Proposed Problem: Large Values of Tan n

David P. Bellamy,
University of Delaware
Newark, DE 19716-2553

Jeffrey C. Lagarias
AT&T Labs – Research
Florham Park, NJ 07932-0971

Felix Lazebnik
University of Delaware
Newark, DE 19716-0001

1. Proposed Problem

(a). *Show that there are infinitely many positive integers n such that*

$$|\tan\ n| > n\ . \tag{1}$$

(b). *Show that there are infinitely many positive integers n such that*

$$\tan\ n > \frac{1}{4} n\ . \tag{2}$$

2. Solution

The proofs require showing that there are integers n sufficiently close to $(2k+1)\frac{\pi}{2}$ for some integer k. We use the following Diophantine approximation lemmas.

(a). **Lemma 1.** *Every irrational real θ has infinitely many Diophantine approximations $q \equiv 1 \pmod 2$ with*

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{2q^2}\ . \tag{3}$$

(b). **Lemma 2.** *Every irrational real θ has infinitely many (one-sided) Diophantine approximations $q \equiv 1 \pmod 2$ with*

$$\frac{2}{q^2} > \theta - \frac{p}{q} > 0 \ . \tag{4}$$

Lemma 1 is sharp in the sense that the constant 2 cannot be improved. Similarly Lemma 2 is sharp in that the constant 2 cannot be improved.

Solution. If $n = (2k+1)\frac{\pi}{2} + y$, where k is an integer and y is small, say $|y| < \frac{1}{4}$, then

$$\begin{aligned} \tan\ n &= \mathit{cot}\ y \\ &= \frac{1 + \frac{y^2}{2!} + O(y^4)}{y - \frac{y^3}{3!} + O(y^5)} \\ &= \frac{1}{y} + y + O(y^3) \ . \end{aligned} \tag{5}$$

Suppose now that

$$|y| = \left| n - (2k+1)\frac{\pi}{2} \right| < \frac{1}{\alpha(2k+1)} \tag{6}$$

for some constant α, then (5) gives

$$\begin{aligned} |\tan\ n| &> \alpha(2k+1) - O(1) \\ &> \frac{2\alpha}{\pi} n - O(1) \ . \end{aligned}$$

To get infinitely many approximations n of the desired quality, we need infinitely many solutions to (6) for fixed α such that:

case (a). $\alpha > \frac{\pi}{2} \cong\ 1.571$.

case (b). $\alpha > \frac{\pi}{8} \cong\ .393$, and n satisfies the *one-sided* approximation condition

$$-\frac{1}{\alpha(2k+1)} < n - (2k+1)\frac{\pi}{2} < 0 \ . \tag{7}$$

The inequalities (6) and (7) can be rewritten as Diophantine approximations to the irrational $\theta = \frac{\pi}{2}$, namely

$$\left| \frac{\pi}{2} - \frac{n}{2k+1} \right| < \frac{1}{\alpha(2k+1)^2} \ , \tag{8}$$

and

$$\frac{1}{\alpha(2k+1)^2} > \frac{\pi}{2} - \frac{n}{2k+1} > 0 \ . \tag{9}$$

These two cases (a) and (b) are thus covered by Lemmas 1 and 2, respectively. □

Proof of Lemma 1. One out of every two convergents $\frac{p_n}{q_n}$ of the continued fraction expansion of $\theta = [a_0, a_1, a_2, \ldots]$ satisfies

$$|\theta - \frac{p_n}{q_n}| < \frac{1}{2q_n^2} \ . \tag{10}$$

(See Hardy and Wright, *The Theory of Numbers,* Theorem 183.) At least one of every two consecutive convergents has an odd denominator, since

$$\det \begin{vmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{vmatrix} = -1 \ . \tag{11}$$

If infinitely many consecutive pairs (q_n, q_{n+1}) are both odd, then one of each pair satisfies (10), so we are done.

The remaining case is where there are finitely many such pairs. In this case the q_n's alternate odd, even, odd, even, ... from some point on. Suppose for convenience that the q_{2n} are all odd, the q_{2n+1} are all even, from some point on. Then

$$q_{2n+1} = a_{2n+1} q_{2n} + q_{2n-1} \ ,$$

whence $a_{2n+1} \equiv 0 \pmod 2$. Thus $a_{2n+1} \geq 2$ and $q_{2n+1} \geq 2q_{2n}$. Now [Hardy and Wright, Theorem 164] gives

$$\left|\theta - \frac{p_{2n}}{q_{2n}}\right| \leq \frac{1}{q_{2n} q_{2n+1}} \ ,$$

and

$$\left|\theta \frac{p_{2n}}{q_{2n}}\right| \leq \frac{1}{q_{2n}(2q_{2n})} \leq \frac{1}{2q_{2n}^2} \ ,$$

as required. A similar argument works if all q_{2n+1} are even and all q_{2n+2} odd, from some point on. □

Proof of Lemma 2. The continued fraction convergents satisfy

$$\frac{p_{2n}}{q_{2n}} > \theta > \frac{p_{2n+1}}{q_{2n+1}} \ , \tag{12}$$

so if infinitely many q_{2n+1} are odd, we are done.

Now suppose all q_{2n+1} are even from some point on, in which case all q_{2n} are odd from that point on, using (11). As in Lemma 1, we have $a_{2n+1} \equiv 0 \pmod 2$ from that point on. Now set

$$\begin{aligned} Q &:= q_{2n+1} - q_{2n} \ , \\ P &:= p_{2n+1} - p_{2n} \ . \end{aligned}$$

Certainly Q is odd, and (12) gives

$$\frac{P}{Q} = \frac{p_{2n+1} - p_{2n}}{q_{2n+1} - q_{2n}} < \frac{p_{2n+1}}{q_{2n+1}} < \theta \ . \tag{13}$$

Next, we have

$$\begin{aligned} |Q\theta - P| &= |(q_{2n+1}\theta - p_{2n+1}) - (q_{2n}\theta - p_{2n})| \\ &\le |q_{2n+1}\theta - p_{2n+1}| + |q_{2n}\theta - p_{2n}| \\ &\le \frac{1}{q_{2n+2}} + \frac{1}{q_{2n+1}} \\ &\le \frac{2}{q_{2n+1} - q_{2n}} = \frac{2}{Q} \ . \end{aligned} \tag{14}$$

Now (13) and (14) give

$$\frac{2}{Q^2} > \theta - \frac{P}{Q} > 0 \ ,$$

as required. □

Remarks.

(1). Extremal θ for Lemma 1 include $\theta = [a_0, a_1, \ldots]$ with all $a_{2n+1} = 2$ and $a_{2n} \to \infty$ as $n \to \infty$.

(2). Extremal θ for Lemma 2 include $\theta = [a_0, a_1, \ldots]$ with all $a_{2n} = 1$, with $a_{2n+1} \equiv 0 \pmod 2$ and $a_{2n+1} \to \infty$ as $n \to \infty$. [It requires a proof to show that $Q = q_{2n+1} - q_{2n}$ are the "best" odd denominator one-sided approximations.]

(3). Presumably for each $\alpha > 0$ there exist infinitely many positive n such that

$$\tan\ n > \alpha n \ .$$

This would be true if $\frac{\pi}{2}$ were a "random" real number.

email: 05789@brahms.udel.edu
jcl@research.att.com
fellaz@math.udel.edu