Sequence 102

Conjecture 1 (102) *The generating function* $f(x) = \sum_{n\geq 1}^{\infty} a_n x^n$ *for the sequence* $\{a_n\}_{n=1}^{\infty}$ *of all positive integers whose binary representation do not begin with* 100 *is*

$$\frac{x}{(1-x)^2} + \frac{x}{(1-x)} \cdot \sum_{k\cdot\geq 0} 2^k x^{3\cdot 2^k} \tag{1}$$

Proof. The integers whose binary representation do NOT begin with 100 reside in the intervals $I_0 = [1,3]$ and $I_M = \left[5\cdot 2^{M-1}, 4\cdot 2^M - 1\right]$ for $M \geq 1$. In particular we have $a_1 = 1$ and for $n \geq 1$

$$a_{n+1} = \begin{cases} a_n + 1 + 2^M & \text{if } a_n = 4\cdot 2^M - 1 \text{ for some } M \geq 0 \\ a_n + 1 & \text{otherwise} \end{cases}.$$

Since the sequence $\{a_n\}_{n\geq 1}$ progresses sequentially through the intervals $\{I_M\}_{M\geq 0}$. The "jumps" where $a_{n+1} = a_n + 1 + 2^M$ is satisfied occur when n indexes the last value of some interval I_k in which case, we have

$$n = |I_0| + \sum_{M=1}^{k} |I_M| = 3 + \sum_{M=1}^{k} \left(\left(4\cdot 2^M - 1\right) - 5\cdot 2^{M-1} + 1\right) = 3\cdot 2^k.$$

We now restate the original recurrence as

$$a_{n+1} = \begin{cases} a_n + 1 + 2^k & \text{if } n = 3\cdot 2^k \text{ for some } k \geq 0 \\ a_n + 1 & \text{otherwise} \end{cases}$$

Finally, multiplying by x^n and summing this relationship we have

$$\begin{aligned} \sum_{n\geq 1} a_{n+1} x^n &= \sum_{n\geq 1} (a_n + 1) x^n + \sum_{k\geq 0} 2^k x^{3\cdot 2^k} \\ \frac{f(x) - x}{x} &= f(x) + \frac{x}{1-x} + \sum_{k\geq 0} 2^k x^{3\cdot 2^k} \end{aligned}$$

Solving $f(x)$ yields (1). ■

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