THE NUMBERS OF SMALL RINGS

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1. Numbers of rings

The textbook [LW80] contains an overview on number theorems for small rings on pages 133 - 141; we quote the main theorems and sources (always let $m, n \in \mathbb{N}$ and p a prime):

First, we notice that it suffices to consider rings of prime power order:

Theorem 1. [Sho38] *Every finite ring R can be uniquely (up to isomorphism) decomposed into a direct sum of rings of prime power order.*

Theorem 2. [Bea48] *There are $n+1$ non-isomorphic rings with additive group $\mathbb{Z}_{p^n}$. Only one of them has an identity.*

Theorem 3. [Rag69] *There are 8 non-isomorphic rings with additive group $\mathbb{Z}_p \oplus \mathbb{Z}_p$.*

From these theorems, it follows that there are $3 + 8 = 11$ non-isomorphic rings of order p^2.

Theorem 4. [FW74, LW80] *There are $(3p+4)n - 4p + 11$ non-isomorphic rings with additive group $\mathbb{Z}_{p^n} \oplus \mathbb{Z}_p$ $(n \geq 2)$ if p is odd and $9n + 2$ if $p = 2$.*

Theorem 5. [FW74, LW80] *There are $p + 27$ non-isomorphic rings with additive group $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ if p is odd and 28 if $p = 2$.*

These two theorems give a total of $3p + 50$ non-isomorphic rings of order p^3 for odd p and 52 for $p = 2$.

This result is also obtained by [AE83].

2. Rings with identity

Theorem 6. [Rag69] *There are 11 rings with identity of order 8 and 12 rings with identity of order p^3 (p an odd prime).*

Theorem 7. [Wie74] *If p is an odd prime, $m \geq 1$, $n \geq 0$, $r = \lfloor \frac{n+1}{2} \rfloor$, and $s = \lfloor \frac{m+1}{2} \rfloor$ then there are*

$$N = \begin{cases} (m - 2(n-r) + 1)p^{n-r} + \frac{p^r - 1}{p-1} + 3\frac{p^{n-r}-1}{p-1} + m & \text{if } n \leq m \text{ and} \\ (m - 2(m-s) + 1)p^{m-s} + \frac{p^s - 1}{p-1} + 3\frac{p^{m-s}-1}{p-1} + m & \text{if } n > m \end{cases}$$

rings with identity with additive group $\mathbb{Z}_{p^{m+n}} \oplus \mathbb{Z}_{p^m}$.

Theorem 8. [Wie85] *If $m \geq 1$, $n \geq 0$, $r = \lfloor \frac{n+1}{2} \rfloor$, and $s = \lfloor \frac{m+1}{2} \rfloor$ then there are*

$$N = \begin{cases} 4m + 3n - 2 & \text{if } n \leq m-2 \text{ and } n \in \{0, 1\} \\ 2^{r+1} + (m - 2(n-r) + 3)2^{n-r+1} + 4m - 17 & \text{if } 2 \leq n \leq m-2 \\ 4m - 1 & \text{if } n > m-2 \text{ and } m \in \{1, 2\} \\ 2^{s+1} + (2s - m + 5)2^{m-s} + 4m - 17 & \text{if } n > m-2 > 0 \end{cases}$$

rings with identity with additive group $\mathbb{Z}_{2^{m+n}} \oplus \mathbb{Z}_{2^m}$.

3. Summary

These theorems allow us to determine the numbers of small rings (as long as their order is not divisible by a fourth prime power – for rings of order p^4, we are only aware of partial solutions, e.g. [KP69, Flo73, Wie88]).

The result on the number of rings with identity on $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ was obtained by computer calculation.

The numbers with the question mark are computer generated and need to be checked.

additive group	total number of rings	rings with identity
$\mathbb{Z}_2$	2	1
$\mathbb{Z}_3$	2	1
$\mathbb{Z}_4$	3	1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	8	3
$\mathbb{Z}_5$	2	1
$\mathbb{Z}_6$	4	1
$\mathbb{Z}_7$	2	1
$\mathbb{Z}_8$	4	1
$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	20	3
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	28	7
$\mathbb{Z}_9$	3	1
$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	8	3
$\mathbb{Z}_{10}$	4	1
$\mathbb{Z}_{11}$	2	1
$\mathbb{Z}_{12}$	6	1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	16	3
$\mathbb{Z}_{13}$	2	1
$\mathbb{Z}_{14}$	4	1
$\mathbb{Z}_{15}$	4	1
$\mathbb{Z}_{16}$	5	1
$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	29	3
$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	66	6
$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	170	15
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	120	25
$\mathbb{Z}_{17}$	2	1
$\mathbb{Z}_{18}$	6	1
$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	16	3
$\mathbb{Z}_{19}$	2	1
$\mathbb{Z}_{20}$	6	1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$	16	3
$\mathbb{Z}_{21}$	4	1
$\mathbb{Z}_{22}$	4	1
$\mathbb{Z}_{23}$	2	1
$\mathbb{Z}_{24}$	8	1
$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	40	3
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	56	7
$\mathbb{Z}_{25}$	3	1
$\mathbb{Z}_5 \oplus \mathbb{Z}_5$	8	3

And so on...

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