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Jacobsthal Numbers and Alternating Sign Matrices

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Abstract

Let $A(n)$ denote the number of $n \times n$ alternating sign matrices and J_m the m^{th} Jacobsthal number. It is known that

$$A(n) = \prod_{\ell=0}^{n-1} \frac{(3\ell+1)!}{(n+\ell)!}.$$

The values of $A(n)$ are in general highly composite. The goal of this paper is to prove that $A(n)$ is odd if and only if n is a Jacobsthal number, thus showing that $A(n)$ is odd infinitely often.

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1 Introduction

In this paper we relate two seemingly unrelated areas of mathematics: alternating sign matrices and Jacobsthal numbers. We begin with a brief discussion of alternating sign matrices.

An $n \times n$ alternating sign matrix is an $n \times n$ matrix of 1s, 0s and -1s such that

- the sum of the entries in each row and column is 1, and
- the signs of the nonzero entries in every row and column alternate.

Alternating sign matrices include permutation matrices, in which each row and column contains only one nonzero entry, a 1.

For example, the seven 3×3 alternating sign matrices are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The determination of a closed formula for $A(n)$ was undertaken by a variety of mathematicians over the last 25 years or so. David Bressoud's text [1] chronicles these endeavors and discusses the underlying mathematics in a very readable way. See also the survey article [2] by Bressoud and Propp.

As noted in [1], a formula for $A(n)$ is given by

$$A(n) = \prod_{\ell=0}^{n-1} \frac{(3\ell+1)!}{(n+\ell)!}. \tag{1}$$

It is clear from this that, for most values of n, $A(n)$ will be highly composite. The following table shows the first few values of $A(n)$ (sequence A005130 in [8]). Other sequences related to alternating sign matrices can also be found in [8].

n	$A(n)$	Prime Factorization of $A(n)$
1	1	1
2	2	2
3	7	7
4	42	$2 \cdot 3 \cdot 7$
5	429	$3 \cdot 11 \cdot 13$
6	7436	$2^2 \cdot 11 \cdot 13^2$
7	218348	$2^2 \cdot 13^2 \cdot 17 \cdot 19$
8	10850216	$2^3 \cdot 13 \cdot 17^2 \cdot 19^2$
9	911835460	$2^2 \cdot 5 \cdot 17^2 \cdot 19^3 \cdot 23$
10	129534272700	$2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17 \cdot 19^3 \cdot 23^2$
11	31095744852375	$3^2 \cdot 5^3 \cdot 7 \cdot 19^2 \cdot 23^3 \cdot 29 \cdot 31$
12	12611311859677500	$2^2 \cdot 3^3 \cdot 5^4 \cdot 19 \cdot 23^3 \cdot 29^2 \cdot 31^2$
13	8639383518297652500	$2^2 \cdot 3^5 \cdot 5^4 \cdot 23^2 \cdot 29^3 \cdot 31^3 \cdot 37$
14	9995541355448167482000	$2^4 \cdot 3^5 \cdot 5^3 \cdot 23 \cdot 29^4 \cdot 31^4 \cdot 37^2$
15	19529076234661277104897200	$2^4 \cdot 3^3 \cdot 5^2 \cdot 29^4 \cdot 31^5 \cdot 37^3 \cdot 41 \cdot 43$
16	64427185703425689356896743840	$2^5 \cdot 3^2 \cdot 5 \cdot 11 \cdot 29^3 \cdot 31^5 \cdot 37^4 \cdot 41^2 \cdot 43^2$
17	358869201916137601447486156417296	$2^4 \cdot 3 \cdot 7^2 \cdot 11 \cdot 29^2 \cdot 31^4 \cdot 37^5 \cdot 41^3 \cdot 43^3 \cdot 47$
18	3374860639258750562269514491522925456	$2^4 \cdot 7^3 \cdot 13 \cdot 29 \cdot 31^3 \cdot 37^6 \cdot 41^4 \cdot 43^4 \cdot 47^2$
19	53580350833984348888878646149709092313244	$2^2 \cdot 7^3 \cdot 13^2 \cdot 31^2 \cdot 37^6 \cdot 41^5 \cdot 43^5 \cdot 47^3 \cdot 53$
20	1436038934715538200913155682637051204376827212	$2^2 \cdot 7^4 \cdot 13^2 \cdot 31 \cdot 37^5 \cdot 41^6 \cdot 43^6 \cdot 47^4 \cdot 53^2$
21	64971294999808427895847904380524143538858551437757	$7^5 \cdot 13 \cdot 37^4 \cdot 41^6 \cdot 43^7 \cdot 47^5 \cdot 53^3 \cdot 59 \cdot 61$
22	4962007838317808727469503296608693231827094217799731304	$2^3 \cdot 3 \cdot 7^6 \cdot 37^3 \cdot 41^5 \cdot 43^7 \cdot 47^6 \cdot 53^4 \cdot 59^2 \cdot 61^2$

Table 1: Values of $A(n)$

Examination of this table and further computer calculations reveals that the first few values of n for which $A(n)$ is odd are

$$1, 3, 5, 11, 21, 43, 85, 171.$$

These appear to be the well–known *Jacobsthal numbers* $\{J_n\}$ (sequence A001045 in [8]). They are defined by the recurrence

$$J_{n+2} = J_{n+1} + 2J_n\ , \tag{2}$$

with initial values $J_0 = 1$ and $J_1 = 1$.

This sequence has a rich history, especially in view of its relationship to the Fibonacci numbers. For examples of recent work involving the Jacobsthal numbers, see [3], [4], [5] and [6].

The goal of this paper is to prove that this is no coincidence: for a positive integer n, $A(n)$ is odd if and only if n is a Jacobsthal number.

2 The Necessary Machinery

To show that $A(J_m)$ is odd for each positive integer m, we will show that the number of factors of 2 in the prime decomposition of $A(J_m)$ is zero. To accomplish this, we develop formulas for the number of factors of 2 in

$$N(n) = \prod_{\ell=0}^{n-1} (3\ell+1)! \qquad \text{and} \qquad D(n) = \prod_{\ell=0}^{n-1} (n+\ell)!\ .$$

Once we prove that the number of factors of 2 is the same for $N(J_m)$ and $D(J_m)$, but not the same for $N(n)$ and $D(n)$ if n is not a Jacobsthal number, we will have our result.

We will make frequent use of the following lemma. For a proof, see for example [7, Theorem 2.29].

Lemma 2.1. *The number of factors of a prime p in $N!$ is equal to*

$$\sum_{k\geq 1} \left\lfloor \frac{N}{p^k} \right\rfloor .$$

It follows that the number of factors of 2 in $N(n)$ is

$$N^{\#}(n) = \sum_{\ell=0}^{n-1} \sum_{k\geq 1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = \sum_{k\geq 1} N_k^{\#}(n)$$

where

$$N_k^{\#}(n) = \sum_{\ell=0}^{n-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor . \tag{3}$$

Similarly, the number of factors of 2 in $D(n)$ is given by

$$D^{\#}(n) = \sum_{\ell=0}^{n-1} \sum_{k\geq 1} \left\lfloor \frac{n+\ell}{2^k} \right\rfloor = \sum_{k\geq 1} D_k^{\#}(n)$$

where

$$D_k^{\#}(n) = \sum_{\ell=0}^{n-1} \left\lfloor \frac{n+\ell}{2^k} \right\rfloor . \tag{4}$$

For use below we note that the recurrence for the Jacobsthal numbers implies the following explicit formula (cf. [9]).

Theorem 2.2. *The m^{th} Jacobsthal number J_m is given by*

$$J_m = \frac{2^{m+1} + (-1)^m}{3}. \tag{5}$$

3 Formulas for $N_k^{\#}(n)$ and $D_k^{\#}(n)$

Lemma 3.1. *The smallest value of* ℓ *for which*

$$\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = m,$$

where m *and* k *are positive integers and* $k \geq 2$*, is*

$$\begin{cases} \frac{m}{3}2^k & \text{if} \quad m \equiv 0 \pmod 3 \\ \frac{m-1}{3}2^k + J_{k-1} & \text{if} \quad m \equiv 1 \pmod 3 \\ \frac{m-2}{3}2^k + J_k & \text{if} \quad m \equiv 2 \pmod 3. \end{cases}$$

Proof. Suppose $m \equiv 0 \pmod 3$ and $\ell = \dfrac{m}{3}2^k$. Then

$$\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = \left\lfloor \frac{3\left(\frac{m}{3}2^k\right)+1}{2^k} \right\rfloor = \left\lfloor \frac{m2^k}{2^k} + \frac{1}{2^k} \right\rfloor = m,$$

and no smaller value of ℓ yields m since the numerators differ by multiples of three.

If $m \equiv 1 \pmod 3$ and $\ell = \dfrac{m-1}{3}2^k + J_{k-1}$, then

$$\begin{aligned} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor &= \left\lfloor \frac{3\left(\frac{m-1}{3}2^k + J_{k-1}\right)+1}{2^k} \right\rfloor \\ &= \left\lfloor \frac{(m-1)2^k + 3\left(\frac{2^k+(-1)^{k-1}}{3}\right)+1}{2^k} \right\rfloor \\ &= \left\lfloor \frac{(m-1)2^k + 2^k + (-1)^{k-1}+1}{2^k} \right\rfloor \\ &= m, \quad \text{if } k \geq 2, \end{aligned}$$

and no smaller value of ℓ yields m.

If $m \equiv 2 \pmod 3$ and $\ell = \dfrac{m-2}{3}2^k + J_k$, then

$$\begin{aligned} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor &= \left\lfloor \frac{3\left(\frac{m-2}{3}2^k + J_k\right)+1}{2^k} \right\rfloor \\ &= \left\lfloor \frac{(m-2)2^k + 3\left(\frac{2^{k+1}+(-1)^k}{3}\right)+1}{2^k} \right\rfloor \\ &= \left\lfloor \frac{(m-2)2^k + 2^{k+1} + (-1)^k+1}{2^k} \right\rfloor \\ &= m, \end{aligned}$$

and no smaller value of ℓ yields m.

□

Lemma 3.2. *For any positive integer* k*,* $J_{k-1} + J_k = 2^k$*.*

Proof. Immediate from (5). □

Lemma 3.3. *For any positive integer k,*

$$\sum_{v=0}^{2^k-1} \left\lfloor \frac{3v+1}{2^k} \right\rfloor = 2^k.$$

Proof. The result is immediate if $k = 1$. If $k \geq 2$, then by Lemma 3.1, J_{k-1} is the smallest value of v for which $\left\lfloor \frac{3v+1}{2^k} \right\rfloor = 1$ and J_k is the smallest value of v for which $\left\lfloor \frac{3v+1}{2^k} \right\rfloor = 2$. Thus

$$\begin{aligned}
\sum_{v=0}^{2^k-1} \left\lfloor \frac{3v+1}{2^k} \right\rfloor &= 0 \times J_{k-1} + 1 \times [(J_k - 1) - (J_{k-1} - 1)] + 2 \times [(2^k - 1) - (J_k - 1)] \\
&= J_k - J_{k-1} + 2(2^k - J_k) \\
&= 2^{k+1} - 2^k \quad \text{by Lemma 3.2} \\
&= 2^k \, .
\end{aligned}$$

□

Theorem 3.4. *Let $n = 2^k q + r$, where q is a nonnegative integer and $0 \leq r < 2^k$. Then*

$$N_k^{\#}(n) = \left(\frac{n-r}{2^{k+1}}\right)(3(n-r) - 2^k) + tail(n) \tag{6}$$

where

$$tail(n) = \begin{cases} 3qr & \text{if} \quad 0 \leq r \leq J_{k-1} \\ 3qr + (r - J_{k-1}) & \text{if} \quad J_{k-1} < r \leq J_k \\ (3q+2)r - 2^k & \text{if} \quad J_k < r < 2^k. \end{cases} \tag{7}$$

Proof. To analyze the sum

$$N_k^{\#}(n) = \sum_{\ell=0}^{n-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor$$

we let $\ell = 2^k u + v$, where $0 \leq v < 2^k$. Then

$$\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = \left\lfloor \frac{3(2^k u + v) + 1}{2^k} \right\rfloor = \left\lfloor \frac{2^k(3u)}{2^k} + \frac{3v+1}{2^k} \right\rfloor = 3u + \left\lfloor \frac{3v+1}{2^k} \right\rfloor.$$

Thus

$$\begin{aligned}
\sum_{\ell=0}^{2^k q-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor &= \sum_{u=0}^{q-1} \sum_{v=0}^{2^k-1} \left(3u + \left\lfloor \frac{3v+1}{2^k} \right\rfloor\right) \\
&= \sum_{u=0}^{q-1} \left((3u)2^k + \sum_{v=0}^{2^k-1} \left\lfloor \frac{3v+1}{2^k} \right\rfloor\right) \\
&= \sum_{u=0}^{q-1} ((3u)2^k + 2^k) \quad \text{by Lemma 3.3} \\
&= 2^k \sum_{u=0}^{q-1} (3u+1) \\
&= 2^k \left(3\left(\frac{(q-1)q}{2}\right) + q\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2^k q \left(3\left(\frac{n-r-2^k}{2^{k+1}}\right)+1\right) \\
&= \left(\frac{q}{2}\right)(3(n-r-2^k)+2^{k+1}) \\
&= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r)-2^k).
\end{aligned}
$$

If $r=0$, we have our result. If $r>0$ and $k=1$, then $r=1$ and we have one extra term in our sum, namely,

$$\left\lfloor \frac{3(2q)+1}{2} \right\rfloor = 3q$$

and again we have our result since $r=1$. If $r>0$ and $k\geq 2$, then by Lemma 3.1, $2^k q$ is the smallest value of ℓ for which $\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = 3q$, $2^k q + J_{k-1}$ is the smallest value of ℓ for which

$$\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = 3q+1,$$

and $2^k q + J_k$ is the smallest value of ℓ for which

$$\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = 3q+2.$$

Hence

$$\sum_{\ell=2^k q}^{2^k q+r-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = \begin{cases} 3qr & \text{if } \quad r \leq J_{k-1} \\ 3qJ_{k-1} + (3q+1)(r-J_{k-1}) & \text{if } J_{k-1} < r \leq J_k \\ 3qJ_{k-1} + (3q+1)(J_k - J_{k-1}) + (3q+2)(r-J_k) & \text{if } J_k < r < 2^k. \end{cases}$$

So, if $n = 2^k q + r$ where $0 \leq r < 2^k$,

$$
\begin{aligned}
N_k^{\#}(n) &= \sum_{\ell=0}^{n-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor \\
&= \sum_{\ell=0}^{2^k q-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor + \sum_{\ell=2^k q}^{2^k q+r-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor \\
&= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r)-2^k) + \mathit{tail}(n),
\end{aligned}
$$

where

$$\mathit{tail}(n) = \begin{cases} 3qr & \text{if } \quad r \leq J_{k-1} \\ 3qJ_{k-1} + (3q+1)(r-J_{k-1}) & \text{if } J_{k-1} < r \leq J_k \\ 3qJ_{k-1} + (3q+1)(J_k - J_{k-1}) + (3q+2)(r-J_k) & \text{if } J_k < r < 2^k. \end{cases}$$

The second expression in $\mathit{tail}(n)$ is clearly equal to $3qr + r - J_{k-1}$. For the third expression, we have

$$
\begin{aligned}
3qJ_{k-1} + (3q+1)(J_k - J_{k-1}) + (3q+2)(r-J_k) &= 3qr + J_k - J_{k-1} + 2r - 2J_k \\
&= (3q+2)r - 2^k \quad \text{by Lemma 3.2.}
\end{aligned}
$$

□

Theorem 3.5. *Let $n = 2^k q + r$ where q is a nonnegative integer and $0 \le r < 2^k$. Then we have*

$$D_k^{\#}(n) = \begin{cases} \left(\dfrac{n-r}{2^{k+1}}\right)(3(n+r)-2^k) & \text{if} \quad 0 \le r \le 2^{k-1} \\ \left(\dfrac{n-(2^k-r)}{2^{k+1}}\right)(3(n-r)+2^{k+1}) & \text{if} \quad 2^{k-1} < r < 2^k. \end{cases} \tag{8}$$

Proof. We may write

$$D_k^{\#}(n) = \sum_{\ell=0}^{2n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor - \sum_{\ell=0}^{n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor .$$

In both sums,

$$\left\lfloor \frac{\ell}{2^k} \right\rfloor = s\,,$$

if $2^k s \le \ell < 2^k(s+1)$, so if $n = 2^k q + r$, where $0 < r \le 2^k$, we have

$$\begin{aligned} \sum_{\ell=0}^{n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor &= 2^k[1+2+\cdots+q-1] + qr \\ &= q\left(\frac{n+r-2^k}{2}\right). \end{aligned}$$

If $0 < r \le 2^{k-1}$, then $2n-1 = 2^k(2q) + (2r-1)$, which means

$$\begin{aligned} \sum_{\ell=0}^{2n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor &= 2^k[1+2+\cdots+(2q-1)] + (2r-1+1)(2q) \\ &= q(2n+2r-2^k). \end{aligned}$$

Hence in this case

$$\begin{aligned} D_k^{\#}(n) &= \sum_{\ell=0}^{n-1} \left\lfloor \frac{n+\ell}{2^k} \right\rfloor \\ &= \sum_{\ell=0}^{2n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor - \sum_{\ell=0}^{n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor \\ &= q(2n+2r-2^k) - q\left(\frac{n+r-2^k}{2}\right) \\ &= \left(\frac{n-r}{2^{k+1}}\right)(3(n+r)-2^k). \end{aligned}$$

If $2^{k-1} < r \le 2^k$, say, $r = 2^{k-1} + s$ where $0 < s \le 2^{k-1}$, then

$$\begin{aligned} 2n-1 &= 2(2^k q + r) - 1 \\ &= 2^k(2q) + 2(2^{k-1}+s) - 1 \\ &= 2^k(2q+1) + 2s - 1. \end{aligned}$$

Thus

$$\begin{aligned}\sum_{\ell=0}^{2n-1}\left\lfloor\frac{\ell}{2^k}\right\rfloor &= 2^k[1+2+\cdots+2q]+(2s-1+1)(2q+1)\\ &= (2q+1)(n+r-2^k).\end{aligned}$$

So in this case

$$\begin{aligned}D_k^{\#}(n) &= \sum_{\ell=0}^{n-1}\left\lfloor\frac{n+\ell}{2^k}\right\rfloor\\ &= \sum_{\ell=0}^{2n-1}\left\lfloor\frac{\ell}{2^k}\right\rfloor-\sum_{\ell=0}^{n-1}\left\lfloor\frac{\ell}{2^k}\right\rfloor\\ &= (2q+1)(n+r-2^k)-q\left(\frac{n+r-2^k}{2}\right)\\ &= \left(\frac{n+r-2^k}{2^{k+1}}\right)(3(n-r)+2^{k+1}).\end{aligned}$$

The reader will note that in the statement of the theorem we have separated the cases according as $0\le r\le 2^{k-1}$ and $2^{k-1}<r<2^k$, whereas in the proof the cases are $0<r\le 2^{k-1}$ and $2^{k-1}<r\le 2^k$. However, these are equivalent since $\frac{n-0}{2^{k+1}}(3(n+0)-2^k)=\frac{n-(2^k-2^k)}{2^{k+1}}(3(n-2^k)+2^{k+1})$. □

4 $A(J_m)$ is odd

Now that we have closed formulas for $N_k^{\#}(n)$ and $D_k^{\#}(n)$ we can proceed to prove that $A(J_m)$ is odd for all Jacobsthal numbers J_m.

Theorem 4.1. *For all positive integers m, $A(J_m)$ is odd.*

Proof. The proof simply involves substituting J_m into (6) and (8) and showing that $N_k^{\#}(J_m)=D_k^{\#}(J_m)$ for all k. This implies that $N^{\#}(J_m)=D^{\#}(J_m)$, and so the number of factors of 2 in $A(J_m)$ is zero. Our theorem is then proved.

We break the proof into two cases, based on whether the parity of k is equal to the parity of m.

- **Case 1:** The parity of m equals the parity of k. Then

$$\begin{aligned}2^k(J_{m-k}-1)+J_k &= 2^k\left(\frac{2^{m-k+1}+(-1)^{m-k}}{3}-1\right)+\frac{2^{k+1}+(-1)^k}{3}\\ &= \frac{2^{m+1}+2^k-3\cdot 2^k+2^{k+1}+(-1)^k}{3} \quad \text{since } (-1)^{m-k}=1\\ &= \frac{2^{m+1}+(-1)^m}{3} \quad \text{since } (-1)^k=(-1)^m\\ &= J_m\end{aligned}$$

  Thus, in the notation of Theorems 3.4 and 3.5, $q=J_{m-k}-1$ and $r=J_k$. We now calculate $N_k^{\#}(J_m)$ and $D_k^{\#}(J_m)$ using Theorems 3.4 and 3.5.

$$
\begin{aligned}
N_k^{\#}(J_m) &= \left(\frac{J_m - J_k}{2^{k+1}}\right)\left(3(J_m - J_k) - 2^k\right) \\
&\quad + 3(J_{m-k} - 1)J_k + (J_k - J_{k-1}) \\
&= \frac{1}{2^{k+1}}\left(\frac{2^{m+1}+(-1)^m}{3} - \frac{2^{k+1}+(-1)^k}{3}\right)\left(3\left(\frac{2^{m+1}+(-1)^m}{3} - \frac{2^{k+1}+(-1)^k}{3}\right) - 2^k\right) \\
&\quad + (3J_{m-k} - 1)J_k - 2^k \quad \text{by Lemma 3.2} \\
&= \frac{1}{3\cdot 2^{k+1}}\left(2^{m+1} - 2^{k+1}\right)\left(2^{m+1} - 2^{k+1} - 2^k\right) \\
&\quad + \left(3\left(\frac{2^{m-k+1}+(-1)^{m-k}}{3}\right) - 1\right)\left(\frac{2^{k+1}+(-1)^k}{3}\right) - 2^k \quad \text{since } (-1)^m = (-1)^k \\
&= \frac{1}{3}\left(2^{2m-k+1} - 2^{m+2} + 2^{k+1} - 2^m + 2^k\right) \\
&\quad + \frac{1}{3}(2^{m-k+1}(2^{k+1} + (-1)^k) - 3\cdot 2^k) \quad \text{since } (-1)^{m-k} = 1 \\
&= \frac{1}{3}\left(2^{2m-k+1} - 2^m + (-1)^k 2^{m-k+1}\right)
\end{aligned}
$$

after much simplification. Next, we calculate $D_k^{\#}(J_m)$, recalling that $2^{k-1} < r = J_k < 2^k$.

$$
\begin{aligned}
D_k^{\#}(J_m) &= \frac{\left(J_m - 2^k + J_k\right)}{2^{k+1}}\left(3(J_m - J_k) + 2^{k+1}\right) \\
&= \frac{1}{2^{k+1}}\left(\frac{2^{m+1}+(-1)^m}{3} + \frac{2^{k+1}+(-1)^k}{3} - 2^k\right)\left(3\left(\frac{2^{m+1}+(-1)^m}{3} - \frac{2^{k+1}+(-1)^k}{3}\right) + 2^{k+1}\right) \\
&= \frac{1}{3\cdot 2^{k+1}}\left(2^{m+1} + 2^{k+1} + 2(-1)^k - 3\cdot 2^k\right)\left(2^{m+1} - 2^{k+1} + 2^{k+1}\right) \quad \text{since } (-1)^m = (-1)^k \\
&= \frac{1}{3}\left(2^{2m-k+1} + 2^{m+1} + 2^{m-k+1}(-1)^k - 3\cdot 2^m\right) \\
&= \frac{1}{3}\left(2^{2m-k+1} - 2^m + (-1)^k 2^{m-k+1}\right)
\end{aligned}
$$

after simplification. We see that $N_k^{\#}(J_m) = D_k^{\#}(J_m)$.

- **Case 2:** The parity of m is not equal to the parity of k. Then

$$
\begin{aligned}
2^k(J_{m-k}) + J_{k-1} &= 2^k\left(\frac{2^{m-k+1}+(-1)^{m-k}}{3}\right) + \frac{2^k+(-1)^{k-1}}{3} \\
&= \frac{2^{m+1} - 2^k + 2^k + (-1)^{k-1}}{3} \\
&= J_m.
\end{aligned}
$$

Thus, in the notation of Theorems 3.4 and 3.5, $q = J_{m-k}$ and $r = J_{k-1}$. We now calculate $N_k^{\#}(J_m)$ and $D_k^{\#}(J_m)$ using Theorems 3.4 and 3.5.

$$
\begin{aligned}
N_k^{\#}(J_m) &= \left(\frac{J_m - J_{k-1}}{2^{k+1}}\right)\left(3(J_m - J_{k-1}) - 2^k\right) \\
&\quad + 3J_{m-k}J_{k-1}
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{2^{k+1}}\left(\frac{2^{m+1}+(-1)^m}{3}-\frac{2^k+(-1)^{k-1}}{3}\right)\left(3\left(\frac{2^{m+1}+(-1)^m}{3}-\frac{2^k+(-1)^{k-1}}{3}\right)-2^k\right) \\
&\quad +3\left(\frac{2^{m-k+1}+(-1)^{m-k}}{3}\right)\left(\frac{2^k+(-1)^{k-1}}{3}\right) \\
&= \frac{1}{3\cdot 2^{k+1}}\left(2^{m+1}-2^k\right)\left(2^{m+1}-2\cdot 2^k\right) \\
&\quad +\frac{1}{3}((2^{m-k+1}-1)(2^k+(-1)^{k-1})) \ \text{ since } (-1)^m=(-1)^{k-1} \text{ and } (-1)^{m-k}=-1 \\
&= \frac{1}{3}(2^{2m-k+1}-2^{m+1}-2^m+2^k+2^{m+1}-2^k+2^{m-k+1}(-1)^{k-1}+(-1)^k) \\
&= \frac{1}{3}(2^{2m-k+1}-2^m+2^{m-k+1}(-1)^{k-1}+(-1)^k)
\end{aligned}$$

after much simplification. Again we find that $N_k^{\#}(J_m)=D_k^{\#}(J_m)$.

Now we calculate $D_k^{\#}(J_m)$, recalling that $0<r<2^{k-1}$.

$$\begin{aligned}
D_k^{\#}(J_m) &= \frac{(J_m-J_{k-1})}{2^{k+1}}(3(J_m+J_{k-1})-2^k) \\
&= \frac{1}{2^{k+1}}\left(\frac{2^{m+1}+(-1)^m}{3}-\frac{2^k+(-1)^{k-1}}{3}\right)\left(3\left(\frac{2^{m+1}+(-1)^m}{3}+\frac{2^k+(-1)^{k-1}}{3}\right)-2^k\right) \\
&= \frac{1}{3\cdot 2^{k+1}}(2^{m+1}-2^k)(2^{m+1}+2(-1)^{k-1}) \ \text{ since } (-1)^m=(-1)^{k-1} \\
&= \frac{1}{3}(2^{2m-k+1}-2^m+2^{m-k+1}(-1)^{k-1}+(-1)^k)
\end{aligned}$$

after simplification. Again we find that $N_k^{\#}(J_m)=D_k^{\#}(J_m)$.

This completes the proof that $A(J_m)$ is odd for all Jacobsthal numbers J_m.

□

5 The Converse

We now prove the converse to Theorem 4.1. That is, we will prove that $A(n)$ is even if n is not a Jacobsthal number. As a guide in how to proceed, we include a table of values for $N_k^{\#}(n)$ and $D_k^{\#}(n)$ for small values of n and k. This table suggests that $N_k^{\#}(n)\geq D_k^{\#}(n)$ for all positive integers n and k. It also suggests that for each value of n, there is at least one value of k for which $N_k^{\#}(n)$ is strictly greater than $D_k^{\#}(n)$ except when n is a Jacobsthal number. (The rows that begin with a Jacobsthal number are indicated in bold-face.)

n	$N_1^{\#}(n)$	$D_1^{\#}(n)$	$N_2^{\#}(n)$	$D_2^{\#}(n)$	$N_3^{\#}(n)$	$D_3^{\#}(n)$	$N_4^{\#}(n)$	$D_4^{\#}(n)$	$N_5^{\#}(n)$	$D_5^{\#}(n)$	$N_6^{\#}(n)$	$D_6^{\#}(n)$
1	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	2	2	1	0	0	0	0	0	0	0	0	0
3	**5**	**5**	**2**	**2**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
4	10	10	4	4	1	0	0	0	0	0	0	0
5	**16**	**16**	**7**	**7**	**2**	**2**	**0**	**0**	**0**	**0**	**0**	**0**
6	24	24	11	10	4	4	1	0	0	0	0	0
7	33	33	15	15	6	6	2	0	0	0	0	0
8	44	44	20	20	8	8	3	0	0	0	0	0
9	56	56	26	26	11	11	4	2	0	0	0	0
10	70	70	33	32	14	14	5	4	0	0	0	0
11	**85**	**85**	**40**	**40**	**17**	**17**	**6**	**6**	**0**	**0**	**0**	**0**
12	102	102	48	48	21	20	8	8	1	0	0	0
13	120	120	57	57	25	25	10	10	2	0	0	0
14	140	140	67	66	30	30	12	12	3	0	0	0
15	161	161	77	77	35	35	14	14	4	0	0	0
16	184	184	88	88	40	40	16	16	5	0	0	0
17	208	208	100	100	46	46	19	19	6	2	0	0
18	234	234	113	112	52	52	22	22	7	4	0	0
19	261	261	126	126	58	58	25	25	8	6	0	0
20	290	290	140	140	65	64	28	28	9	8	0	0
21	**320**	**320**	**155**	**155**	**72**	**72**	**31**	**31**	**10**	**10**	**0**	**0**
22	352	352	171	170	80	80	35	34	12	12	1	0
23	385	385	187	187	88	88	39	37	14	14	2	0
24	420	420	204	204	96	96	43	40	16	16	3	0
25	456	456	222	222	105	105	47	45	18	18	4	0

Table 2: Values for $N_k^{\#}(n)$ and $D_k^{\#}(n)$

(We note in passing that the values of $N_1^{\#}(n)$ form sequence A001859 in [8].)

In order to prove the first assertion (that $N_k^{\#}(n) \geq D_k^{\#}(n)$), we separate the functions defined by the cases in equations (6) and (8) into individual functions denoted by $N_k^{\#(1)}(n), N_k^{\#(2)}(n), \ldots, D_k^{\#(2)}(n)$. That is,

$$
\begin{aligned}
N_k^{\#(1)}(n) &:= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r)-2^k)+3qr \\
N_k^{\#(2)}(n) &:= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r)-2^k)+3qr+(r-J_{k-1}) \\
N_k^{\#(3)}(n) &:= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r)-2^k)+(3q+2)r-2^k \\
D_k^{\#(1)}(n) &:= \left(\frac{n-r}{2^{k+1}}\right)(3(n+r)-2^k) \\
D_k^{\#(2)}(n) &:= \left(\frac{n-(2^k-r)}{2^{k+1}}\right)(3(n-r)+2^{k+1})
\end{aligned}
$$

For a given value of n, $N_k^{\#}(n)$ will equal $N_k^{\#(i)}(n)$ for some $i \in \{1,2,3\}$ and $D_k^{\#}(n)$ will be $D_k^{\#(j)}(n)$ for some $j \in \{1,2\}$ depending on the value of r. Note that not all combinations of i and j are possible (for example, there is no value of n such that $i=1$ and $j=2$). In Lemmas 5.1 through 5.4 we show that $N_k^{\#(i)}(n) \geq D_k^{\#(j)}(n)$ for all possible combinations of i and j (that correspond to some integer n) which implies that $N_k^{\#}(n) \geq D_k^{\#}(n)$ for all positive integers n.

Lemma 5.1. *For all integers n and k, $N_k^{\#(1)}(n) = D_k^{\#(1)}(n)$.*

Proof. We first note that, in the notation of Theorem 3.4, $\frac{n-r}{2^{k+1}} = \frac{2^k q}{2^{k+1}} = \frac{q}{2}$. Then

$$
N_k^{\#(1)}(n) = \left(\frac{n-r}{2^{k+1}}\right)(3(n-r)-2^k)+3qr
$$

$$
\begin{aligned}
&= \left(\frac{n-r}{2^{k+1}}\right)\left(3(n-r) - 2^k + 3qr\left(\frac{2}{q}\right)\right) \quad \text{since } \frac{n-r}{2^{k+1}} = \frac{q}{2} \\
&= \left(\frac{n-r}{2^{k+1}}\right)(3n + 3r - 2^k) \\
&= D_k^{\#(1)}(n).
\end{aligned}
$$

□

Lemma 5.2. *For all integers k and all integers n such that $r > J_{k-1}$ (in the notation of Theorem 3.4),*

$$N_k^{\#(2)}(n) > D_k^{\#(1)}(n).$$

Proof.

$$
\begin{aligned}
N_k^{\#(2)}(n) &= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r) - 2^k) + 3qr + (r - J_{k-1}) \\
&> \left(\frac{n-r}{2^{k+1}}\right)(3(n-r) - 2^k) + 3qr \quad \text{since } r > J_{k-1} \\
&= N_k^{\#(1)}(n) \\
&= D_k^{\#(1)}(n) \quad \text{by Lemma 5.1.}
\end{aligned}
$$

This proves our result. □

Lemma 5.3. *For all integers k and all integers n such that $r \le J_k$ (in the notation of Theorem 3.4),*

$$N_k^{\#(2)}(n) \ge D_k^{\#(2)}(n).$$

Proof. We see that $r \le J_k = 2^k - J_{k-1}$ by Lemma 3.2. Thus, $2^k q + r \le 2^k(q+1) - J_{k-1}$. This implies $n \le 2^k(q+1) - J_{k-1}$, so $2n - 2^k(q+1) \le n - J_{k-1}$. Hence,

$$
\begin{aligned}
N_k^{\#(2)}(n) &= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r) - 2^k) + 3qr + (r - J_{k-1}) \\
&= \frac{q}{2}(3(2^k q) - 2^k) + 3q(n - 2^k q) + n - 2^k q - J_{k-1} \\
&= q^2(-3(2^{k-1})) + q(-3(2^{k-1}) + 3n) + n - J_{k-1} \\
&\ge q^2(-3(2^{k-1})) + q(-3(2^{k-1}) + 3n) + 2n - 2^k(q+1) \quad \text{by the above argument} \\
&= \frac{2n - 2^k - 2^k q}{2^{k+1}}(3(2^k q) + 2^{k+1}) \\
&= D_k^{\#(2)}(n).
\end{aligned}
$$

□

Lemma 5.4. *For all positive integers n and k, $N_k^{\#(3)}(n) = D_k^{\#(2)}(n)$.*

Proof.

$$\begin{aligned} N_k^{\#(3)}(n) &= \binom{n-r}{2^{k+1}}(3(n-r)-2^k)+(3q+2)r-2^k \\ &= \binom{q}{2}(3(2^kq)-2^k)+3q(n-2^kq)+2(n-2^kq)-2^k \\ &= \frac{n-2^k+n-2^kq}{2^k+1}(3(2^kq)+2^{k+1}) \\ &= D_k^{\#(2)}(n). \end{aligned}$$

□

Remark 5.5. *To summarize, Lemmas 5.1 through 5.4 tell us that for any positive integer n,*

$$N_k^{\#}(n) \geq D_k^{\#}(n).$$

For Propositions 5.6 through 5.9 we make the assumption that $J_\ell < n < J_{\ell+1}$ for some positive integer ℓ.

Proposition 5.6. *For ℓ and n, as given above, $N_{\ell+1}^{\#}(n) = n - J_\ell$.*

Proof. By Lemma 3.1,

$$\begin{aligned} \sum_{i=0}^{n-1} \left\lfloor \frac{3i+1}{2^{\ell+1}} \right\rfloor &= 0 \times (J_\ell) + 1 \times ((n-1)-(J_\ell - 1)) \\ &= n - J_\ell. \end{aligned}$$

□

Proposition 5.7. *$D_k^{\#}(n) = 0$ if $n < 2^{k-1}$. In particular, $D_{\ell+1}^{\#}(n) = 0$ if $n < 2^\ell$.*

Proof. If $n < 2^k$ then, in the notation of Theorem 3.5, $n = r$ and $q = 0$, so by Theorem 3.5, $D_k^{\#}(n) = 0$. □

Proposition 5.8. *$D_{\ell+1}^{\#}(n) = 2(n-2^\ell)$ if $2^\ell \leq n < J_{\ell+1}$.*

Proof. If $2^\ell \leq n < J_{\ell+1}$ then, in the notation of Theorem 3.5, $q = 0$ and $r = n$. Since $n \geq 2^\ell$, we are in the second case of Theorem 3.5 so

$$D_{\ell+1}^{\#}(n) = \frac{n - 2^{\ell+1} + n}{2^{\ell+2}}(0 + 2^{\ell+2}) = 2(n-2^\ell).$$

□

Proposition 5.9. *For n and ℓ as given above, $2(n-2^\ell) < n - J_\ell$.*

Proof. We begin by showing that $J_{\ell+1} - 2^\ell = 2^\ell - J_\ell$. We have

$$\begin{aligned} J_{\ell+1} - 2^\ell &= \frac{2^{\ell+2} + (-1)^{\ell+1}}{3} - 2^\ell \\ &= 2^\ell - \frac{2^{\ell+1} + (-1)^\ell}{3} \\ &= 2^\ell - J_\ell, \end{aligned}$$

and hence

$$\begin{aligned} 2(n-2^\ell) &= n-2^\ell+n-2^\ell \\ &< n-2^\ell+J_{\ell+1}-2^\ell \\ &= n-2^\ell+2^\ell-J_\ell \quad \text{from the above argument} \\ &= n-J_\ell \end{aligned}$$

so we have our result. □

We are now ready to prove our theorem.

Theorem 5.10. *$A(n)$ is even if n is not a Jacobsthal number.*

Proof. Our goal is to show that there is some k such that $N_k^{\#}(n)$ is strictly greater than $D_k^{\#}(n)$ since, by Remark 5.5, we have shown that $N_k^{\#}(n) \geq D_k^{\#}(n)$ for all positive integers k and n.

Given n, not a Jacobsthal number, there exists a positive integer ℓ such that $J_\ell < n < J_{\ell+1}$. Then $N_{\ell+1}^{\#}(n) = n - J_\ell$ by Proposition 5.6, and since $n > J_\ell$, $N_{\ell+1}^{\#}(n) > 0$. On the other hand, by Proposition 5.7, if $n < 2^\ell$, then $D_{\ell+1}^{\#}(n) = 0$. If $2^\ell \leq n < J_{\ell+1}$, then by Proposition 5.8, $D_{\ell+1}^{\#}(n) = 2(n-2^\ell)$ which is strictly less than $n - J_\ell = N_{\ell+1}^{\#}(n)$ by Proposition 5.9. Hence, in every case, $N_{\ell+1}^{\#}(n)$ is strictly greater than $D_{\ell+1}^{\#}(n)$ so there is at least one factor of two in $A(n)$ and we have our result. □

6 A Closing Remark

We close by noting that we can prove a stronger result than Theorem 5.10. If $J_\ell < n < J_{\ell+1}$, then

$$N_{\ell+1}^{\#}(n) - D_{\ell+1}^{\#}(n) = \begin{cases} n - J_\ell & \text{if} \quad J_\ell < n \leq 2^\ell \\ J_{\ell+1} - n & \text{if} \quad 2^\ell \leq n < J_{\ell+1} \end{cases}$$

by Propositions 5.6, 5.7, 5.8 and Lemma 3.2.

Let $ord_2(n)$ be the highest power of 2 that divides n. By Remark 5.5, $N_k^{\#}(n) - D_k^{\#}(n) \geq 0$ for all n and for all k, so that

$$ord_2(A(n)) \geq \begin{cases} n - J_\ell & \text{if} \quad J_\ell < n \leq 2^\ell \\ J_{\ell+1} - n & \text{if} \quad 2^\ell \leq n < J_{\ell+1} \end{cases},$$

which strengthens Theorem 5.10.

Finally, we see that $ord_2(A(2^\ell)) = J_{\ell-1}$ since, for all $k < \ell+1, N_k^{\#}(2^\ell) = N_k^{\#(1)}(2^\ell) = D_k^{\#(1)}(2^\ell) = D_k^{\#}(2^\ell)$, and $2^\ell - J_\ell = J_{\ell+1} - 2^\ell = J_{\ell-1}$. So, for example, we know that $A(2^{10})$ is divisible by 2^{J_9}, which equals 2^{341}, and that $A(2^{10})$ is not divisible by 2^{342}.

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