Some Predictable Pierce Expansions

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I. Introduction.

In 1929, T. A. Pierce discussed an algorithm for expanding real numbers $x \varepsilon (0, 1)$ in the form

$$x = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} \cdots \qquad (1)$$

where the a_i form a strictly increasing sequence of positive integers.

He showed that these expansions (which we call **Pierce expansions**) are essentially unique. The Pierce expansion for x terminates if and only if x is rational. See [Pie] or [Sha] for details.

In this note, we give formulas for the a_i in the case where

$$x = \frac{c - \sqrt{c^2 - 4}}{2}$$

and $c \geq 3$ is an integer. For these numbers, Pierce expansions provide extremely rapidly converging series.

II. Finding Real Roots of Polynomials.

To save space, we will sometimes write the equation (1) in the form

$$x = \{ a_1, a_2, a_3, \cdots \}$$

where the curly brackets denote a Pierce expansion.

Let

$$p_1(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

be a polynomial with integers coefficients and a single real zero α in the interval (0, 1). We want to find the first term in the Pierce expansion of α. From equation (1) it is easy to see that $a_1 = \lfloor 1/\alpha \rfloor$. Consider the polynomial $q_1(x) = x^n \; p_1(1/x)$; this is a polynomial with integer co-efficients that has $1/\alpha$ as a zero. Through a simple binary search procedure, it is easy to find d_1 such that

$$\mathrm{sign}(q(d_1)) \neq \mathrm{sign}(q(d_1 + 1));$$

this shows that $d_1 = \lfloor 1/\alpha \rfloor$ and so we can take $\alpha_1 = d_1$.

Now consider the polynomial

$$p_2(x) = a_1^n \ p_1(\frac{1-x}{a_1})$$

This again is a polynomial with integer coefficients. It is easily verified that if β is a zero of $p_2(x)$ then

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1}\beta$$

so

$$\beta = \frac{1}{a_2} - \frac{1}{a_2 a_3} + \cdots$$

By repeating this procedure on the polynomial $p_2(x)$, we generate the co-efficient a_2 in the Pierce expansion of α. And by continuing in the same fashion, we can generate as many terms of the Pierce expansion for α as desired:

$$\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots$$

Now let us specify our polynomial to be

$$p_1(x) = x^2 - cx + 1$$

where $c \geq 3$ is an integer. Let α be the smaller positive zero, so

$$\alpha = \frac{c - \sqrt{c^2 - 4}}{2} . \tag{2}$$

Now $q_1(x) = x^2 p_1(1/x) = x^2 - cx + 1$. We find $q_1(c-1) = 2 - c$, which is negative, and $q_1(c) = 1$ which is positive. Hence we see that $a_1 = c - 1$.

Now $p_2(x) = (c-1)^2 p_1(\frac{1-x}{c-1})$; hence

$$p_2(x) = x^2 + (c^2 - c - 2)x + 2 - c.$$

We find

$$q_2(x) = x^2 p_2(1/x) = (2-c)x^2 + (c^2 - c - 2)x + 1$$

Now $q_2(c+1) = 1$ which is positive; but $q_2(c+2) = 5 - c^2$ which is negative. Hence we see that $a_2 = c + 1$.

Now $p_3(x) = x^2 p_2(\frac{1-x}{c+1})$ so we see

$$p_3(x) = x^2 - (c^3 - 3c)x + 1.$$

So far we have been following the algorithm. But now we notice that $p_3(x)$ is essentially just $p_1(x)$ with $c^3 - 3c$ playing the role of c. We have found

$$\alpha = \frac{1}{c-1} - \frac{1}{(c-1)(c+1)} + \frac{1}{(c-1)(c+1)}\gamma$$

where γ is the root of $x^2 - (c^3 - 3c)x + 1 = 0$. By continuing this process, we get

Theorem.

Let α be as in equation (2). Then

$$\alpha = \{\ c_0 - 1,\ c_0 + 1,\ c_1 - 1,\ c_1 + 1,\ c_2 - 1,\ c_2 + 1,\ \cdots \}$$

where $c_0 = c$, $c_{k+1} = c_k^3 - 3c_k$.

For example, let $c = 3$. Then we find

$$\frac{3 - \sqrt{5}}{2} \{\ 2,\ 4,\ 17,\ 19,\ 5777,\ 5779,\ \cdots\ \}$$

Another example: let $c = 6$. Then, after some manipulation, we find:

$$\sqrt{2} - 1 = \{\ 2,\ 5,\ 7,\ 197,\ 199,\ 7761797,\ 7761799,\ \cdots\ \}$$

Ironically, both Pierce and Salzer [Sal] gave the first four terms of this expansion, but apparently neither detected the general pattern!

III. The Coefficients c_k.

The recurrence $c_{k+1} = c_k^3 - 3c_k$ is an interesting one which has been previously studied [AhSl], [Esc]. Some brief comments are in order.

If we let α and β be the roots of the quadratic

$$x^2 - cx + 1 = 0$$

and define

$$V(n) = \alpha^n + \beta^n;\ U(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

then it is easy to show by induction that

$$V(n) = cV(n-1) - V(n-2);\ \ U(n) = cU(n-1) - U(n-2)$$

where

$$V(0) = 2,\ V(1) = c;\ \ U(0) = 0,\ U(1) = 1$$

We can also show that $V(3k) = V(k)^3 - 3V(k)$; hence by induction $c_k = V(3^k)$. This gives the following closed form for the c_k:

$$c_k = \left(\frac{c + \sqrt{c^2 - 4}}{2}\right)^{3^k} + \left(\frac{c - \sqrt{c^2 - 4}}{2}\right)^{3^k}$$

Similarly, it is easy to show by induction that

$$\frac{U(3^k - 1)}{U(3^k)} = \{\ c_0 - 1,\ c_0 + 1,\ c_1 - 1,\ c_1 + 1,\ \cdots,\ c_{k-1} - 1,\ c_{k-1} + 1\ \}$$

which gives an alternative proof of our Theorem.

References

[AhSl] A. V. Aho and N. J. A. Sloane, *Some Doubly Exponential Sequences*, Fib. Quart. **15** (1973) 429-437.

[Esc] E. B. Escott, *Rapid Method for Extracting a Square Root*, Am. Math. Monthly **44** (1937) 644-646.

[Pie] T. A. Pierce, *On an algorithm and its use in approximating roots of polynomials*, Am. Math. Monthly **36** (1929) 523-525.

[Sal] H. E. Salzer, *The Approximation of Numbers as Sums of Reciprocals*, Am. Math. Monthly **54** (1947) 135-142.

[Sha] J. O. Shallit, *Metric Theory of Pierce Expansions*, to appear.