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A SURVEY ON INTEGRAL GRAPHS

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A graph whose spectrum consists entirely of integers is called an integral graph. We present a survey of results on integral graphs and on the corresponding proof techniques.

Throughout this paper a graph G is assumed to be simple, i.e. a finite undirected graph without loops or multiple edges. Therefore, the characteristic polynomial of (the adjacency matrix of) G, denoted by $P_G(\lambda)$, has only real zeroes and this family of eigenvalues (the spectrum of G) will be represented as $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ or in the form $\mu_1^{k_1}, \mu_2^{k_2}, \ldots, \mu_m^{k_m}$, where $\mu_1, \mu_2, \ldots, \mu_m$ are distinct eigenvalues of G in decreasing order and $k_1, k_2, \ldots, k_m$ are the corresponding multiplicities. The sum $\sum_{i=1}^{n} \lambda_i^k$ is called the k-th spectral moment and is equal to the number of closed walks of length k of G.

The characteristic polynomial of a graph is monic (i.e. its leading coefficient is 1), and hence the rational eigenvalues are integers. A graph whose spectrum consists entirely of integers is called an *integral* graph. Since there is no general characterization (besides the definition) of these graphs, the problem of finding (or characterizing) integral graphs has to be treated in some special classes of graphs.

This text gives a survey of former investigations and main results concerning this topic. The paper is based on a chapter on the same subject of the book [**54**]. For all notation and terminology see [**20, 54**].

1. OPENING THE PROBLEM

Which graphs have integral spectra? This question was posed in 1973 by F. Harary and A.J. Schwenk (see [**34**]), with the immediate remark that the general problem appears intractable. Indeed, the number of integral graphs is not only infinite, but one can find them in all classes of graphs and among graphs of all orders. However, they are very rare and difficult to be found.

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Also, there are comparatively huge (possibly infinite) classes of graphs containing a very restricted (finite) number of integral graphs. For example, if we regard only graphs with a given maximum vertex degree, we get that the number of such integral graphs is finite ([**15**]; see also Theorem 2 and its consequences).

Since the spectrum of a disconnected graph is the union of the spectra of its components, in any investigation of integral graphs it is sufficient to consider connected graphs only.

An immediate example of a set consisting entirely of integral graphs is the set of complete graphs K_n, whose eigenvalues are: $n-1,\ (-1)^{n-1}$. The like occurs with cocktail-party graph $CP(n)\ (=\overline{nK_2})$; the eigenvalues of $CP(n)$ are: $2n-2,\ 0^n,\ (-2)^{n-1}$. Also, the complete multipartite graph $K_{n/k,n/k,\ldots,n/k}$, on n vertices and k colour classes of sizes n/k, is always integral; the eigenvalues are: $n-n/k,\ 0^{n-k},\ (-n/k)^{k-1}$. But all these graphs are in fact the complements of some disconnected regular graphs: in particular, $n\cdot K_1$, $n\cdot K_2$ and $k\cdot K_{n/k}$, respectively. Now, since the characteristic polynomial $P_{\overline{G}}(\lambda)$ of the complement $\overline{G}$ of a regular graph G on n vertices of degree r can be expressed as

$$P_{\overline{G}}(\lambda)=(-1)^n\,\frac{\lambda-n+r+1}{\lambda+r+1}\,P_G(-\lambda-1),$$

we see that the complement of an integral regular graph must be integral, too.

There are many other simple examples of integral graphs (some of them are given in [**34**]). Thus, in the set of graphs P_n (the path with n vertices, i.e. of the length $n-1$) the only integral path is P_2 because the spectrum of P_n consists of the numbers $2\cos(\pi i/(n+1))$ $(i=1,\ldots,n)$. Similarly, the eigenvalues of C_n (the circuit on n vertices) are determined by the expression $2\cos(2\pi i/n)$ $(i=1,\ldots,n)$, and therefore the only integral circuits are C_3, C_4 and C_6. Also, since the complete bipartite graph $K_{m,n}$ has $\sqrt{mn},\ 0^{m+n-2},\ -\sqrt{mn}$ as its eigenvalues, it is integral if and only if mn is a perfect square. Thus, if we take all stars $K_{1,n}$ with $n=p^2$ $(p=1,2,3,\ldots)$ we get an infinite series of integral graphs.

A regular graph of degree $r>0$ which is not the complete graph is called *strongly regular* if there exist non-negative integers e and f such that any two adjacent vertices have exactly e common neighbours, while any two non-adjacent vertices have exactly f such neighbours (see [**20**], p. 103 and 194). It is known that if a strongly regular graph G with parameters r,e,f exists, a sufficient condition for G to be integral is that $(e-f)^2-4(f-r)=s^2$ for some positive integer s.

Some of the well known graph operations, when applied to integral graphs, result in new integral graphs and thus can be used in generating an arbitrary number of them. Let us look at some operations based on the Cartesian product of the sets of vertices. If λ_{1i_1} $(i_1=1,2,\ldots,n_1)$ and λ_{2i_2} $(i_2=1,2,\ldots,n_2)$ are the eigenvalues of the graphs G_1 and G_2, respectively, then

1° the sum G_1+G_2 has eigenvalues $\lambda_{1i_1}+\lambda_{2i_2}$;

2° the product $G_1\times G_2$ has eigenvalues $\lambda_{1i_1}\cdot\lambda_{2i_2}$;

3° the strong product of G_1 and G_2 has eigenvalues $\lambda_{1i_1} \cdot \lambda_{2i_2} + \lambda_{1i_1} + \lambda_{2i_2}$,

(in all these cases $i_1 = 1, \dots, n_1$, $i_2 = 1, \dots, n_2$). (For the definitions of the sum, the product and the strong product of graphs see, e.g. [**20**] pp. 65–66.) Thus, these three operations preserve the integrality. For example, the so called bipartite product $G \times K_2$ has eigenvalues $\pm\lambda_i$, where λ_i $(i = 1, \dots, n)$ are the eigenvalues of G (the fact to be used later).

Also, in the case of the *non-complete extended p-sum of graphs* (shortly NEPS, see [**20**] p. 66), whose spectrum is determined as all possible values

$$\Lambda_{i_1,\dots,i_n} = \sum_{\beta \in B} \lambda_{1i_1}^{\beta_1} \cdots \lambda_{ni_n}^{\beta_n},$$

$(i_k = 1, \dots, n_k;\ k = 1, \dots, n;\ 0^0 = 1)$, where B is the basis of NEPS of graphs $G_1, \dots, G_n$, we see that if G_i $(i = 1, \dots, n)$ are all integral, their NEPS also gives an integral graph. (In fact, the sum and the product of two graphs are the NEPSs for the basis $\{(0,1),(1,0)\}$ and $\{(1,1)\}$, respectively). It is interesting that for some graphs the converse statement holds as well (see [**65**]).

Theorem 1. *Suppose* $G = \mathrm{NEPS}(G_1, \dots, G_n;\ B)$*, and let* $G_1, \dots, G_n$ *be connected graphs. Then* G *is regular integral graph if and only if* G_i $(i = 1, \dots, n)$ *are regular integral graphs.*

If G_i are regular graphs on n_i vertices and of the degree r_i $(i = 1, 2)$, the characteristic polynomial of their *complete product* (*join*) $G_1 \nabla G_2$ (the graph obtained by joining each vertex of G_1 to all vertices of G_2) is given by the expression

$$P_{G_1 \nabla G_2}(\lambda) = \frac{P_{G_1}(\lambda) P_{G_2}(\lambda)}{(\lambda - r_1)(\lambda - r_2)} \big((\lambda - r_1)(\lambda - r_2) - n_1 n_2\big),$$

implying that the complete product of two regular graphs is integral if and only if both G_1 and G_2 are integral and $(r_1 - r_2)^2 + 4n_1n_2$ is a perfect square.

The line graph $L(G)$ of a regular integral graph G is also integral because the characteristic polynomial of the line graph of a regular graph of degree r with n vertices and $m = nr/2$ edges can be expressed as

$$P_{L(G)}(\lambda) = (\lambda + 2)^{m-n} P_G(\lambda - r + 2).$$

Thus, in all previously mentioned cases of integral regular graphs we can obtain new classes of integral graphs by taking their line graphs. It was shown in [**16**] that $L_2(G) = L(S(G))$ is integral if and only if G is the (disjoint) union of complete graphs all having a fixed number s (≥ 2) of vertices (here S denotes the *subdivision* of some graph obtained by inserting only a single vertex into each edge).

In [**57**] one can find an interesting construction of an infinite family of integral graphs in the class of complete tripartite graphs K_{n_1,n_2,n_3}.

One of the first more general results on integral graphs is given in [**15**] and it considers the question of the finiteness of the number of integral graphs inside an infinite set of graphs.

Theorem 2. *The set $\mathcal{I}_r$ of all regular, connected, integral graphs of a fixed degree r is finite.*

To see this, notice that besides the maximum vertex degree, the diameter of any graph of $\mathcal{I}_r$ is bounded (namely, since the spectrum of a regular graph of degree r lies in the segment $[-r, r]$, the number of distinct integral eigenvalues is at most $2r+1$, and hence the diameter is at most $2r$ – see, for example, [**20**] p. 88). In the same way one can prove that the set of all non-regular, connected, integral graphs with a given maximum vertex degree $\triangle$ is finite. Some further generalizations of Theorem 2 can be found in [**67**] and [**68**] (see also [**69**]).

One of the first results on integral graphs is obtained in [**23**] and is given in the following theorem.

Theorem 3. *The only connected, integral graphs which are not* 3*-regular and whose maximum vertex degrees are at most three are:*

$$K_1,\ K_2,\ K_3,\ C_4,\ C_6,\ K_2 \circ 2K_1,\ S(K_{1,3}),$$

where $\circ$ denotes the corona of two graphs.

(For the definition of *corona* see [**33**], p. 167.)

2. TREES

In the initial paper of F. Harary and A.J. Schwenk integral trees were mentioned as well, while first considerable results on this topic were published by M. Watanabe and A.J. Schwenk in [**75**] and [**76**]. Then, after a several years pause and having started by the article [**44**] of X.L. Li and G.N. Lin, a group of Chinese mathematicians began to present their results. Unfortunately the majority of these papers were written in Chinese, as well as their authors were not always aware of the results of their colleagues of other countries, which led to some overlapping of results of Chinese and other authors. In any case, the problem of integral trees appeared to be not at all an easy one and that is why, besides some general results, in several papers the authors have been engaged in constructing various necessary or sufficient conditions or particular cases and examples. (The authors of this paper were not able to have insight into all papers concerning integral trees that are cited in our references; these items were given in the form in which they have been quoted in other papers.)

One of the first and very general results is the following theorem of M. Watanabe [**75**].

Theorem 4. *No integral tree except K_2 has a perfect matching.*

Among the results on integral trees an important position is held by so called balanced trees, i.e. trees which are symmetric with respect to a vertex (the root) or an edge. Starting with the notion of the eccentricity $ecc(v)$ of a vertex v in a connected graph G, defined as $ecc(v) = \max d(v, w)$ for all vertices w of G, we can define a central vertex (of G) as a vertex of minimal eccentricity; the centre of G is a set of its central vertices. But if G is a tree, according to a well-known theorem of D. König, its center consists of either one vertex or two adjacent vertices. Now, a tree T is called *balanced* if all the vertices at the same distance from the centre $Z(T)$ are of the same degree.

A balanced tree is uniquely determined by the parity of its diameter and the sequence $(n_k, n_{k-1}, \ldots, n_1)$, where k is the radius of T and $n_j (1 \leq j \leq k)$ are the numbers of successors of a vertex at distance $k - j$ from the centre $Z(T)$. If $diam(G)$ is odd, this sequence may be modified to the form $(1; n_k, n_{k-1}, \ldots, n_1)$. These sequences are called *integral* if the corresponding balanced trees are integral.

The following three theorems of [**38**] contain important general results on balanced integral trees.

Theorem 5. *A sequence* $(n_k, n_{k-1}, \ldots, n_1)$ *of positive integers is integral if and only if for every* $q \in \mathbf{N}$ *the sequence* $(q^2 n_k, q^2 n_{k-1}, \ldots, q^2 n_1)$ *is integral.*

Theorem 6. *If a sequence* $(n_k, n_{k-1}, \ldots, n_1)$ *is integral, then* $(n_j, n_{j-1}, \ldots, n_1)$ *is integral for every* $1 \leq j \leq k - 1$.

A branch of a tree T is a subtree T' of T such that every end-vertex of T' is an end-vertex of T.

Theorem 7. *Let* T *be an integral tree. If the balanced tree defined by the sequence* $(2, n_k, \ldots, n_1)$ *is a branch of* T, *then the sequence* $(n_k, n_{k-1}, \ldots, n_1)$ *is integral.*

We say that a tree T is *star-like* if it is homeomorphic to a star $K_{1,m}$, which means that T has a unique vertex v of degree $m \geq 3$ such that $T - v$ is the (disjoint) union of m paths.

Theorem 8. [**76**] *A star-like tree* T *is integral if and only if* T *is one of these trees:*

1° $T = K_1$;

2° $T - v = k^2 P_1 \quad (k \in \mathbf{N})$;

3° $T - v = (k^2 + 2k) P_2 \quad (k \in \mathbf{N})$.

In the cases 2^o and 3^o the corresponding characteristic polynomials are $P_T(\lambda) = (\lambda^2 - k^2)\lambda^{k^2-1}$ and $P_T(\lambda) = (\lambda^2 - k^2)\lambda(\lambda^2 - 1)^{k^2+2k-1}$, respectively.

The next result concerns the trees homeomorphic to a double star, i.e. a tree obtained by joining the centres of two stars with an edge. Let a tree T have exactly two vertices u and v of degree greater than two, let them be adjacent and let T have m_i paths of length i at u and n_j paths of length j at v (then the number of vertices is clearly $n = 2 + \sum i m_i + \sum j n_j$).

Theorem 9. [**76**] *If* T *is an integral tree having exactly two vertices* u *and* v *of degree exceeding two, and if* u *and* v *are adjacent, then* T *is either*

1° *a double star such that* $T - u - v = (m_1 + n_1) P_1$ *where the polynomial* $x^4 - (m_1 + n_1 + 1)x^2 + m_1 n_1$ *has only integral roots, or*

2° *a tree determined by* $T - u - v = m_1 P_1 + n_2 P_2$ *where the polynomial* $x^4 - (m_1 + n_2 + 2)x^2 + m_1 n_2 + m_1 + 1$ *has only integral roots.*

For example, if $m_1 = r_1 = a(a+1)$ $(a \in N)$, we get a whole family of solutions. The problem of finding all solutions was solved by R. L. Graham in 1978 (see also [**76**]).

Another family of integral trees of diameter four can be constructed as follows. Let us join the centres of r copies of $K_{1,m}$ to a new vertex v (i.e. $T - v = rK_{1,m}$) and let such a tree with $n = mr + r + 1$ vertices (mr of them being end-vertices) be denoted by $T(r,m)$, or, according to the previous notation of balanced trees, simply by (r,m).

Theorem 10. [**76**] *$T(r,m)$ is integral if and only if both m and $r+m$ are perfect squares.*

For $m = 1$ we get just what we have had in Theorem 8 (3°), while $m = 4$, $r = 5$ is the smallest case for $m > 1$. Since these trees are balanced, in accordance with Theorem 5 the set of the solutions is infinite.

As a generalization of this case, suppose that, instead of r copies of $K_{1,m}$, we take r stars $K_{1,m_1}, K_{1,m_2}, \dots K_{1,m_r}$ and form the tree $T(r, m_1, m_2, \dots, m_r)$ by joining their centres with a new vertex v.

Theorem 11. [**45**] *A tree $T(r, m_i)$ is integral if and only if the equation*

$$(x^2 - m_1 - 1)(x^2 - m_2) \cdots (x^2 - m_r) - \sum_{j=2}^{r} \prod_{i=1,\ i \neq j}^{r} (x^2 - m_i) = 0$$

has only integral roots.

One more family of integral trees of diameter four can be formed by joining the centres of r copies of K_1 and s copies of $K_{1,t}$ to a new vertex v and let us denote it by $R(r,s,t)$.

Theorem 12. [**75**] *$R(r,s,t)$ is integral if and only if s is a perfect square and the polynomial $x^4 - (r+s+t)x^2 + rt$ has only integral roots.*

For $r = t = 4$, $s = 9$, we have the smallest member of this family provided $r > 0$ (case $r = 0$ equals to that in Theorem 10). The general problem of determining $r,\ s,\ t$ is equivalent to turning the polynomial of Theorem 12 into the form $(x^2 - a^2)(x^2 - b^2)$ $(a, b \in \mathbf{N})$. It has been proved that r and t can be expressed as

$$r = \frac{1}{4}(A^2 + B^2 - 2s) + \frac{1}{2}C, \quad t = \frac{1}{4}(A^2 + B^2 - 2s) - \frac{1}{2}C,$$

where integers A, B, C satisfy $(A^2 - s)(B^2 - s) = C^2$, and that there are infinitely many solutions. The authors of [**45**] showed by construction that even in case $r = t$ the number of solutions is infinite. Some sufficient conditions for $R(r,s,t)$ to be integral can also be found in [**70**]. Also, the following theorem holds [**74**].

Theorem 13. *If $R(r,s,t)$ is integral, then for every $n \in \mathbf{N}$ $R(rn^2, sn^2, tn^2)$ is integral, too.*

A generalization of the case $R(r,s,t)$ (instead of taking s copies of $K_{1,t}$ we take s stars K_{1,t_i} $i=1,\dots,s$) can be found in [**74**] (and some other papers). Among others, it contains the results analogous to Theorem 12, and some results which generalize Theorem 13.

A lot of other more or less particular results, in the form of necessry or sufficient conditions, on integral trees of diameter four can be found in [**13**, **38**, **45**, **47**, **48**, **49**, **56**, **70**, **73**, **74**, **77**, **78**, **80**, **81**, **82**].

Integral trees with diameter five were mentioned for the first time in [**45**], where the authors observed the graph obtained by joining the centres of $T(r,m_i)$ and $T(s,n_i)$ and got a theorem in the form of a necessary and sufficient condition that such a tree be integral, but were not able to find any example. The first integral tree with diameter five was constructed in [**14**], while in [**46**] it was proved that there are infinitely many such trees. It is interesting that none of them is balanced.

Theorem 14. [**38**] *There is no balanced integral tree of diameter* $4k+1$ $(k\in \mathbf{N})$.

As for diameter $4k-1$, we have so far the following result.

Theorem 15. [**38**] *There is no balanced integral tree of diameter seven.*

The question of finding an integral tree of diameter six was touched for the first time in [**76**]: it was the observation of C. Godsil that one can construct integral trees of diameter six by attaching t new end-vertices to each vertex of the tree $T(r,m)$. The parameters $t,\ r,\ m$ must be chosen so that $m,\ m+r,\ t,\ m+4t$ and $m+r+4t$ are perfect squares, and it can be made by taking

$$m=(a^2-b^2)^2,\quad r=(c^2-d^2)^2-(a^2-b^2)^2,\quad t=a^2b^2=c^2d^2.$$

For example, $a=3,\ b=2,\ c=6,\ d=1$ gives an integral tree of diameter six with 1 123 236 vertices.

Balanced trees of diameter six can be imagined as to have been constructed by joining the centres of n_3 copies of $T(n_2,n_1)$.

Theorem 16. [**45**] *The sequence* (n_3,n_2,n_1) *is integral if and only if* n_1 *and* n_1+n_2 *are perfect squares and the polynomial*

$$x^4-(n_1+n_2+n_3)x^2+n_1n_3$$

has only integral roots, that is, can be factorized as $(x^2-a^2)(x^2-b^2)$.

For example, let $p,q\in\mathbf{N},\ p>q$, and put $n_1=4p^2q^2,\ n_2=(p^2-q^2)^2$, $n_3=(p^2+q^2)^2$. Then if $2\,(p^2+q^2)$ is a perfect square, the sequence (n_3,n_2,n_1) is integral. Thus, for $p=7,\ q=1$ we have one such example and by Theorem 5 the number of integral sequences (n_3,n_2,n_1) is infinite.

A somewhat different form of the same result can be found in [**38**].

Theorem 17. *A sequence* (n_3,n_2,n_1) *is integral if and only if* $n_1=k^2,\ n_2=m^2+2mk,\ n_3=a^2b^2/k^2$, *where* $a,\ b,\ k,\ m$ *are positive integers satisfying*

(1) $$(k^2-b^2)(a^2-k^2)=k^2(m^2+2mk),\quad b<k<a.$$

A generalization of previous cases of diameter six is given in [**70**].

Theorem 18. *Let T be a tree obtained by identifying the centre of $K_{1,s}$ and the centre (root) of a balanced tree defined by the sequence (n_3, n_2, n_1), which for this occasion will be denoted by $T(r, m, t)$. Then T is integral if and only if t and $m+t$ are perfect squares and*

$$x^4 - (r+m+t+s)x^2 + rt + s(m+t)$$

can be factorized as $(x^2 - a^2)(x^2 - b^2)$.

Particularly, if $s = t$ we have the following.

Corollary. *For $s = t$ the tree T is integral if and only if t, $m+t$ and $m+t+r$ are perfect squares.*

The authors of [**70**] produced also a list of examples of such integral trees.

A theorem equivalent to the previous one is given in [**74**].

Theorem 19. *Let T be a tree obtained by identifying the centre of $K_{1,s}$ and the centre of balanced tree $T(r, m, t)$. Such a tree of diameter six is integral if and only if $t = k^2$, $m = n^2 + 2nk$, $s = k^2 + \dfrac{(a^2-k^2)(b^2-k^2)}{n^2+2nk}$ (≥ 1) and $r = a^2 + b^2 - (n+k)^2 - k^2 - \dfrac{(a^2-k^2)(b^2-k^2)}{n^2+2nk}$ (≥ 1), where a, b, k, n are positive integers.*

An interesting result (analogous to a previous one) on such a type of trees is the following theorem.

Theorem 20. [**74**] *For any positive integer n, if the tree of the previous theorem (let it be denoted by $K_{1,s} \cdot T(r, m, t)$) is integral, then $K_{1,sn^2} \cdot T(rn^2, mn^2, tn^2)$ is integral, too.*

Besides these general facts, there are also many particular results which all together make up an exhaustive discussion about trees of diameter six described in previous theorems. In the majority of such cases we have a construction of a set of sufficient conditions for such a tree to be integral, combined with a computer search which provides examples. Various results on integral trees of diameter six can be found in [**13**, **14**, **36**, **38**, **45**, **46**, **47**, **48**, **71**, **72**, **73**, **74**].

Finally, there is a characterization of balanced integral trees of diameter eight analogous with the case of diameter six expressed by Theorem 17 [**38**].

Theorem 21. *A sequence (n_4, n_3, n_2, n_1) is integral if and only if $n_1 = k^2$, $n_2 = m^2 + 2mk$, $n_3 = \dfrac{a^2b^2}{k^2}$, $n_4 = \dfrac{c^2d^2 - a^2b^2}{(m+k)^2}$, where a, b, c, d, k, m are positive integers satisfying the equality* (1) *and*

$$(c^2 + d^2)(m+k)^2k^2 = (m+k)^4k^2 + a^2b^2(m^2 + 2mk) + c^2d^2k^2, \tag{2}$$

where $a^2b^2 < c^2d^2$.

The authors of [**38**] have managed to find 182 “small” solutions of (1) and (2). Some examples and sufficient conditions for integral trees of diameter eight have been given in [**71**, **74**], too.

In fact, for every k a system (S_k) of diophantine equations can be found such that every solution of (S_k) gives an integral sequence $(n_k, n_{k-1}, \ldots, n_1)$ and vice versa, but at the moment no solution of (S_k) is known for $k \geq 5$. Moreover, no integral tree of diameter seven and greater than eight has been found so far and these problems remain open.

Recently some results have appeared which treat interrelations among integral trees of various diameters. Let us give an example (for details see [**74**].

Theorem 22. *For any positive integer n, if a balanced tree of diameter eight determined by the sequence (s, r, m, t) is integral, then $K_{1,sn^2} \cdot T(rn^2, mn^2, tn^2)$ of diameter six, is integral, too.*

In the last several years the topic of integral trees resulted in many new papers which contain a lot of particular or somewhat more general results. At present, these problems seem to come in a more mature stage, when some extent of systematization of results and open problems becomes possible (see [**74**].

3. CUBIC GRAPHS

A cubic graph is a 3-regular graph.

“There are exactly 13 connected, cubic, integral graphs.”

It was the title of the paper of F.C. Bussemaker and D.M. Cvetković [**8**] published in 1976, which announced the first significant result in the quest for integral graphs (in fact, the first part of this investigation was presented in [**15**], while the rest was given in [**8**]). At the same time and independently, the same result was reported (and published a bit later) by A.J. Schwenk [**60**]. It is interesting that the research techniques used by different authors were also somewhat different; among others, F.C. Bussemaker and D. Cvetković combined the aid of a computer with theoretical reasoning, while A.J. Schwenk achieved the result completely “by hand and pencil”.

The initial idea in the first case was to list the all possible sets of distinct eigenvalues, then to find the possible multiplicities of them (subject to several restrictions resulting from the connections between spectral moments and the numbers of vertices, edges and triangles, and also from the Hoffman polynomial, see [**20**] p. 95), and, finally, to deduce whether a graph (possibly more than one) with a considered spectrum exists. The result, attained through a discussion of numerous particular cases and by combining theoretical reasoning with the aid of a computer, is as follows ([**15**, **8**, **60**]):

Theorem 23. *There are exactly thirteen connected cubic integral graphs. They are:*
K_4, $K_{3,3}$, $C_3 + K_2$, $C_4 + K_2$, $C_6 + K_2$, the Petersen graph, $L(S(K_4))$, the Tutte's

8-cage, the graph on 10 *vertices obtained from* $K_{3,3}$ *by specifying a pair of non-adjacent vertices and replacing each of them by a triangle, Desargues' graph and its cospectral-mate, the graph obtained from two (disjoint) copies of* $K_{2,3}$ *by adding three edges between vertices of degree two in different copies of* $K_{2,3}$*, and a bipartite graphs on* 24 *vertices (with girth* 6*).*

These graphs are displayed in original articles in another order (see also Fig. 5.2 in [**54**]).

As we have already pointed out, the product of two integral graphs G_1 and G_2 is integral itself. Particularly, if one of these two graphs is K_2 (whose eigenvalues are ± 1), the product, whose eigenvalues are all possible products of the eigenvalues of G_1 and G_2, has a symmetric spectrum, which means that it is bipartite. Moreover, the product $G_1 \times K_2$ is connected if and only if G_1 is non-bipartite (in the case of bipartite G_1, its spectrum is being duplicated by multiplying by K_2, which results in two disjoint copies of G_1). Finally, if G_1 is cubic, it is obvious that $G_1 \times K_2$ is also a cubic graph. These were the starting facts on which A.J. Schwenk leaned his approach to finding all connected, cubic, integral graphs. His idea was to begin with identifying all resulting bipartite graphs, and then to see which of them can be decomposed as $G \times K_2$. Using a similar set of restrictions as F.C. Bussemaker and D. Cvetković, he found eight bipartite connected cubic integral graphs and then, by decomposing all of them in the form $G \times K_2$, managed to obtain the rest.

4. NON-REGULAR GRAPHS WITH MAXIMUM VERTEX DEGREE FOUR

If G is a non-regular integral graph with maximum vertex degree four, then $\lambda_1 \leq 3$ (since $1 < \overline{d} < \lambda_1 < \triangle = 4$). Therefore the spectrum of G lies in the segment $[-3, 3]$. If $\lambda_1 = 2$, G is one of Smith graphs (connected graphs with the largest eigenvalue equal to 2, see [**20**], pp. 78-79), and hence $G = K_{1,4}$. So we have $\lambda_1 = 3$, while -3 may or may not be the eigenvalue of G. Obviously, if -3 is contained in the spectrum, G is bipartite; otherwise the graph is non-bipartite, and it has -2 as its least eigenvalue (if it were -1 (or 0), G would be a complete graph), and since its eigenvalues are in the set $\{-2, -1, 0, 1, 2, 3\}$, we have $diam(G) \leq 5$.

"There are just thirteen connected, non-regular, non-bipartite, integral graphs having maximum vertex degree four" - it was the title of the report published in 1986 by Z. Radosavljević and S. Simić ([**55**], the full version in [**62**]). These graphs are displayed in Fig. 1.

A graph whose spectrum is bounded from below by -2 is either a generalized line graph (a graph representable in the root system D_n for some n) or an exceptional graph (a graph representable in the exceptional root system E_8); see, e.g., [**19**], p. 4). Recall, a *generalized line graph*, denoted by $L(H; a_1, a_2, \ldots, a_n)$, is a graph obtained from some labelled graph H having labels (i.e. non-negative integers) $a_1, a_2, \ldots, a_n$ assigned to its vertices $v_1, v_2, \ldots, v_n$, respectively. It is obtained in the following way: we take the disjoint union of $L(H)$ (the line graph of H) and n copies of $CP(a_i)$ (the cocktail party graph on $2a_i$ vertices), and then (for each

i) we add edges between vertices of $L(G)$ that correspond to edges in G incident with v_i and vertices of $CP(a_i)$ (for each i). An *exceptional graph* is a connected graph whose least eigenvalue is smaller than or equal to -2, but which is not a generalized line graph.

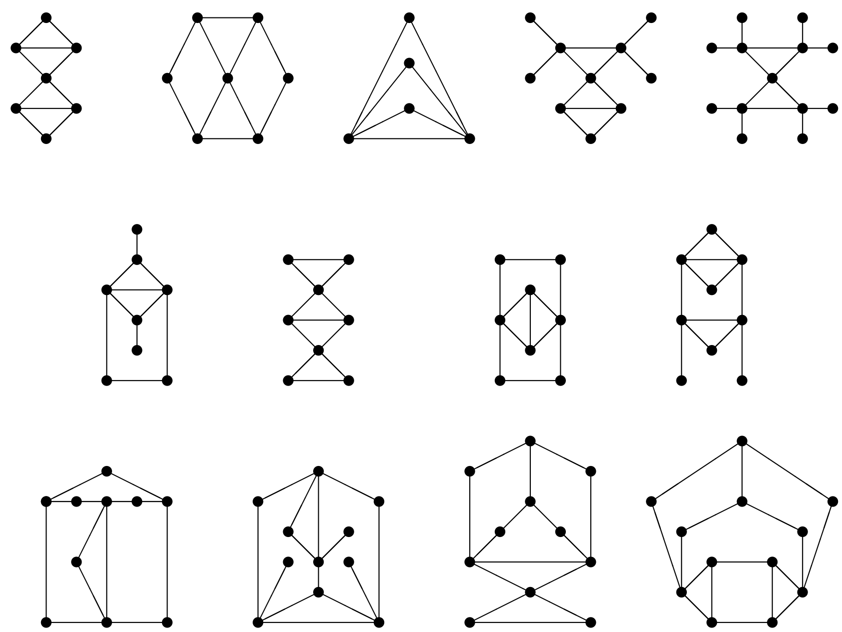

Fig. 1

Let $\mathcal{S}'$ be the subset of $\mathcal{S}$ which contains all generalized line graphs. If $G \in \mathcal{S}'$, then $G = L(H; a_1, a_2, \dots, a_n)$ for some graph H and some labels $a_1, a_2, \dots, a_n$ of its vertices. Notice that any subgraph of H, together with the induced labelling, produces an induced subgraph of G. Therefore, a possibility appears of finding all so-called root graphs H (together with the labellings) of the graphs of $\mathcal{S}'$, which is a significant advantage since a root graph H has a simpler structure than G. (For instance, the root graph has the maximum degree three, all vertices are labelled by 0 or 1, etc. – for more details see [**62**].) By considering the cyclic structure of the root graph and some spectral restrictions on G, the following result was proved in [**62**]:

Theorem 24. *All graphs of $\mathcal{S}'$ are the first five graphs of Fig.* 1.

Let now $\mathcal{S}''$ be the set of non-regular, non-bipartite, connected, integral graphs with maximum vertex degree four which are exceptional graphs. Since the graphs of $\mathcal{S}''$ have the average degree smaller than three, this bound can be significantly decreased by virtue of the inequality (see [**20**], p. 115)

$$\frac{n_1(d-\lambda_n)}{n} + \lambda_n \leq \overline{d_1},$$

n and n_1 being the number of vertices of a regular graph G and its induced subgraph G_1 (not necessarily regular), respectively, d the degree of G, λ_n its least eigenvalue and $\overline{d_1}$ the average degree of G_1. Putting $n = 120$, $d = 56$, $\lambda_n = -4$ and $\overline{d_1} < 3$, we get $n_1 \leq 13$. Thus, we know that the graphs of $\mathcal{S}''$ have at most 13 vertices. In [**62**] these graphs were produced starting from the 31 graphs that are forbidden for generalized line graphs [**22**], and by extending them in accordance with the mentioned structural and spectral restrictions. These graphs can now be found in the lists of all (connected) integral graphs up to 13 vertices (recently found by K. T. Balińska et. al.). Anyhow we have:

Theorem 25. *All graphs of $\mathcal{S}''$ are the last eight graphs of Fig.* 1.

The problem of finding all bipartite graphs G in the set of non-regular integral graphs with maximum degree four appears to be much more difficult than in the non-bipartite case. So far only some partial results concerning these graphs are known (some serious investigations are in progress). Basic results are contained in [**5**, **6**, **7**].

In [**5**] many structural restrictions have been proved on graphs G (e.g., the eccentricity of each vertex of degree four is at most five; the vertices at distance at least three from the fixed vertex of degree four have Smith graphs (or their subgraphs) as components, etc.). Based on these results the following bound on n, the order of G, is deduced:

$$n \leq \begin{cases} 36 & \text{if } s(r) = 3, \\ 52 & \text{if } s(r) = 2, \\ 68 & \text{if } s(r) = 1, \\ 84 & \text{if } s(r) = 0. \end{cases}$$

Here r is a vertex (of G) of degree four, while $s(r)$ denotes the number of the Smith graphs in the graph obtained from G by deleting r and all vertices at distance at most two from r. It is also proved that these bounds can be slightly improved (for example, the largest one less than 80). On the other hand, the largest graph G constructed so far is of order 29.

In [**6**, **7**] bipartite non-regular graphs with maximum degree four with ± 2 (resp. ± 1) excluded from the spectra were investigated. In [**6**] all (integral) graphs from the above set with ± 2 excluded were found (two in total, with 12 and 15 vertices). Other results of [**6**] are related to graphs with small cyclomatic number (trees, unicyclic and bicyclic graphs). They refer to all bipartite graphs considered in [**5**] (so, without above restrictions). More complicated situation appears with the observed graphs if ± 1 is excluded from their spectra [**7**]. In this case it was proved that the corresponding graphs have at most 29 vertices (besides, their least degree, as also pointed out in [**6**], is two; some potentially feasible degree sequences of these graphs were found).

5. 4-REGULAR GRAPHS

As in Section 3, we shall first consider bipartite graphs G, and later find non-bipartite graphs H from the decompositions of bipartite ones in the form $G = H \times K_2$.

A regular bipartite graph has the same number of vertices in each part so that we may assume that G has $p = 2n$ vertices. As usual, we write its spectrum as $4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4$. Let q and h denote the numbers of quadrilaterals and hexagons in G, respectively. Since the sum of the k-th powers of the eigenvalues is just the number of closed walks of length k, the parameters n, x, y, z, w, q, h satisfy the following Diophantine equations:

$$\begin{aligned}
\frac{1}{2}\sum \lambda_i^0 &= 1 + x + y + z + w = n,\\
\frac{1}{2}\sum \lambda_i^2 &= 16 + 9x + 4y + z = 4n,\\
\frac{1}{2}\sum \lambda_i^4 &= 256 + 81x + 16y + z = 28n + 4q,\\
\frac{1}{2}\sum \lambda_i^6 &= 4096 + 729x + 64y + z = 232n + 72q + 6h.
\end{aligned}$$

Based on these facts and dividing the search for possible spectra in cases depending on the greatest integer less than 4 avoided in spectrum, D. Cvetković, S. Simić and D. Stevanović [**28**] found 1888 possible spectra of 4-regular bipartite integral graphs. (Due to the space limit, spectra with 9 distinct eigenvalues and more than 20 vertices are not shown in [**28**]; for the complete list see [**65**]).

The above equations may be generalized by using graph angles (cf. [**17**], [**18**] and [**26**]). The matrix of graph angles is indexed by the set of distinct eigenvalues and the vertex set of G. Let $\alpha_{\mu,j}$ be the angle which corresponds to the eigenvalue μ and vertex j. In [**26**] it is proved that $w_j^s = \sum_\mu \alpha_{\mu,j}^2 \mu^s$ is the number of closed walks of length s starting at vertex j. It is also proved there that in regular graphs all angles corresponding to the index are equal to $1/\sqrt{2n}$, while in bipartite graphs for each eigenvalue μ and each vertex j it holds that $\alpha_{-\mu,j} = \alpha_{\mu,j}$. Further and more comprehensive information on graph angles may also be found in the monograph [**27**].

Let q_j and h_j denote the number of quadrilaterals and hexagons to which j belongs, respectively. If the neighbors of j are $u_1, \ldots, u_4$, then let $Q_j = \sum_{i=1}^4 q_{u_i}$. Identifying w_j^s for $s = 0, 2, 4, 6$ with the expression obtained by using q_j, h_j and Q_j, we get the following system of equations

$$\begin{aligned}
\alpha_{0,j}^2 + 2\alpha_{1,j}^2 + 2\alpha_{2,j}^2 + 2\alpha_{3,j}^2 + 2\alpha_{4,j}^2 &= 1,\\
2\alpha_{1,j}^2 + 4\cdot 2\alpha_{2,j}^2 + 9\cdot 2\alpha_{3,j}^2 + 16\cdot 2\alpha_{4,j}^2 &= 4,\\
2\alpha_{1,j}^2 + 16\cdot 2\alpha_{2,j}^2 + 81\cdot 2\alpha_{3,j}^2 + 256\cdot 2\alpha_{4,j}^2 &= 28 + 2q_j,\\
2\alpha_{1,j}^2 + 64\cdot 2\alpha_{2,j}^2 + 729\cdot 2\alpha_{3,j}^2 + 4096\cdot 2\alpha_{4,j}^2 &= 232 + 28q_j + 2Q_j + 2h_j.
\end{aligned}$$

Using these equations, Stevanović [**64**] has shown the non-existence of graphs with more than 500 of the spectra from [**28**]. The technique of obtaining the non-existence results is an extension of the technique with spectral moments used in previous work (cf. [**15**], [**28**]).

Since there are only five possible spectra with $n > 630$ and $q = h = 0$, we have the following theorem.

Theorem 26. ([**64, 65**]) *A connected* 4*-regular bipartite integral graph has at most* 1260 *vertices, unless it has (if exists) one of the following spectra:*

1° $[4, 3^{208}, 2^{172}, 1^{304}, 0^{70}, -1^{304}, -2^{172}, -3^{208}, -4]$ *and* 1440 *vertices;*

2° $[4, 3^{244}, 2^{196}, 1^{364}, 0^{70}, -1^{364}, -2^{196}, -3^{244}, -4]$ *and* 1680 *vertices;*

3° $[4, 3^{370}, 2^{280}, 1^{574}, 0^{70}, -1^{574}, -2^{280}, -3^{370}, -4]$ *and* 2520 *vertices;*

4° $[4, 3^{496}, 2^{364}, 1^{784}, 0^{70}, -1^{784}, -2^{364}, -3^{496}, -4]$ *and* 3360 *vertices;*

5° $[4, 3^{748}, 2^{532}, 1^{1204}, 0^{70}, -1^{1204}, -2^{532}, -3^{748}, -4]$ *and* 5040 *vertices.*

If G is a connected non-bipartite 4-regular integral graph, then $G \times K_2$ is a connected bipartite 4-regular integral graph, and we obtain the following corollary.

Corollary. *A connected* 4*-regular non-bipartite integral graph* G *has at most* 630 *vertices, unless* $G \times K_2$ *has one of the spectra* 1° – 5° *of Theorem* 26.

4-regular integral graphs avoiding ±3 in the spectrum have been considered in [**65**]. The possible spectra of 4-regular bipartite integral graphs with $x = 0$ have been found in [**65**] by D. Stevanović and they are shown in Table 5.1. while these graphs are depicted in Figs. 5.7. and 5.8. of [**54**].

There are 16 bipartite 4-regular integral graphs avoiding ±3 in the spectrum and they are shown in Fig. 5.7 of [**54**]. The smallest one is $D_1 = K_{4,4}$, while D_{11} and D_{16} with 32 and 30 vertices, respectively, are the largest such graphs. There are two triples of cospectral nonisomorphic graphs: (D_{13}, D_{14}, D_{15}) with 20 vertices and (D_{10}, D_{11}, D_{12}) with 18 vertices. There are also two pairs of cospectral nonisomorphic graphs: (D_7, D_8) with 24 vertices and (D_5, D_6) with 16 vertices.

Non-bipartite 4-regular integral graphs avoiding ±3 in the spectrum were found from the decompositions of graphs D_1–D_{16} in the form $H \times K_2$.

n	x	y	z	w	q	h
4	0	2	0	3	36	96
6	0	2	0	3	30	112
8	0	4	0	3	24	128
12	0	8	0	3	12	120
16	0	12	0	3	0	192

n	x	y	z	w	q	h
5	0	0	4	0	30	130
6	0	1	4	0	27	138
9	0	4	4	0	18	162
10	0	5	4	0	15	170
12	0	7	4	0	9	186
15	0	10	4	0	0	210

Table 1: Possible integral graph spectra with $x = 0$

There are 8 non-bipartite 4-regular integral graphs avoiding ±3 in the spectrum and they are shown in Fig. 5.8 of [**54**]. The smallest such graph is $E_1 = K_5$, while the largest one is E_8 having 15 vertices. Among these graphs there is one pair of cospectral nonisomorphic graphs with 12 vertices (E_4, E_5). Graphs E_2 and E_7 are strongly regular, while E_7 is also self-complementary. Graphs $E_1, \ldots, E_8$ have the least eigenvalue equal to -2 and they are either line graphs or cocktail-party graphs, except E_4 (which is one of the graphs found in [**9**]).

We have already seen (Theorem 1) that NEPS of graphs is regular and integral if and only if each of its factors is regular and integral. Further, D. Stevanović [**65**] proved that NEPS of graphs which are themselves representable as NEPS is isomorphic to NEPS of their factors with suitable basis, so that when looking for factors of 4-regular integral NEPS, we can consider only those which are not representable as NEPS of graphs.

Let $G = \mathrm{NEPS}(G_1, \ldots, G_n; \mathcal{B})$, where $G_1, \ldots, G_n$ are connected regular integral graphs which are not representable as NEPS of graphs. Let r_i be the degree of G_i, $i = 1, \ldots, n$ and suppose that $r_1 \geq \cdots \geq r_n$. G is regular with degree $\sum_{\beta \in \mathcal{B}} \prod_{i=1}^{n} r_i^{\beta_i}$ from which we see that $r_1 \leq 4$.

If $r_1 \leq 2$, the possible factors of NEPS are C_3 and K_2. There are fourteen non-isomorphic 4-regular NEPS with these factors:

$$
\begin{aligned}
N_1 &= \mathrm{NEPS}(K_2, K_2, K_2; \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}) = D_1\\
N_2 &= \mathrm{NEPS}(K_2, K_2, K_2; \{(1,0,0), (0,1,0), (0,0,1), (1,1,0)\}) = E_3\\
N_3 &= K_2 + K_2 + K_2 + K_2 = D_5\\
N_4 &= C_3 \times C_3\\
N_5 &= C_3 \times C_3 \times K_2\\
N_6 &= K_2 \oplus C_3\\
N_7 &= \mathrm{NEPS}(C_3, K_2, K_2; \{(1,0,1), (1,1,0)\})\\
N_8 &= \mathrm{NEPS}(C_3, C_3, K_2; \{(1,0,0), (0,1,1)\})\\
N_9 &= \mathrm{NEPS}(C_3, C_3, K_2; \{(1,0,1), (0,1,1)\})\\
N_{10} &= \mathrm{NEPS}(C_3, C_3, K_2, K_2; \{(1,0,0,1), (0,1,1,0)\})\\
N_{11} &= C_3 + K_2 + K_2\\
N_{12} &= \mathrm{NEPS}(C_3, K_2, K_2; \{(1,0,1), (0,1,0), (0,0,1)\})\\
N_{13} &= \mathrm{NEPS}(C_3, K_2, K_2; \{(1,0,1), (0,1,0), (0,1,1)\})\\
N_{14} &= \mathrm{NEPS}(C_3, K_2, K_2, K_2; \{(1,0,0,1), (0,1,0,0), (0,0,1,0)\})
\end{aligned}
$$

Here, $\oplus$ denotes NEPS with basis $\{(1,1), (0,1)\}$.

If $3 \leq r_1$, one of the factors is an r_1-regular integral graph and remaining factors are isomorphic to K_2.

Theorem 27. ([**65**]) *Let G be 4-regular integral* NEPS *of graphs. Then one of the following holds*:

1° *G is isomorphic to one of the graphs N_1–N_{14};*

2° $G = H + K_2$, *where* H *is a* 3*-regular integral graph;*

3° $G = H \oplus K_2$, *where* H *is a* 3*-regular non-bipartite integral graph;*

4° $G = H \times K_2$, *where* H *is a* 4*-regular non-bipartite integral graph.*

6. GRAPHS UP TO TWELVE VERTICES

"There are exactly 150 connected integral graphs up to ten vertices". The paper [**1**] under this title by K. Balińska, D. Cvetković, M. Lepović and S. Simić has appeared recently, finalizing the research whose partial results had already been obtained formerly (see [**4**]). The aid of a computer was essential for the completing of this list: namely, while integral graphs up to 7 vertices can easily be identified in the already published tables of graph spectra, finding the rest was enabled by the fact that the files of all graphs on 8, 9 and 10 vertices have recently been generated by a computer.

The numbers i_n of connected integral graphs on n $leq10$ vertices are given in the following table, along with those on 11 and 12 vertices (see below).

n	1	2	3	4	5	6	7	8	9	10	11	12
i_n	1	1	1	2	3	6	7	22	24	83	236	325

Cases $n \leq 5$ are obtained easily from the tables of graph spectra of [**20**], the six graphs on 6 vertices can be extracted from the table of [**25**], while the seven graphs on 7 vertices are selected form the Table of [**19**].

Besides these graphs, [**1**] contains a complete list of connected integral graphs on 8, 9 and 10 vertices, together with their spectra (in the sets of cospectral graphs such graphs are ordered by their angles).

Completing the list of connected integral graphs up to ten vertices enabled also some interesting observations and conclusions concerning cospectral connected integral graphs, integral complementary pairs, self-complementary graphs, cospectral complements of cospectral integral graphs, etc.

Let us also note that, among the other results in [**4**], one can find the automorphism groups of all integral graphs up to nine vertices and a part of the graphs on ten vertices. A fact that may be interesting is that all those graphs have at least one non-trivial automorphism.

These results have been recently extended to integral graphs up to 12 vertices [**2**, **3**]. The evolutionary algorithm was deviced for that purpose, and the obtained results were verified by the brute force method (on a supercomputer). The generation of integral (connected) graphs on 13 vertices is still in progress.

7. THE LAPLACIAN SPECTRUM

Let us consider now the Laplacian matrix $C = D - A$, D being the diagonal matrix of the vertex degrees and A the adjacency matrix. The matrix C is positive semidefinite and its rank is $n - w(G)$, where $w(G)$ is the number of connected components of a graph G. For the corresponding Laplacian spectrum $(\lambda_1, \lambda_2, \dots, \lambda_n)$ we shall assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$. Of course, $\lambda_n = 0$ always holds because of the rank of C.

Graphs with integral Laplacian eigenvalues will be called *Laplacian integral.*

When considering integral and Laplacian integral graphs, one can see great differences. A good example is the set of all 112 connected graphs on six vertices. As we saw in Section 6, there are six of them which are integral, five of them being regular. These five regular graphs are also Laplacian integral, while the only non-regular one is not, and, on the other hand, there are 37 other connected graphs on six vertices which are Laplacian integral.

As for regular graphs, since in that case $C + A = rI$, λ is an eigenvalue of C if and only if $r - \lambda$ is an eigenvalue of A. That means that a regular graph is Laplacian integral if and only if it is integral.

However, the situation with trees is quite different. In Section 2 we saw the state of matter with integral trees. But if we consider Laplacian spectrum of a tree, it turns out that $\lambda_{n-1} < 1$ unless we have a star $K_{1,n-1}$ (whose spectrum is $(n, 1, 1, \dots, 1, 0)$). Thus, a tree is Laplacian integral if and only if it is a star.

Another great difference concerns complements. Since $C(G)+C(\overline{G}) = nI - J$ (J consisting entirely of 1's), the eigenvalues of $C(\overline{G})$ are $\lambda_i(\overline{G}) = n-\lambda_{n-i}(G) \quad (1 \leq i \leq n-1)$, and 0, which means that G and $\overline{G}$ can only together be Laplacian integral. For example, if we make the graph G_n by subdividing an edge of $K_{n-1} (n > 2)$, we immediately know it is Laplacian integral, since $\overline{G_n}$ consists of one copy of K_2 and one copy of $K_{1,n-3}$.

Some graph operations, when applied to integral graphs, can also in the Laplacian case give rise to integral graphs. Thus, $G_1 \bigtriangledown G_2 = \overline{(\overline{G_1} \cup \overline{G_2})}$, i.e. the complete product of graphs, being the complement of the disjoint union (direct sum) of their complements, is one of such operations. In particular, if n_1 and n_2 are the respective numbers of vertices of G_1 and G_2, the eigenvalues of $C(G_1 \bigtriangledown G_2)$ are: 0, $n_1 + n_2$, $n_2 + \lambda_i(G_1)$ $(1 \leq i < n_1)$ and $n_1 + \lambda_i(G_2)$ $(1 \leq i < n_2)$. Another such example is the sum of graphs, having as the eigenvalues all possible sums $\lambda_i(G_1) + \lambda_j(G_2)$ $(1 \leq i \leq n_1,\ 1 \leq j \leq n_2)$.

Some interesting additional results can be found in [**31**] and [**53**].

Theorem 28. [**31**] *Let G be a connected, r-regular, Laplacian integral graph on n vertices. Then its subdivision graph $S(G)$ is Laplacian integral if and only if $G = K_n$.*

Theorem 29. [**53**] *Let G be a connected, (r, s)-semiregular, Laplacian integral graph. Then its line graph $L(G)$ is Laplacian integral.*

The most interesting and remarkable result concerning Laplacian integral spectra is expressed by a theorem about the so called maximal graphs.

Let $d(G) = (d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n$ are the degrees of the vertices of a graph G. Conversely, an arbitrary partition of $2m$ is said to be *graphic* if there exists a graph (on m edges) with such vertex degrees.

Given an arbitrary partition $(a) = (a_1, a_2, \ldots, a_n)$, where $a_1 \geq a_2 \geq \cdots \geq a_n$, let $f(a) = |\{i : a_i \geq i\}|$ be the trace of (a). Let $(a^*) = (a_1^*, a_2^*, \ldots)$, where $a_i^* = |\{j : a_j \geq i\}|$, be the conjugate of (a). It is known that a partition (a) of $2m$ is graphic if and only if the following condition holds:

$$\sum_{i=1}^{k}(a_i + 1) \leq \sum_{i=1}^{k} a_i^*.$$

If (d) is a graphic partition and (d) majorizes (a) (see, e.g. [**54**]), it follows that (a) is graphic, too.

A graph G is said to be *maximal* if there is no other graphic partition that majorizes $d(G)$. According to the above condition, G is maximal if and only if $d(G)$ satisfies $d_i + 1 = d_i^* \ \big(i = 1, 2, \ldots, f(d(G))\big)$.

Apart from isolated vertices, maximal graphs are characterized by their Laplacian spectra in a really impressive way.

Theorem 30. [**51**] *Let G be a graph with no isolated vertices. Then G is maximal if and only if the conjugate of its degree sequence is identical to its nonzero Laplacian spectrum.*

Some conditions under which a Laplacian integral graph preserves this property when adding an edge are studied in [**79**].

8. OTHER TOPICS

In this section we shall consider problems with integral eigenvalues in two-graphs and digraphs.

A *two-graph* $(X, \triangle)$ is a pair of a vertex set X and a set $\triangle$ of 3-subsets of X such that each 4-subset of X contains an even number of triples of $\triangle$.

The *Seidel spectrum* or the *S-spectrum* of a graph is the spectrum of its $(-1, 1, 0)$ adjacency matrix (having -1 as its (i, j)-entry if vertices i and j are adjacent, 1 if they are not adjacent and 0 if $i = j$).

If A is the usual adjacency matrix, then $S = J - I - 2A$, J being a square matrix whose all entries are equal to 1. Suppose now U is a diagonal matrix whose diagonal entries are only 1 or -1, which means that U is self-inverse. Then USU is similar and cospectral with S. Let V_1 be the set of vertices of a graph G corresponding to those diagonal entries which are equal to -1 and V_2 the rest of vertices, and we see that now USU is the $(-1, 1, 0)$ adjacency matrix of the graph in which two vertices in V_1 or in V_2 are adjacent if and only if they were adjacent in G, while now two vertices of different subsets are adjacent if and only if they were not adjacent in G. This way of forming a graph cospectral with G is called *Seidel switching.* It generates an equivalence relation in the set of graphs. Given n, there is a one-to-one correspondence between the two-graphs and the switching classes of graphs on n vertices (for the proof see e.g. [**20**], p. 200).

Therefore, the spectrum of a two-graph is naturally defined as the S-spectrum of its corresponding switching class of graphs.

The report [**12**] contains a lot of facts on two-graphs (obtained mainly by a computer) and some of them concern integral two-graphs (there are 22 integral two-graphs up to nine vertices).

Next we consider digraphs.

Contrary to (non-oriented) graphs, whose spectra are real, the eigenvalues of digraphs are complex numbers. A complex number $\lambda = \alpha + i\beta$ is called a *Gaussian integer* if α and β are integers. A digraph is called *Gaussian* if its spectrum consists only of Gaussian integers. Of course, if it comes about that all of them are real integers, such digraph will be called integral. Some results on Gaussian and integral digraphs one can find in [**30**].

As in the case of integral graphs, once having two Gaussian digraphs (e.g. C_4 and its complement) we can produce arbitrarily large families of Gaussian digraphs by means of well known graph operations (for the definition of the NEPS for digraphs see [**24**]).

As for integral digraphs, we note that there is an interesting example of two cospectral integral digraphs with four vertices (Fig. 5.11. of [**54**]), which are the smallest such digraphs.

Also, the following theorem holds.

Theorem 31. [**30**] *For any positive integer n we can find n cospectral strongly connected non-symmetric digraphs which are integral.*

Added in proof: P. Híc and S. Pokorný have announced that they have found an infinite family of integral trees of diameter ten.

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