ON A PARTITION FUNCTION OF RICHARD STANLEY

by
George E. Andrews*

Abstract

In this paper, we examine partition π classified according to the number $r(\pi)$ of odd parts in π and $s(\pi)$ the number of odd parts in π', the conjugate of π. The generating function for such partitions is obtained when the parts of π are all $\leqq N$. From this a variety of corollaries follow including a Ramanujan type congruence for Stanley's partition function $t(n)$.

1 Introduction

Let π denote a partition of some integer and π' its conjugate. For definition of these concepts see[1; Ch.1]. Let $\mathcal{O}(\pi)$ denote the number of odd parts of π. For example if π is $6+5+4+2+2+1$. Then the Ferrers graph if π is

```
. . . . . .
. . . . .
. . . .
. .
. .
.
```

*Partially supported by National Science Foundation Grant DMS-0200047

Reading columns we see that π' is $6+5+3+3+2+1$. Hence $\mathcal{O}(\pi)$ and $\mathcal{O}(\pi')=4$.

Richard Stanley [4], [5], has shown that if $t(n)$ denotes the number of partitions π of n for which $\mathcal{O}(\pi) \equiv \mathcal{O}(\pi') (\mod 4)$, then

$$t(n) = \frac{1}{2}\bigg(p(n) + f(n)\bigg), \tag{1.1}$$

where $p(n)$ is the total number of partitions of n [1; p.1], and

$$\sum_{n=0}^{\infty} f(n)q^n = \Pi_{i \geqq 1} \frac{(1+q^{2i-1})}{(1-q^{4i})(1+q^{4i-2})^2}. \tag{1.2}$$

$t(n)$ is Stanley's partition function referred to in the title of this paper.

Stanley's result for $t(n)$ is related nicely to a general study of sign-balanced, labeled possets [5]. In this paper, we shall restrict our attention to $S_N(n,r,s)$, the number of partition π of n where each part of π is $\leqq N$, $\mathcal{O}(\pi)=r$, $\mathcal{O}(\pi')=s$. In Section 2, we shall prove our main result:

Theorem 1.1.

$$\sum_{n,r,s \geqq 0} S_{2N}(n,r,s) q^n z^r y^s \tag{1.3}$$

$$= \Bigg(\sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}; q^4\Bigg] (-zyq;q^4)_j (-zy^{-1}q;q^4)_{N-j} (yq)^{2N-2j}\Bigg) \Big/ \Bigg(q^4;q^4)_N (z^2q^2;q^4)_N\Bigg),$$

and

$$\sum_{n,r,s \geqq 0} S_{2N+1}(n,r,s) q^n z^r y^s \tag{1.4}$$

$$= \Bigg(\sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}; q^4\Bigg] (-zyq;q^4)_{j+1} (-zy^{-1}q;q^4)_{N-j} (yq)^{2N-2j}\Bigg) \Big/ \Bigg(q^4;q^4)_N (z^2q^2;q^4)_{N+1}\Bigg),$$

where

$$\left[{N \atop j}; q\right] = \begin{cases} 0 \ \textit{if}\ j < 0 \ \textit{or}\ j > N \\ \\ \dfrac{(1-q^N)(1-q^{N-1})\dots(1-q^{N-j+1})}{(1-q^j)(1-q^{j-1})\dots(1-q)} \ \textit{for}\ 0 \leqq j \leqq N, \end{cases} \tag{1.5}$$

and

$$(A;q)_M = (1-A)(1-Aq)\dots(1-Aq^{M-1}). \tag{1.6}$$

¿From Theorem 1 follows an immediate lovely corollary:

Corollary 1.1.

$$\sum_{n,r,s\geqq 0} S_\infty(n,r,s)q^n z^r y^s \tag{1.7}$$
$$= \Pi_{j=1}^{\infty} \frac{(1+yzq^{2j-1})}{(1-q^{4j})(1-z^2q^{4j-2})(1-y^2q^{4j-2})}.$$

¿From Corollary 1.1, we shall see in Section 3, that

Corollary 1.2.

$$t(5n+4) \equiv 0 \pmod 5. \tag{1.8}$$

Also

Corollary 1.3.

$$\sum_{n=0}^{\infty} t(n)q^n = \frac{Q(q^2)^2Q(q^{16})^4}{Q(q)Q(q^4)^5Q(q^{32})^2}, \tag{1.9}$$

where

$$Q(q) = (q;q)_\infty = \Pi_{j=1}^{\infty}(1-q^j). \tag{1.10}$$

We conclude with some open questions.

2 The Main Theorem

We begin with some preliminaries about partitions and their conjugates. For a given partition π with parts each $\leqq N$, we denote by $f_i(\pi)$ the number of appearances of i as a part of π. The parts of π' in non-increasing order are thus

$$\sum_{i=1}^{N} f_i(\pi),\ \sum_{i=2}^{N} f_i(\pi),\ \sum_{i=3}^{N} f_i(\pi), \ldots, \sum_{i=N}^{N} f_i(\pi). \tag{2.1}$$

Note that some of the entries in this sequence may well be zero; the non-zero entries make up the parts of π'. However in light of the fact that 0 is even, we see that $\mathcal{O}(\pi')$ is the number of odd entries in the sequence (2.1) while

$$\mathcal{O}(\pi) = f_1(\pi) + f_3(\pi) + f_5(\pi) + \ldots. \tag{2.2}$$

We now define

$$\sigma_N(q,z,y) = \left(\sum_{n,r,s \geqq 0} S_N(n,r,s) q^n z^r y^s \right) (q^4;q^4)_{\lfloor \frac{N}{2} \rfloor} (z^2q^2;q^4)_{\lfloor \frac{N+1}{2} \rfloor}. \tag{2.3}$$

Lemma 2.1. $\sigma_0(q,z,y) = 1$, *and for* $N \geqq 1$

$$\sigma_{2N}(q,z,y) = \sigma_{2N-1}(q,z,y) + y^{2N} q^{2N} \sigma_{2N-1}(q,z,y^{-1}) \tag{2.4}$$

$$\sigma_{2N-1}(q,z,y) = \sigma_{2N-2}(q,z,y) + z y^{2N-1} q^{2N-1} \sigma_{2N-2}(q,z,y^{-1}). \tag{2.5}$$

Proof. We shall in the following be dealing with partitions whose parts are all $\leqq$ some given N. We let $\bar{\pi}$ be that partition made up of the parts of π that are $< N$. In light of (2.1) we see that if N is a part of π an even number of times, then $\mathcal{O}(\pi') = \mathcal{O}(\bar{\pi}')$ and if N appears an odd number of times in π then $\mathcal{O}(\bar{\pi}') = N - \mathcal{O}(\pi)$ (because the removal of $f_N(\pi)$ from each sum in (2.1) reverses parity).

Initially we note that the only partition with at most zero parts is the empty partition of 0; hence $\sigma_0(q,z,y)=1$.

Next for $N \geqq 1$

$$
\begin{aligned}
\frac{\sigma_{2N}(q,z,y)}{(q^4;q^4)_N(z^2q^2;q^4)_N} &= \sum_{\pi,\text{parts}\leqq 2N} q^{\sum i f_i(\pi)} z^{f_1(\pi)+f_3(\pi)+\ldots+f_{2N-1}(\pi)} y^{\mathcal{O}(\pi')} \\
&= \sum_{\substack{\pi,\text{parts}\leqq 2N\\ f_{2N}(\pi)\text{even}}} q^{\sum i f_i(\bar\pi)+2Nf_{2N}(\pi)} z^{f_1(\pi)+f_3(\pi)+\ldots+f_{2N-1}(\pi)} y^{\mathcal{O}(\bar\pi')} \\
&+ \sum_{\substack{\pi,\text{parts}\leqq 2N\\ f_{2N}(\pi)\text{odd}}} q^{\sum i f_i(\bar\pi)+2Nf_{2N}(\pi)} z^{f_1(\pi)+f_3(\pi)+\ldots+f_{2N-1}(\pi)} y^{2N-\mathcal{O}(\pi')} \\
&= \frac{1}{(1-q^{4N})}\frac{\sigma_{2N-1}(q,z,y)}{(q^4;q^4)_{N-1}(z^2q^2;q^4)_N} \\
&+ \frac{y^{2N}q^{2N}}{(1-q^{4N})}\frac{\sigma_{2N-1}(q,z,y^{-1})}{(q^4;q^4)_{N-1}(z^2q^2;q^4)_N},
\end{aligned}
$$

which is equivalent to (2.4). Finally

$$
\begin{aligned}
\frac{\sigma_{2N+1}(q,z,y)}{(q^4;q^4)_N(z^2q^2;q^4)_{N+1}} &= \sum_{\pi,\text{parts}\leqq 2N+1} q^{\sum i f_i(\pi)} z^{f_1(\pi)+f_3(\pi)+\ldots+f_{2N+1}(\pi)} y^{\mathcal{O}(\pi')} \\
&= \sum_{\substack{\pi,\text{parts}\leqq 2N+1\\ f_{2N+1}(\pi)\text{even}}} q^{\sum i f_i(\bar\pi)+(2N+1)f_{2N+1}(\pi)} z^{f_1(\pi)+\ldots+f_{2N+1}(\pi)} y^{\mathcal{O}(\bar\pi')} \\
&+ \sum_{\substack{\pi,\text{parts}\leqq 2N+1\\ f_{2N+1}(\pi)\text{odd}}} q^{\sum i f_i(\bar\pi)+(2N+1)f_{2N+1}(\pi)} z^{f_1(\pi)+\ldots+f_{2N-1}(\pi)+f_{2N+1}(\pi)} \\
& y^{2N+1-\mathcal{O}(\bar\pi')} \\
&= \frac{1}{(1-z^2q^{4N+2})}\frac{\sigma_{2N}(q,z,y)}{(q^4;q^4)_N(z^2q^2;q^4)_N} \\
&+ \frac{y^{2N+1}q^{2N+1}z}{(1-z^2q^{4N+2})}\frac{\sigma_{2N}(q,z,y)}{(q^4;q^4)_N(z^2q^2;q^4)_N},
\end{aligned}
$$

which is equivalent to (2.5) with N replaced by $N+1$. $\square$

Proof. Proof of Theorem 1.

We let $\tau_{2N}(q,z,y)$ denote the numerator on the right-hand side of (1.3) and $\tau_{2N+1}(q,z,y)$ denote the numerator on the right-hand side of (1.4). If we can show that $\tau_N(q,z,y)$ satisfies (2.4) and (2.5), then notify immediately that $\tau_0(q,z,y)=1$, we will have proved that $\sigma_N(q,z,y)=\tau_N(q,z,y)$ for each $N \geqq 0$ (by mathematical induction) and will then prove Theorem 1 once we recall (2.3). First

$$
\begin{aligned}
&\tau_{2N-1}(q,z,y)+y^{2N}q^{2N}\tau_{2N-1}(q,z,y^{-1})\\
&=\sum_{j\geqq 0}\begin{bmatrix} N-1 \\ j \end{bmatrix}; q^4\Big](-zyq;q^4)_{j+1}(-zy^{-1}q;q^4)_{N-j-1}(yq)^{2(N-1-j)}\\
&+y^{2N}q^{2N}\sum_{j\geqq 0}\Big[\begin{matrix} N-1 \\ j \end{matrix}; q^4\Big](-zy^{-1}q;q^4)_{N-j}(-zyq;q^4)_{j}(y^{-1}q)^{2j}\\
&\qquad\qquad\text{(where } j\to N-1-j \text{ in the second sum)}\\
&=\sum_{j\geqq 0}\Big[\begin{matrix} N-1 \\ j-1 \end{matrix}; q^4\Big](-zyq;q^4)_{j}(-zy^{-1}q;q^4)_{N-j}(yq)^{2(N-j)}\\
&+y^{2N}q^{2N}\sum_{j\geqq 0}\Big[\begin{matrix} N-1 \\ j \end{matrix}; q^4\Big](-zy^{-1}q;q^4)_{N-j}(-zyq;q^4)_{j}(y^{-1}q)^{2N}\\
&\qquad\qquad\text{(where } j\to j-1 \text{ in the first sum)}\\
&=\sum_{j\geqq 0}(-zyq;q^4)_{j}(-zy^{-1}q;q^4)_{N-j}(yq)^{2(N-j)}\times\left\{\Big[\begin{matrix} N-1 \\ j-1 \end{matrix}; q^4\Big]+q^{4j}\Big[\begin{matrix} N-1 \\ j \end{matrix}; q^4\Big]\right\}\\
&=\sum_{j\geqq 0}(-zyq;q^4)_{j}(-zy^{-1}q;q^4)_{N-j}(yq)^{2(N-j)}\Big[\begin{matrix} N \\ j \end{matrix}; q^4\Big]\\
&\qquad\qquad\text{(by [1; p.35, eq.(3.3.4)])}\\
&=\tau_{2N}(q,z,y).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\tau_{2N}(q,z,y) + zy^{2N+1}q^{2N+1}\tau(q,z,y^{-1}) \\
&= \sum_{j=0} \begin{bmatrix} N \\ j \end{bmatrix}; q^4 \Big] (-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j} \\
&+ zq^{2N+1}y^{2N+1}\sum_{j=0}^{N} \Big[{N \atop j}; q^4 \Big] (-zy^{-1}q;q^4)_{N-j}(-zyq;q^4)_j(qy^{-1})^{2j}
\end{aligned}
$$

(where $j \to N-j$ in the second sum)

$$
\begin{aligned}
&= \sum_{j=0}^{N} \Big[{N \atop j}; q^4 \Big] (-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}(1+zyq^{4j+1}) \\
&= \sum_{j=0}^{N} \Big[{N \atop j}; q^4 \Big] (-zyq;q^4)_{j+1}(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j} \\
&= \tau_{2N+1}(q,z,y).
\end{aligned}
$$

□

Proof. Proof of Corollary 1.1.

¿From Theorem 1 (either (1.3) or (1.4) with $j \to N-j$)

$$
\begin{aligned}
&\sum_{n,r,s\geqq 0} S_\infty(n,r,s)q^n z^r y^s \qquad (2.6)\\
&= \frac{1}{(q^4;q^4)_\infty(z^2q^2;q^4)_\infty}\sum_{j=0}^{\infty}\frac{1}{(q^4;q^4)_j}(-zyq;q^4)_\infty(-zy^{-1}q;q^4)_j(yq)^{2j} \\
&= \frac{(-zyq;q^4)_\infty}{(q^4;q^4)_\infty(z^2q^2;q^4)_\infty}\frac{(-zyq^3;q^4)_\infty}{(y^2q^2;q^4)_\infty}
\end{aligned}
$$

(by [1; p.17, eq.(2.2.1)])

$$
= \frac{(-zyq;q^2)_\infty}{(q^4;q^4)_\infty(z^2q^2;q^4)_\infty(y^2q^2;q^4)_\infty},
$$

which is Corollary 1.1. □

Corollary 1.2. Identity (1.1) is valid.

Proof. We note that $\mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \mod 2$ because each is clearly congruent $\mod 2$ to the number being partitioned. Hence

$$\begin{aligned}\sum_{n\geqq 0} t(n)q^n &= \sum_{\substack{n,r,s\geqq 0\\ \frac{r-s}{2}\text{even}}} S_\infty(n,r,s)q^n \qquad (2.7)\\ &= \frac{1}{2}\sum_{n,r,s\geqq 0} S_\infty(n,r,s)q^n(1+i^{r-s})\\ &= \frac{1}{2}\left(\frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(q^2;q^4)_\infty^2} + \frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(-q^2;q^4)_\infty^2}\right)\\ &= \frac{1}{2}\left(\frac{1}{(q;q)_\infty} + \frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(-q^2;q^4)_\infty^2}\right)\\ &= \frac{1}{2}\sum_{n=0}^{\infty}(p(n)+f(n))q^n,\end{aligned}$$

and comparing coefficients of q^n in the extremes of this identity we deduce (1.1). □

3 Further Properties of $t(n)$

Corollary 1.3. $t(5n+4) \equiv 0 \pmod 5$.

Proof. Ramanujan proved [3; p.287, Th. 359] that

$$p(5n+4) \equiv 0 \pmod 5.$$

So it follows from (1.1) that to prove $5|t(5n+4)$ we need only prove that $5|f(5n+4)$.

By (1.2)

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n)q^n &= \frac{(-q;q^2)}{(q^4;q^4)_\infty(-q^2;q^4)_\infty^2} \qquad (3.1)\\
&= \frac{(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty}{(q^4;q^4)_\infty^2(-q^2;q^4)_\infty^2}\\
&= \frac{\sum_{n=-\infty}^{\infty} q^{2n^2-n}}{(-q^2;-q^2)_\infty^2} \text{ (by [1; p.21, eq.(2.2.10)])}\\
&= \frac{\sum_{n=-\infty}^{\infty} q^{2n^2-n}(-q^2;-q^2)_\infty^3}{(-q^2;-q^2)_\infty^5}\\
&\equiv \frac{\sum_{n=-\infty}^{\infty} q^{2n^2-n}\sum_{j=0}^{\infty}(-1)^{j+j(j+1)/2}(2j+1)q^{j^2+j}}{(-q^{10};-q^{10})_\infty} \pmod 5\\
&\qquad \text{(by [3; p.285, Th.357]).}
\end{aligned}$$

Now the only time an exponent of q in the numerator is congruent to 4 mod 5 is when $n \equiv 4 \pmod 5$ and $j \equiv 2 \pmod 5$. But then $(2j+1) \equiv 0 \pmod 5$. I.e. the coefficient of q^{5m+4} in the numerator must be divisible by 5. Given that the denominator is a function of q^5, it cannot possibly affect the residue class of any term when it is divided into the numerator. So

$$f(5n+4) \equiv 0 \pmod 5.$$

Therefore

$$t(5n+4) \equiv 0 \pmod 5.$$

□

Corollary 1.4.

$$\sum_{n\geqq 0} t(n)q^n = \frac{Q(q^2)^2Q(q^{16})^4}{Q(q)Q(q^4)^5Q(q^{32})^2}, \qquad (3.2)$$

where

$$Q(q) = (q;q)_\infty. \tag{3.3}$$

Proof. By (2.7)

$$\begin{aligned}
\sum_{n\geqq 0} t(n)q^n &= \frac{1}{2}\left(\frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(q^2;q^4)_\infty^2} + \frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(-q^2;q^4)_\infty^2}\right) \\
&= \frac{(-q;q^2)_\infty}{2(q^4;q^4)_\infty^2(q^2;q^4)_\infty^2(-q^2;q^4)_\infty^2}\left((q^4;q^4)_\infty(-q^2;q^4)_\infty^2 + (q^4;q^4)_\infty(q^2;q^4)_\infty^2\right) \\
&= \frac{(-q;q^2)_\infty}{2(q^4;q^4)_\infty^2(q^4;q^8)_\infty^2}\left(\sum_{n=-\infty}^{\infty} q^{2n^2} + \sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}\right) \\
&\qquad \text{(by [1; p.21, eq.(2.2.10)])} \\
&= \frac{(-q;q^2)_\infty \sum_{n=-\infty}^{\infty} q^{8n^2}}{(q^4;q^4)_\infty^2(q^4;q^8)_\infty^2} \\
&= \frac{(-q;q^2)_\infty(q^{16};q^{16})_\infty(-q^8;g^{16})_\infty^2}{(q^4;q^4)_\infty^2(q^4;q^8)_\infty^2} \\
&= \frac{Q(q^2)^2Q(q^{16})^5}{Q(q)Q(q^4)^5Q(q^{32})^2},
\end{aligned}$$

where the last line follows from several applications of the two identities

$$(q;q^2)_\infty = \frac{Q(q)}{Q(q^2)},$$

and

$$(-q;q^2)_\infty = \frac{Q(q^2)^2}{Q(q)Q(q^4)}.$$

□

Corollary 1.4 allows us to multisect the generality function for $t(n)$ modulo 4.

Corollary 1.5

$$\sum_{n\geqq 0} t(4n)q^n = (q^{16};q^{16})_\infty(-q^7;q^{16})_\infty(-q^9;q^{16})_\infty W(q), \tag{3.4}$$

$$\sum_{n\geqq 0} t(4n+1)q^n = (q^{16};q^{16})_\infty(-q^5;q^{16})_\infty(-q^{11};q^{16})_\infty W(q), \tag{3.5}$$

$$\sum_{n\geqq 0} t(4n+2)q^n = q(q^{16};q^{16})_\infty(-q;q^{16})_\infty(-q^{15};q^{16})_\infty W(q), \tag{3.6}$$

$$\sum_{n\geqq 0} t(4n+3)q^n = q(q^{16};q^{16})_\infty(-q^3;q^{16})_\infty(-q^{13};q^{16})_\infty W(q), \tag{3.7}$$

where

$$W(q) = \frac{Q(q^4)^5}{Q(q)^5 Q(q^8)^2}. \tag{3.8}$$

Proof. We begin with Gauss's special case of the Jacobi Triple Product Identity [1; p.23, eq.(2.2.13)]

$$\sum_{n=-\infty}^{\infty} q^{2n^2-n} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{Q(q^2)^2}{Q(q)} \tag{3.9}$$

Therefore by Corollary 1.4, we see that

$$\sum_{n\geqq 0} t(n)q^n = W(q^4)\sum_{n=-\infty}^{\infty} q^{2n^2-n}. \tag{3.10}$$

Now $2n^2 - n \equiv n \pmod 4$. So to obtain (3.4)-(3.7) we multisect the right-hand series in (3.10) by setting $n = 4m + j$ $(0 \leqq j \leqq 3)$, so

$$\sum_{n\geqq 0} t(n)q^n = W(q^4)\sum_{j=0}^{3}\sum_{m=-\infty}^{\infty} q^{2(4m+j)^2-(4m+j)}.$$

One then obtains four identities arising from the four residue classes mod 4. We carry out the full calculations in the case $j = 0$:

$$\sum_{n\geqq 0} t(4n)q^{4n} = W(q^4)\sum_{m=-\infty}^{\infty} q^{32m^2-4m}$$
$$= W(q^4)(q^{64};q^{64})_\infty(-q^{28};q^{64})_\infty(-q^{36};q^{64})_\infty,$$

a result equivalent to (3.4) once q is replaced by $q^{1/4}$. The remaining results are proved similarly. □

4 Conclusion

As is obvious, Theorem 1 is easily proved once it is stated, but the sums appearing in (1.3) and (1.4) seem to arise from nowhere.

I note that by considering the cases $N = 1, 2, 3, 4$, I discovered empirically that

$$\sum_{n,r,s\geqq 0} S_{2N}(n,r,s)q^n z^r y^s \tag{4.1}$$
$$= \frac{1}{(q^4;q^4)_N}\sum_{j=0}^{N}\frac{(-zyq;q^2)_{2j}}{(z^2q^2;q^4)_j}\begin{bmatrix}N\\j\end{bmatrix}; q^4\Big](y^2q^2)^{N-j}$$

and

$$\sum_{n,r,s\geqq 0} S_{2N+1}(n,r,s)q^n z^r y^s \tag{4.2}$$
$$= \frac{1}{(q^4;q^4)_N}\sum_{j=0}^{N}\frac{(-zyq;q^2)_{2j+1}}{(z^2q^2;q^4)_{j+1}}\begin{bmatrix}N\\j\end{bmatrix}; q^4\Big](y^2q^2)^{N-j}.$$

One can then pass to (1.3) and (1.4) by means of a ${}_3\phi_2$ transformation [3; p.242, eq.(III.13)], and the proof of Theorem 1 is easiest using (1.3) and (1.4).

There are many mysteries surrounding many of the identities in this paper.

Problem 1. Is there a partition statistic that will divide the partitions enumerated by $t(5n + 4)$ into five equinumerous classes? Dyson's rank (largest part - number of parts) provides such a division at least for $n = 0$ and 1 (cf. [1; p.175]).

Problem 2. Identity (1.7) cries out for combinatorial proof.

References

[1] G.E. Andrews. *The Theory of Partitions*, Addison-Wesley, Reading, 1976 (Reissued: Cambridge University Press, Cambridge, 1998).

[2] G. Gasper and M. Rahman. *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[3] G.H. Hardy and E.M. Wright. *An Introduction to the Theory of Numbers*, 4 ed., Oxford University Press, Oxford, 1960.

[4] R.P. Stanley. Problem 10969, *Amer. Math. Monthly*, **109** (2002), 760.

[5] R.P. Stanley. Some remarks on sign-balanced and maj-balanced possets (to appear).

The Pennsylvania State University
University Park, PA 16802
andrews@math.psu.edu