have $n/4 - 2 = -2 \Rightarrow n = 0$.

Case II: a versus c: For $n \equiv 2 \mod 4$, we have $(n-2)/4 - 2 = n - 1 \Rightarrow (n-2)/4 = n+1 \Rightarrow n - 2 = 4n + 4 \Rightarrow 3n = -6 \Rightarrow n = -2$.

For $n \equiv 0 \mod 4$, we have $n/4 - 2 = n - 1 \Rightarrow 3n/4 = -1 \Rightarrow n = -4/3$.

Case III: a versus d: For $n \equiv 2 \mod 4$, we have $n/2 + 2i - 1 - kn + i = (n-2)/4 - 2 \Rightarrow 3i = (-n-2)/4 - i + kn \Rightarrow i = (-n - 6 + 4kn)/12$ For $k = 0$, $i = (-n-6)/12 < 0$. For $k = 1$, $i = (3n-6)/12 < n/2$.

For $n \equiv 0 \mod 4$, we have $n/2 + 2i - 1 - kn + i = n/4 - 2 \Rightarrow 3i = n/4 - 2 - n/2 + 1 + kn \Rightarrow i = (-n - 4 + 4kn)/12$. For $k = 0$, $i = (-n-4)/12 < 0$. For $k = 1$, $i = (3n-4)/12 < n/2$.

Case IV: a versus e: For $n \equiv 2 \mod 4$, we have $n/2 + 2i + 2 - kn + i = (n-2)/4 - 2 \Rightarrow 3i = (-n-2)/4 - 4 + kn \Rightarrow i = (-n - 18 + 4kn)/12$. Since $k = 1$, $i = (3n - 18)/12 < n/2$.

For $n \equiv 0 \mod 4$, we have $n/2 + 2i + 2 - kn + i = n/4 - 2 \Rightarrow 3i = -n/4 - 4 + kn \Rightarrow i = (-n - 16 + 4kn)/12$. Since $k = 1$, $i = (3n - 16)/12 < n/2$.

Case V: b versus c: $-2 = n - 1 \Rightarrow n = -1$.

Case VI: b versus d: $n/2 + 2i - 1 - kn + i = -2 \Rightarrow 3i = -2 - n/2 + 1 + kn \Rightarrow i = (-2 - n + 2kn)/6$. For $k = 0$, $i = (-2 - n)/6 < 0$. For $k = 1$, $i = (n - 2)/6$. However, $n/2 + 2((n - 2)/6 - 1 = n/2 + n/3 - 5/3 < n$ for $n > -10$.

Case VII: b versus e: $n/2 + 2i + 2 - kn + i = -2 \Rightarrow 3i = -4 - n/2 + kn \Rightarrow i = (-8 - n + 2kn)/6$. For $k = 1$, $i = (n - 8)/6$, which is less than $n/2$ for $n > -4$. For $k = 2$, $i = (3n - 8)/6 < n/2$.

Case VIII: c versus d: $n/2 + 2i - 1 - kn + i = n - 1 \Rightarrow 3i = kn - n/2 \Rightarrow i = (2kn - n)/6$. Since $k = 0$, this implies $i = -n/6 < 0$.

Case IX: c versus e: $n/2 + 2i + 2 - kn + i = n - 1 \Rightarrow 3i = kn - n/2 - 3 \Rightarrow i = (2kn - n)/6 - 1$. For $k = 1$, $i = n/6 - 1 < n/2$. For $k = 2$, $i = 3n/6 - 1 < n/2$.

Case X: d versus d: $n/2 + 2i - 1 - k_i n + i = n/2 + 2j - 1 - k_j n + j$. The values of k are either 0 or 1. If $k_i = k_j$, then $3i = 3j$ so $i = j$. If $k_i = 0$ and $k_j = 1$, then $3i = 3j - n \Rightarrow i = j - n/3$ which is not an integer since $n \not\equiv 0 \mod 6$. Symmetrically, $j = i - n/3$ is not an integer.

Case XI: d versus e: $n/2 + 2i - 1 - k_i n + i = n/2 + 2j + 2 - k_j n + j$. The values of k are either 0 or 1 for d and either 1 or 2 for e. If $k_i = k_j$, then $3i - 1 = 3j + 2 \Rightarrow i = j + 1$. But $j > i$. If $k_j - k_i = 1$, then $3i - 1 = 3j + 2 - n \Rightarrow i = j + 1 - n/3$ which is not an integer since $n \not\equiv 0 \mod 6$. If $k_i = 0$ and $k_j = 2$, then $3i - 1 = 3j + 2 - 2n \Rightarrow i = j + 1 - 2n/3$ which is also not an integer.

Case XII: e versus e: $n/2 + 2i + 2 - k_i n + i = n/2 + 2j + 2 - k_j n + j \Rightarrow 3i - k_i n = 3j - k_j n$. If $k_i = k_j$, then $i = j$. If $k_i = 1$ and $k_j = 2$, then $3i - n = 3j - 2n \Rightarrow 3i = 3j - n \Rightarrow i = j - n/3$ which is not an integer. Symmetrically, $j = i - n/3$ is not an integer.

Case XIII: a' versus b': For $n \equiv 2 \mod 4$, we have $(n - 2)/4 + 2 = 0 \Rightarrow (n - 2)/4 = -2 \Rightarrow n - 2 = -8 \Rightarrow n = -6$.

For $n \equiv 0 \mod 4$, we have $n/4 + 2 = 0 \Rightarrow n/4 = -2 \Rightarrow n = -8$.

Case XIV: a' versus c': For $n \equiv 2 \mod 4$, we have $(n - 2)/4 + 2 = n + 1 \Rightarrow (n - 2)/4 = n - 1 \Rightarrow n - 2 = 4n - 4 \Rightarrow 3n = 2 \Rightarrow n = 2/3$.

For $n \equiv 0 \mod 4$, we have $n/4 + 2 = n + 1 \Rightarrow 3n/4 = 1 \Rightarrow n = 4/3$.

Case XV: a' versus d': For $n \equiv 2 \mod 4$, $n/2 + 2i - 1 - kn - i = (n - 2)/4 + 2 \Rightarrow i = (4kn - n + 10)/4$. For $k = 0$, $i = (-n + 10)/4 < 0$ since $n > 10$. For $k = 1$,

$i = (3n + 10)/4 > n/2$.

For $n \equiv 0 \mod 4$, $n/2 + 2i - 1 - kn - i = n/4 + 2 \Rightarrow i = (4kn - n + 12)/4$. For $k = 0$, $i = (-n + 12)/4 < 0$ since $n > 12$. For $k = 1$, $i = (3n + 12)/4 > n/2$.

Case XVI: a' versus e': For $n \equiv 2 \mod 4$, $n/2 + 2i + 2 - kn - i = (n-2)/4 + 2 \Rightarrow i = (4kn - n - 2)/4$. Since $k = 2$, $i = (7n - 2)/4$ which is greater than n for $n > 2/3$.

For $n \equiv 0 \mod 4$, $n/2 + 2i + 2 - kn - i = n/4 + 2 \Rightarrow i = (4kn - n)/4$. Since $k = 2$, $i = 7n/4 > n$ for $n > 0$.

Case XVII: b' versus c': $0 = n + 1 \Rightarrow n = -1$.

Case XVIII: b' versus d': $n/2 + 2i - 1 - kn - i = 0 \Rightarrow i = kn - n/2 + 1$. For $k = 0$, $i = -n/2 + 1$ which is less than 0 for $n > 2$. For $k = 1$, $i = n/2 + 1 > n/2$.

Case XIX: b' versus e': $n/2 + 2i + 2 - kn - i = 0 \Rightarrow i = kn - n/2 - 2$. For $k = 1$, $i = n/2 - 2 < n/2$. For $k - 2$, $i = 3n/2 - 2$ which is greater than n for $n > 4$.

Case XX: c' versus d': $n/2 + 2i - 1 - kn - i = n + 1 \Rightarrow i = n/2 + kn + 2$. For $k = 0$, $= n/2 + 2 > n/2$. For $k = 1$, $i = 3n/2 + 2 > n$.

Case XXI: c' versus e': $n/2 + 2i + 2 - kn - i = n + 1 \Rightarrow i = n/2 + kn - 1$. For $k = 1$, this implies $i = 3n/2 - 1$. For $k = 2$, $i = 5n/2 - 1$. In both cases, $i > n$ for $n > 2$.

Pattern II leaves an empty main diagonal, so it can be used to obtain solutions for some odd order chessboards.

Pattern III. Given an $(n+3) \times (n+3)$ board with even $n \not\equiv 0 \mod 6$, number the rows $1, 0, \ldots, n+1$ and the columns $-2, 0, 1, \ldots, n$. Place Pawn and Queens as in Pattern II, with an additional Queen at $(n+1, n)$.

proof:

Add the following row to the table in Pattern II.

	row + column		*row − column*
f	$2n+1$	f'	1

Case I: a versus f: For $n \equiv 2 \mod 4$, we have $(n-2)/4 - 2 = 2n + 1 \Rightarrow -3 = 2n - n/4 + 1/2 \Rightarrow -7/2 = 7n/8 \Rightarrow n = -4$. But n is not negative.

For $n \equiv 0 \mod 4$, we have $n/4 - 2 = 2n + 1 \Rightarrow -3 = 7n/4 \Rightarrow n = -12/7$.

Case II: b versus f: $-2 = 2n + 1 \Rightarrow -3 = 2n \Rightarrow n = -3/2$.

Case III: c versus f: $n - 1 = 2n + 1 \Rightarrow n = -2$.

Case IV: d versus f: $n/2 + 2i - 1 - kn + i = 2n + 1 \Rightarrow 3i = 2n - n/2 + 2 + kn \Rightarrow i = (3n + 4 + 2kn)/6 > n/2$. But $i < n/2$.

Case V: e versus f: $n/2+2i+2-kn+i=2n+1 \Rightarrow i=2n-n/2-1+kn \Rightarrow i=kn+3n/2-1>n$.

Case VI: a' versus f': For $n \equiv 2 \mod 4$, we have $(n-2)/4+2=1 \Rightarrow (n-2)/4 = -1 \Rightarrow n-2=-4 \Rightarrow n=-2$.

For $n \equiv 0 \mod 4$, we have $n/4+2=1 \Rightarrow n/4=-1 \Rightarrow n=-4$.

Case VII: b' versus f': $0=1$ does not need to be checked.

Case VIII: c' versus f': $n+1=1 \Rightarrow n=0$.

Case IX: d' versus f': $n/2+2i-1-kn-i=1 \Rightarrow i=2+kn-n/2$. If $k=0$, then $i=2-n/2<0$ for $n>4$. If $k=1$, then $i=2+n/2>n/2$. But $0 \leq i \leq n/2$.

Case X: e' versus f': $n/2+2i+2-kn-i=1 \Rightarrow i=-1-n/2+kn$. For $k=1$, $i=-1-n/2+n \Rightarrow i=n/2-1<n/2$. For $k=2$, $i=-1-n/2+2n \Rightarrow i = 3n/2-1>n$ for $n>2$.

Pattern IV. Given an $(n+3)\times(n+3)$ board with $n \equiv 0 \mod 6$ and $n \geq 12$, number the rows $-1,0,\ldots,n+1$ and the columns $-3,-2,\ldots,n-1$. Place the Pawn at $(n/2,2)$ and the Queens at $(n/2,-3)$, $(n+1,-2)$, $(-1,-1)$, $(n,2)$, $(2i+1,i)$ for $0 \leq i < n/2$, and $(2i-n,i)$ for $n/2 \leq i < n$.
proof:

	row + column		*row − column*
a	$n/2-3$	a'	$n/2+3$
b	$n-1$	b'	$n+3$
c	-2	c'	0
d	$n+2$	d'	$n-2$
e	$3i+1$	e'	$i+1$
f	$3i-n$	f'	$i-n$

The cases b versus c, b versus e, b versus f, c versus e, c versus f, e versus e, e versus f, and f versus f, along with the corresponding prime versions of each, were checked in Pattern I.

Case I: a versus b: $n/2-3=n-1 \Rightarrow -2=n/2 \Rightarrow n=-4$. But $n \geq 12$.

Case II: a versus c: $n/2-3=-2 \Rightarrow n/2=1 \Rightarrow n=2$.

Case III: a versus d: $n/2-3=n+2 \Rightarrow -5=n/2 \Rightarrow n=-10$.

Case IV: a versus e: $n/2-3=3i+1 \Rightarrow n/2-4=3i \Rightarrow i=n/6-4/3=(n-8)/6$, which is not an integer since $n \not\equiv 2 \mod 6$.

Case V: a versus f: $n/2-3=3i-n \Rightarrow 3n/2-3=3i \Rightarrow 1=n/2-1<n/2$.

Case VI: b versus d: $n-1=n+2$ does not result in a diagonal conflict.

Case VII: c versus d: $-2=n+2 \Rightarrow n=-4$.

Case VIII: d versus e: $n+2=3i+1 \Rightarrow n+1=3i \Rightarrow i=(n+1)/3$, which is not an integer since $n \equiv 0 \mod 6$.

Case IX: d versus f: $n+2=3i-n \Rightarrow 2n+2=3i \Rightarrow i=2(n+1)/3$, which is not an integer since $n \equiv 0 \mod 6$.

Case X: a' versus b': $n/2+3=n+3 \Rightarrow n/2=n \Rightarrow n=0$.

Case XI: a' versus c': $n/2+3=0 \Rightarrow n=-6$.

Case XII: a' versus d': $n/2+3=n-2 \Rightarrow n=10$. But $n \geq 12$.

Case XIII: a' versus e': $n/2+3=i+1 \Rightarrow i>n/2$.

Case XIV: a' versus f': $n/2+3=i-n \Rightarrow i=3n/2+3 \Rightarrow i>n$.

Case XV: b' versus d': $n+3=n-2$ does not result in a diagonal conflict.

Case XVI: c' versus d': $0=n-2 \Rightarrow n=2$.

Case XVII: d' versus e': $n-2=i+1 \Rightarrow i=n-3 \Rightarrow i>n/2$ since $n>6$.

Case XVIII: d' versus f': $n-2=i-n \Rightarrow i=2n-2>n$ for $n>2$.

The cases of $n=7$ and $n=9$ must be considered separately, but Figures 2 and 3 show that both $s_Q(8,7)=1$ and $s_Q(10,9)=1$. ■