SOLUTIONS TO ASSIGNMENT #7

1. Find the linearizations $L(x)$ of the following functions $f(x)$ near $x = 0$.

(a) $f(x) = \sqrt{25 + x^2 + x}$.

(b) $f(x) = (1 - 2x)^\beta$, where β is some constant.

(c) $f(x) = \ln(x + \sqrt{1 - x^2})$.

Solution: In all cases the linearization near $x = 0$ is $L(x) = f(0) + f'(0)x$.

(a) $f'(0) = \dfrac{2x+1}{2\sqrt{25+x^2+x}}\Big|_{x=0} = \dfrac{1}{10}$ and $f(0) = 5 \Longrightarrow L(x) = 5 + \dfrac{1}{10}x$.

(b $f'(0) = -2\beta(1-2x)^{\beta-1}\Big|_{x=0} = -2\beta$ and $f(0) = 1 \Longrightarrow L(x) = 1 - 2\beta x$.

(c) $f'(0) = \left(\dfrac{1 + \frac{1}{2}(1-x^2)^{-1/2}(-2x)}{x + \sqrt{1-x^2}}\right)\Big|_{x=0} = 1$ and $f(0) = 0 \Longrightarrow L(x) = x$.

2. Two towns on the trans Canada Hiway are 100 km apart. Two cars leave the first town at 1:00 pm and both arrive at the second town one hour later. Show that at sometime between 1:00 pm and 2:00 pm they had the same velocity.

Solution: Let the positions of the 2 cars be $x_1(t)$ and $x_2(t)$ respectively, and set $x(t) = x_1(t) - x_2(t)$. Then $x(0) = 0$ and $x(1) = 0$, and so by the Mean Value Theorem there exists t such that $0 < t < 1$ and $x'(t) = 0$. Therefore the cars have the same speed at this time.

3. Find the length of the longest ladder that can be carried horizontally around a corner, from a corridor a m wide to one that is b m wide.

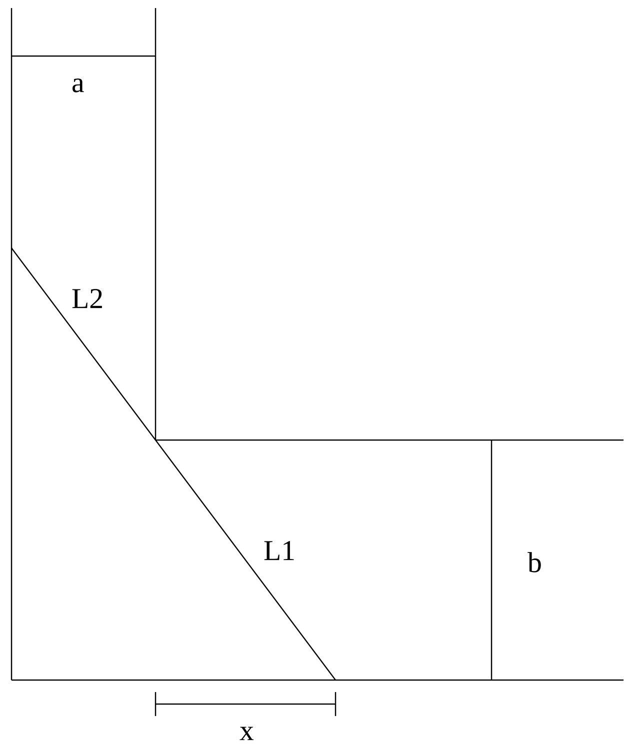
Solution:
The problem is equivalent to minimizing $L(x) = L_1 + L_2$, see the diagram below. Now $L_1 = \sqrt{b^2 + x^2}$ and $L_2 = aL_1/x$, where $0 < x < \infty$. Thus $L(x) = \sqrt{b^2+x^2} + \dfrac{a\sqrt{b^2+x^2}}{x}$.

$$\begin{aligned} L'(x) &= \frac{x}{\sqrt{b^2+x^2}} + \frac{a}{\sqrt{b^2+x^2}} - \frac{a\sqrt{b^2+x^2}}{x^2} = 0 \\ &\Longleftrightarrow x + a - \frac{a(b^2+x^2)}{x^2} = 0 \Longleftrightarrow x = (ab^2)^{1/3} \end{aligned}$$

Therefore

$$\begin{aligned} L_{min} &= \sqrt{b^2+x^2}\left(1 + \frac{a}{x}\right)\Big|_{x=(ab^2)^{1/3}} = \sqrt{b^2 + (ab^2)^{2/3}}\left(1 + \frac{a}{(ab^2)^{1/3}}\right) \\ &= b^{2/3}\sqrt{b^{2/3} + a^{2/3}}\left(1 + \frac{a}{(ab^2)^{1/3}}\right) = \left(b^{2/3} + a^{2/3}\right)^{3/2} \end{aligned}$$

Figure 1: Diagram for question 3



4. Let $f(x) = x^3 - 3x^2 + 1,\ -\infty < x < \infty$.

(a) Determine all critical points of $f(x)$.

(b) Determine all intervals of increase of $f(x)$.

(c) Determine all intervals of decrease of $f(x)$.

(d) Determine all intervals where $f(x)$ is concave up.

(e) Determine all intervals where $f(x)$ is concave down.

Solution:

(a) $f'(x) = 3x^2 - 6x = 3x(x-2) = 0 \iff x = 0, 2$.

(b) The intervals of increase are $-\infty < x < 0$ and $2 < x < \infty$.

(c) The interval of decrease is $0 < x < 2$.

(d) $f''(x) = 6x - 6$, and therefore $f(x)$ is concave up for $1 < x < \infty$.

(e) $f(x)$ is concave down for $-\infty < x < 1$.

5. Let $f(x) = \dfrac{x^2+1}{x^2+2},\ -\infty < x < \infty$.

(a) Determine all critical points of $f(x)$.

(b) Determine all intervals of increase of $f(x)$.

(c) Determine all intervals of decrease of $f(x)$.

(d) Determine all intervals where $f(x)$ is concave up.

(e) Determine all intervals where $f(x)$ is concave down.

Solution:

(a) $f'(x) = \dfrac{2x}{(x^2+2)^2} = 0 \Longleftrightarrow x = 0.$

(b) $f(x)$ is increasing for $0 < x < \infty$.

(c) $f(x)$ is decreasing for $-\infty < x < 0$.

(d) $f''(x) = \dfrac{4-6x^2}{(x^2+2)^3} > 0 \Longleftrightarrow -\sqrt{\dfrac{2}{3}} < x < \sqrt{\dfrac{2}{3}}.$

(e) $f(x)$ is concave down for $-\infty < x < -\sqrt{\dfrac{2}{3}}$ and $\sqrt{\dfrac{2}{3}} < x < \infty$.

6. Let $f(x) = \dfrac{e^x - 1}{e^{2x}+1},\ -\infty < x < \infty.$

(a) Determine all critical points of $f(x)$.

(b) Determine all intervals of increase of $f(x)$.

(c) Determine all intervals of decrease of $f(x)$.

Solution:

(a) $f'(x) = -e^x \dfrac{e^{2x} - 2e^x - 1}{(e^{2x}+1)^2} = 0 \Longleftrightarrow x = \ln(1+\sqrt{2}).$

(b) $f(x)$ is increasing for $-\infty < x < \ln(1+\sqrt{2})$.

(c) $f(x)$ is decreasing for $\ln(1+\sqrt{2}) < x < \infty$.

7. Let $f(x) = \dfrac{\ln x}{x},\ x > 0.$

(a) Determine all critical points of $f(x)$.

(b) Determine all intervals of increase of $f(x)$.

(c) Determine all intervals of decrease of $f(x)$.

(d) Determine all intervals where $f(x)$ is concave up.

(e) Determine all intervals where $f(x)$ is concave down.

Solution:

(a) $f'(x) = \dfrac{1-\ln x}{x^2} = 0 \Longleftrightarrow x = e.$

(b) $f(x)$ is increasing for $0 < x < e$.

(c) $f(x)$ is decreasing for $e < x < \infty$.

(d) $f''(x) = \dfrac{2\ln x - 3}{x^3} = 0 \Longleftrightarrow x = e^{3/2}$. Thus $f(x)$ is concave up for $e^{3/2} < x < \infty$.

(e) $f(x)$ is concave down for $0 < x < e^{3/2}$.

8. Use implicit differentiation to find $\dfrac{dy}{dx}$ for the following:

(a) $\dfrac{\arcsin x}{\pi} + y^2 = \dfrac{7}{6}$.

(b) $\arctan(x^2) + \arctan y = \dfrac{7\pi}{12}$.

(c) $x^3 + y^3 = 6xy - 3$

(d) $xy = e^{-y^2+y}$.

Solution:

(a) $\dfrac{1}{\pi}\dfrac{1}{\sqrt{1-x^2}} + 2yy' = 0 \Longrightarrow y' = -\dfrac{1}{2\pi y\sqrt{1-x^2}}$.

(b) $\dfrac{2x}{1+x^4} + \dfrac{y'}{1+y^2} = 0 \Longrightarrow y' = -\dfrac{2x(1+y^2)}{1+x^4}$.

(c) $3x^2 + 3y^2y' = 6y + 6xy' \Longrightarrow y' = \dfrac{3x^2 - 6y}{6x - 3y^2}$.

(d) $y + xy' = e^{-y^2+y}(-2yy' + y') \Longrightarrow y' = \dfrac{y}{e^{-y^2+y}(-2y+1) - x}$.

9. Find the equation of the tangent lines for each of the curves in question 8, at the given point (a, b).

(a) $(a, b) = (1/2, 1)$.

(b) $(a, b) = (-1, \sqrt{3})$.

(c) $(a, b) = (2, 1)$.

(d) $(a, b) = (1, 1)$.

Solution: In each case we evaluate y' at the indicated value to compute the slope. The tangent line will have the equation $y = b + y'(a)(x - a)$.

(a) $y = 1 - \dfrac{1}{\pi\sqrt{3}}(x - 1/2)$.

(b) $y = \sqrt{3} + 4(x + 1)$.

(c) $y = 1 + \dfrac{2}{3}(x - 2)$.

(d) $y = 1 - \dfrac{1}{2}(x - 1)$.

10. Show that $\cos x > 1 - \dfrac{1}{2}x^2$ for $x > 0$.

Solution: Let $f(x) = \cos x - 1 + \frac{1}{2}x^2$. Then $f(0) = 0$ and $f'(x) = -\sin x + x > 0$ for $x > 0$, and therefore $f(x) > f(0) = 0$ for $x > 0$.

11. Show that $\arcsin x = \arctan \dfrac{x}{\sqrt{1-x^2}}$.

Solution: Let $f(x) = \arcsin x - \arctan \dfrac{x}{\sqrt{1-x^2}}$. Then $f(0) = 0$ and

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{1+\frac{x^2}{1-x^2}} \times \frac{\sqrt{1-x^2} + x^2(1-x^2)^{-1/2}}{1-x^2} = 0 \text{ (by some algebra.)}$$

Therefore $f(x)$ is identically zero (on any interval on which it is defined).