New coverings of t-sets with $(t+1)$-sets

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Abstract

The minimum number of k-subsets out of a v-set such that each t-set is contained in at least one k-set is denoted by $C(v, k, t)$. In this paper, a computer search for finding good such covering designs, leading to new upper bounds on $C(v, k, t)$, is considered. The search is facilitated by predetermining automorphisms of desired covering designs. A stochastic heuristic search (embedded in the general framework of tabu search) is then used to find appropriate sets of orbits. A table of upper bounds on $C(v, t+1, t)$ for $v \leq 28$ and $t \leq 8$ is given, and the new covering designs are listed.

1 Introduction

A t-(v, k, λ) *covering design* is a pair $(X, \mathcal{B})$, where X is a v-set of points and $\mathcal{B}$ is a family of k-subsets of X, called blocks, such that each t-subset of X is contained in at least λ blocks. The minimum number of blocks in such a family $\mathcal{B}$ is denoted by $C_\lambda(v, k, t)$. If $\lambda = 1$, we omit λ and write $C(v, k, t)$. (A more general definition of covering designs is discussed, for example, in [7, 19, 25]. The methods in this paper can straightforwardly be generalized to that approach.) For recent surveys of covering designs, see [12, 17, 26].

The problem of determining values of $C_\lambda(v, k, t)$ is highly nontrivial even for relatively small values of the parameters. Hence, exact values are known only in special cases. For other sets of parameters, there is a gap between the best known lower and upper bounds. Upper bounds are obtained by explicitly constructing covering designs. In some cases, the best known designs can be obtained by combinatorial or algebraic methods, whereas in other cases, the record-holders have been found by computer search. In [19], for example, the authors used a stochastic heuristic called simulated annealing to find good covering designs.

Many of the bounds in [19] are still the best currently known. Unfortunately, the performance of the method used in that study deteriorates with increasing value of v. Here we shall see how we can attack this problem by predetermining a structure (automorphism group) of the covering design. Furthermore, another stochastic method apparently superior to simulated annealing, tabu search, is used in the search. Note that due to the heuristic nature of the search method, if there exists a (not necessarily optimal) covering design with given parameters and automorphism group, we may still fail to find it. Also, the choice of automorphisms affects the outcome of the search. Similar results for packing designs (constant weight error-correcting codes) have recently been reported in [18].

Covering designs with predetermined automorphisms are discussed in Section 2 and the search method is outlined in Section 3. Old bounds and constructions that give new bounds from old ones are considered in Section 4. Finally, the search results are presented in Section 5, where the new covering designs are listed and an updated table of $C(v, t+1, t)$ for $v \leq 28$, $3 \leq t \leq 8$ is given.

2 Covering Designs with Given Automorphisms

The first extensive results on computer search for covering designs were presented by Stanton and Bate [25], who used exhaustive search. By using exhaustive search, we are guaranteed to find the exact value of $C_\lambda(v, k, t)$. Unfortunately, this approach is too time-consuming to be useful for all but the smallest parameters. When exhaustive search is not fast enough, we can try a partial search, as in this work. If such a search is successful, a covering design is found, and the number of blocks gives an upper bound on $C_\lambda(v, k, t)$.

The idea of using a stochastic heuristic was used by Nurmela and Östergård in [19], where simulated annealing was applied to finding covering designs. Several of the bounds that were obtained (for $v \leq 13$) are still record-holders, as can be seen from Table II to be presented later. Unfortunately, not even that approach—in which one tries to explicitly find all blocks of the covering design—is a panacea, and there is a point where it is not effective any more. Namely, if there are too many blocks in the desired covering design, then the search needs an unreasonably long time to find solutions (if these are found at all)—use of shorter running times in these cases will lead to convergence to configurations that are not coverings. One remedy, which will be developed here, is to predetermine a structure (automorphism group), and, instead of searching for a set of blocks, search for a set of orbits under this automorphism group.

The approach of predetermining an automorphism group is a well-known approach used extensively in searches for t-designs since it was introduced by Kramer and Mesner [15]. It is known that it can be used to search for packing designs (see, for example, [2, 18]). Here it is analogously carried over to the problem of finding covering designs, introducing some novel ideas in the way the search is carried out.

In the search, we first take a permutation group G acting on the point set X. The blocks of a design are derived from base blocks $B_1, \ldots, B_s$ by taking the blocks B_j^α for all $1 \leq j \leq s$ and $\alpha \in G$. Let X_i $(i \leq n)$ be the set of all i-subsets of X. Given a permutation group G, we first construct a so-called A_{tk}-matrix. This is done by first labeling the rows and columns of this matrix with all orbits under the induced action of G on X_t and X_k, respectively (note that these orbits partition all t-sets and k-sets, respectively). Then we set $a_{ij} = u$ to denote that the blocks in the ith orbit of t-sets are covered u times ($u \geq 0$) by the blocks in the jth orbit of k-sets. Furthermore, to each column we associate a value c_j giving the number of k-sets in the corresponding orbit. We let $c = (c_1\ c_2\ \ldots)$.

Having obtained the A_{tk}-matrix in the aforementioned way, we now want to find a column vector x (with non-negative integer entries) such that cx (which gives the number of blocks of the covering design) is minimized for $A_{tk}x \geq \lambda J$ (J is the all-one column vector).

If we are using a permutation group of large order so that the size of the A_{tk}-matrix is small, the problem of minimizing cx for vectors x such that $A_{tk}x \geq \lambda J$ can be solved exactly. However, for the cases considered in this work, the size of the A_{tk}-matrix has been as large as 4466×9897, so a stochastic heuristic search (an implementation of tabu search), which will be discussed in the next section, was adopted in the search.

The permutation groups G used in this work are mainly cyclic groups and affine groups. In several designs we have one or more positions that are not permuted; these are called fixed positions. Since groups may have several permutation representations, the generators used are explicitly listed in Table III.

3 Search Method

Although only the case $\lambda = 1$, $k = t+1$ is considered in Table II, in this paper we shall discuss a search method that works for any $t < k$ and is easily generalized for $\lambda \geq 1$.

Tabu search [9] is a general framework for a class of local search algorithms, where we start from an initial solution and then repeatedly modify the solution slightly, in hope of finally finding a globally optimal solution. In tabu search we always move to the best (lowest cost for minimization problems) solution in the neighborhood of the current solution, unless the move is forbidden, tabu.

To prevent looping around a local optimum, a tabu condition is used. A (recency based) tabu condition prevents the inverses of the latest few moves from being used. The selections of a cost function, possible moves applicable to a solution (that is, a neighborhood), and tabu conditions are rather tricky in difficult optimization problems, and they are usually based on some experimentation and problem specific heuristics.

If there are k-set orbits of only one size r, we can try to find a covering with s orbits such that $sr < C$, where C is the number of blocks in the best previously known design. A solution is now a collection of s orbits of k-sets. The cost of a solution is the number of orbits of t-sets that are not covered by any orbits of the solution. This cost function is easily generalized for $\lambda > 1$ using [19, Eq. (5)]. If the cost reaches zero, we have found a desired covering design.

In most cases, however, there are k-set orbits of several sizes. Then we can find all maximal combinations of orbit sizes that would improve the current bound. We then try a search for each orbit size combination. In some cases there is a too large number of possible orbit size combinations; then we have to concentrate only on some of them.

In the search we use a neighborhood closely resembling that in [20]. To find the neighbors of the current solution we check which orbits of t-sets are not covered by the current solution. In the final solution each of these orbits should be covered. We select one of the uncovered orbits and find all solutions such that they

1. cover the selected uncovered orbit, and

2. differ from the current solution so that one orbit in the solution is changed to another orbit of the same size.

Of these neighbors we select the one with lowest cost. If there are several equally good neighbors, we pick one of them at random. The number of orbits and blocks stays constant during one search process.

Usually there are more than one uncovered orbit and we have to select one of them to compute the neighborhood of the current solution. The selection of the uncovered orbit to consider can be done in many different ways. We have used an index variable that is incremented cyclically through all the orbits of t-sets. The first uncovered orbit encountered is selected.

To complete our tabu search implementation we specify a tabu condition: when a move introduces an orbit to the current solution, then that orbit cannot be changed to another orbit during the next l moves. Suitable values of l vary depending on the parameters of the optimization instance, but, for example $l = s/10$ often works reasonably well.

4 Bounds and Constructions

4.1 Old Bounds

We shall now discuss some of the old bounds. We restrict Table II to $t \geq 3$, because trivially $C(v,2,1) = \lceil v/2 \rceil$, and Fort, Jr. and Hedlund [8] showed already in the 1950s that $C(v,3,2) = \lceil \frac{v}{3} \lceil \frac{v-1}{2} \rceil \rceil$.

In the two aforementioned cases, the so-called *Schönheim lower bound* is met. This bound says that

$$C_\lambda(v,k,t) \geq L_\lambda(v,k,t) = \lceil \frac{v}{k} \lceil \frac{v-1}{k-1} \cdots \lceil \frac{v-t+1}{k-t+1} \lambda \rceil \cdots \rceil \rceil$$

(where we write $L(v,k,t)$ if $\lambda = 1$). The case $C(v,4,3)$ is settled except for a finite number of cases: Hartman, Mills, and Mullin [13] proved that $C(v,4,3) = L(v,4,3)$ for $v \geq 52423$. The following result is due to Mills [16]:

$$C(v,4,3) = \lceil \frac{v}{4} \lceil \frac{v-1}{3} \lceil \frac{v-2}{2} \rceil \rceil \rceil \text{ for } v \not\equiv 7 \pmod{12}. \quad (1)$$

When k is close to v, it is a straightforward task to prove that

$$C(v, v, t) = 1 \text{ and} \tag{2}$$

$$C(v, v-1, t) = t+1, \tag{3}$$

but the case $k = v-2$ is already more complicated. Turán [27] was able to prove that

$$C(v, v-2, t) = L(v, v-2, t). \tag{4}$$

This case was actually settled for $t = v-3$ by Mantel in 1907 (see [24]). Schönheim [22] showed that if there is a t-$(v, k, 1)$ design (that is, a Steiner system), then in addition to $C(v, k, t)$, also $C(v+1, k, t)$ meets the Schönheim bound.

Theorem 1 *If there exists a t-$(v, k, 1)$ design, then $C(v+1, k, t) = L(v+1, k, t)$.*

4.2 New Designs from Old Ones

In the sequel, some results are discussed where upper bounds (for example, the new bounds from Table III) are used to get other upper bounds. For example, it is not difficult to show that

$$C(v, k, t) \leq C(v-1, k, t) + C(v-1, k-1, t-1). \tag{5}$$

In the next few bounds (cf. [12]), the distribution of points affects the result. For a covering design with M blocks, let a point occur in M_1 blocks and let a (possibly different) point be absent from M_0 blocks. For any design, by suitable point selection the following averaging formulas hold:

$$M_0 \leq \lfloor (v-k)M/v \rfloor, \;\; M_1 \leq \lfloor kM/v \rfloor. \tag{6}$$

If there is a covering design giving an upper bound on $C(v+1, k+1, t+1)$ that has a point occurring in M_1 blocks, then

$$C(v, k, t) \leq M_1. \tag{7}$$

Sidorenko (see [12]) developed a construction leading to the following result. We give the proof, since this construction is also used in a theorem to be presented later. Assume that there is a $(t-1)$-$(v-1, k-1, 1)$ covering design with M blocks and a point that is absent from M_0 blocks.

Theorem 2 $C(v, k, t) \leq M + M_0 + C(v-2, k, t)$.

Proof. Let D_1 be a $(t-1)$-$(v-1, k-1, 1)$ covering design and let D_2 be a t-$(v-2, k, 1)$ covering design, where the point set of D_2 is a subset of the point set of D_1. By considering the single point p that is not in the point set of D_2 (any point of D_1 can be made this point by permuting the points) and a new point q, create the following new block set D_1' from D_1: adjoin q to all blocks which contain p, and replace the other blocks (which do not contain p) by two blocks obtained by adjoining p only and q only, respectively. The covering design $D_1' \cup D_2$ is then a desired covering design, the parameters of which are obvious (we can take D_2 to be an optimal covering design); the covering property is shown by a case-by-case argument depending on whether a given t-set contains the points p and q or not.

From Theorem 2 and (6), we get the following result.

Corollary 1 $C(v,k,t) \leq \lfloor (2v-k-1)C(v-1,k-1,t-1)/(v-1) \rfloor + C(v-2,k,t)$.

Occasionally, good bounds can be obtained by repeatedly applying Theorem 2. In that case, the possible values of M and M_0 in subsequent combinations may be affected by permuting the coordinates of the combined designs. This observation leads to the following two-step (clearly, it can be generalized to any number of steps) Sidorenko construction.

For two given points, let M_{ij} $(i,j \in \{0,1\})$ denote the number of blocks in which these points occur (subindex 1) or do not occur (0) in a $(t-2)$-$(v-2,k-2,1)$ covering design with M blocks. So, for example, M_{10} is the number of blocks in which the first point occurs, but not the second one. Furthermore, let M'_0 be the number of blocks in which a given point does not occur in a $(t-1)$-$(v-3,k-1,1)$ covering design with M' blocks.

Theorem 3 $C(v,k,t) \leq M + 3M_{00} + M_{01} + M_{10} + M' + M'_0 + C(v-2,k,t)$.

Proof. Let D_1 be a $(t-2)$-$(v-2,k-2,1)$ covering design, let D_2 be a $(t-1)$-$(v-3,k-1,1)$ covering design, and let D_3 be a t-$(v-2,k,1)$ covering design.

We essentially apply Theorem 2 twice. In the first step, we combine D_1 and D_2 to get a $(t-1)$-$(v-1,k-1,1)$ covering design, and then finally combine this with D_3 in the same way. From the proof of Theorem 2 it is clear that we get a t-$(v,k,1)$ covering design. In the second step, we take neither p nor q (see proof of Theorem 2) as the new p. Direct counting shows that the total number of blocks is as stated.

To apply (7), Theorem 2, and Theorem 3, we need to know how the points are distributed in known covering designs. If these have even point distribution, we can get the necessary parameters by (6). In Table I, we give the smallest number of blocks in which some point in a known covering design occurs, $\min\{M_1\}$, and in which some point does not occur, $\min\{M_0\}$ (the total number of blocks is M); the minimum value of an expression needed in Theorem 3 is also listed.

INSERT TABLE I HERE.

We conclude this section by discussing a case that needs more detailed consideration.

Example 1. From Theorem 2, we have that $C(25,9,8) \leq M + M_0 + C(23,9,8)$, where M is known (56337). Equation (6) gives $M_0 \leq 37558$, but we can do slightly better. The upper bound for $C(24,8,7)$ is obtained by (5), where two covering designs giving the upper bounds $C(23, 7, 6) \leq 17375$ and $C(23, 8, 7) \leq 38962$ are taken and the new point is added to all blocks of the former covering design. For the new point of the covering design constructed this way, clearly there are 38962 blocks in which the new point does not occur; already this, as $38962 > 37558$, indicates that the average among the rest of the points must be smaller than 37558 blocks. If this average for the other points is calculated it is $56337 - (56337 \cdot 8 - 17375)/23 = 37496.9\ldots$ and we get that $C(25, 9, 8) \leq 168996$.

5 The Results

The new covering designs found by the computer search are listed in Table III. The base blocks are given in hexadecimal notation so that 0, 1, 2,…, 9, A, B,…, F correspond to the binary sequences 0000, 0001, 0010,…, 1001, 1010, 1011,…, 1111, respectively. Leading 0's are not shown, and the rightmost position is numbered 0, the leftmost $n-1$. The size of the orbit of a base block is shown as a superscript. The permutation group is given in brackets. For the groups, we have adopted the notation used in [5]. The generating permutations (in the cyclic notation) for the group are also given, since they can be constructed in many different ways. Fixed positions (if any) are those that are not listed in the generating permutations.

In some rare cases new covering designs obtained by the search method in Section 3 can be further improved by removing single blocks. In a preliminary version of this paper we presented a covering design showing that $C(14, 9, 8) \leq 472$. Gordon [11] then pointed out that it is possible to remove a block to further tighten the bound to $C(14, 9, 8) \leq 471$. Indeed, there are two orbits of five blocks (orbits $1FF^5$ and $2FF^5$ in Table III) such that any single block of these orbits can be removed.

An updated table of upper bounds on $C(v, t+1, t)$ for $v \leq 28$, $3 \leq t \leq 8$ is given in Table II, where exact values are indicated by a period. For exact values, the reference is to a proof of this value. For upper bounds, the reference is to a general construction giving this bound, or, if no such construction exists, to the paper where the result was first published in the scientific literature. Several of the bounds with keys *ui* and *vi* were obtained by starting from codes in a preliminary version of this paper and slightly improving these using heuristic computer algorithms.

INSERT TABLE II HERE.

INSERT TABLE III HERE.

Acknowledgments

We are grateful to Dan Gordon and John van Rees for valuable suggestions and comments on an earlier version of this paper. We would like to thank Rado Belić, Iliya Bluskov, Adolf Mühl, and Dietmar Pree for informing us about unpublished, record-breaking covering designs. The research was supported by the Academy of Finland.

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TABLE CAPTIONS LIST & KEYS

TABLE I. Point distributions for covering designs

TABLE II. Upper bounds for $C(v, t+1, t)$, $v \leq 28$, $3 \leq t \leq 8$

TABLE III. New covering designs

Key to Table II.

. — Exact value
a — This paper
b — Eq. (2)
c — Eq. (3)
d — Eq. (4)
e — Katona, Nemetz, and Simonovits [14]
f — Computer proofs (see [17, 24] and their references)
g — Sidorenko [23] and de Caen et al. [3]
h — Etzion, Wei, and Zhang [7]
i — Eq. (1)
j — de Caen, Kreher, and Wiseman [4]
k — Corollary 1
$k1$ — Theorem 2
$k2$ — Theorem 3
m — Eq. (7)
n — Nurmela and Östergård [19]
p — Theorem 1
s — Steiner system [6, 28]
t — Eq. (5)
ui — Unpublished, various constructors: 1) Rade Belić [1], 2) Dietmar Pree [21]
vi — Various constructors [10]: 1) Rade Belić, 2) Dan Gordon, 3) Dietmar Pree, 4) Rade Belić and Dietmar Pree, 5) Adolf Mühl and Dietmar Pree

TABLE I.

Covering design	M	Reference	$\min\{M_0\}$	$\min\{M_1\}$	$\min\{3M_{00}+M_{01}+M_{10}\}$
8-$(15,9,1)$	789	[10]	315	471	741
3-$(24,4,1)$	510	[10]	425	85	1201
7-$(25,8,1)$	78012	This paper	48576	24761	134123

TABLE II.

$v\backslash t$	3	4	5	6	7	8
4	$1.^{b}$					
5	$4.^{c}$	$1.^{b}$				
6	$6.^{d}$	$5.^{c}$	$1.^{b}$			
7	$12.^{e}$	$9.^{d}$	$6.^{c}$	$1.^{b}$		
8	$14.^{i}$	$20.^{e}$	$12.^{d}$	$7.^{c}$	$1.^{b}$	
9	$25.^{i}$	$30.^{g}$	$30.^{e}$	$16.^{d}$	$8.^{c}$	$1.^{b}$
10	$30.^{i}$	51^{h}	$50.^{g}$	$45.^{f}$	$20.^{d}$	$9.^{c}$
11	$47.^{i}$	$66.^{s}$	100^{n}	84^{j}	$63.^{f}$	$25.^{d}$
12	$57.^{i}$	$113.^{p}$	$132.^{s}$	176^{h}	126^{j}	$84.^{f}$
13	$78.^{i}$	157^{n}	$245.^{p}$	264^{n}	297^{n}	185^{j}
14	$91.^{i}$	230^{u1}	371^{a}	508^{v1}	471^{m}	471^{a}
15	$124.^{i}$	295^{v1}	580^{v2}	825^{v1}	979^{t}	789^{v2}
16	$140.^{i}$	405^{v3}	808^{h}	1329^{v1}	1722^{v3}	1768^{t}
17	$183.^{i}$	492^{h}	1213^{t}	2048^{v3}	3040^{a}	3355^{v1}
18	$207.^{i}$	664^{v5}	1547^{v1}	3261^{t}	4690^{v3}	6098^{u2}
19	258^{v5}	846^{a}	2175^{v3}	4608^{v4}	7949^{u1}	10641^{u1}
20	$285.^{i}$	1083^{a}	2900^{h}	6765^{u1}	12134^{u2}	18590^{t}
21	$352.^{i}$	1251^{h}	3979^{v1}	9338^{u1}	18894^{u1}	29961^{u1}
22	$385.^{i}$	1573^{h}	4687^{u1}	13244^{a}	27624^{u1}	48855^{t}
23	$466.^{i}$	$1771.^{s}$	6169^{u1}	17375^{u1}	38962^{a}	75163^{k}
24	$510.^{i}$	$2237.^{p}$	$7084.^{s}$	23276^{a}	56337^{t}	113227^{k}
25	$600.^{i}$	2706^{k}	$9321.^{p}$	29759^{u1}	78012^{a}	168996^{k1}
26	$650.^{i}$	3306^{t}	11952^{k2}	39080^{t}	105000^{a}	239815^{k1}
27	$763.^{i}$	3848^{a}	15210^{a}	50895^{a}	144080^{t}	334854^{a}
28	$819.^{i}$	4550^{a}	18369^{a}	65286^{a}	193595^{k}	470340^{a}

TABLE III.

Bound [Group] Generators	Base Blocks
$C(14,6,5) \leq 371$ $[\mathbb{Z}_{14}]$ (0,1,2,3,4,5,6,7,8,9,10,11,12,13)	3F^{14}, D7^{14}, EB^{14}, 15B^{14}, 173^{14}, 19D^{14}, 1B5^{14}, 21F^{14}, 279^{14}, 29B^{14}, 2CD^{14}, 32B^{14}, 353^{14}, 365^{14}, 45D^{14}, 46B^{14}, 48F^{14}, 4E5^{14}, 539^{14}, 547^{14}, 5A3^{14}, 633^{14}, 927^{14}, 92D^{14}, 955^{14}, 387^{7}, 993^{7}, A95^{7}
$C(14,9,8) \leq 471$ $[\mathbb{Z}_{5}]$ (0,2,4,6,8)(1,3,5,7,9)	1FF^{5}, 2FF^{5}, CBF^{5}, CEF^{5}, CFB^{5}, D5F^{5}, D77^{5}, DAF^{5}, DBE^{5}, DFA^{5}, 147F^{5}, 14F7^{5}, 14FD^{5}, 157B^{5}, 159F^{5}, 15B7^{5}, 16AF^{5}, 16BB^{5}, 18DF^{5}, 18FE^{5}, 196F^{5}, 197E^{5}, 19BB^{5}, 19DE^{5}, 19EB^{5}, 19EE^{5}, 1C7D^{5}, 1CAF^{5}, 1CDE^{5}, 1D57^{5}, 1D6E^{5}, 1EAB^{5}, 24DF^{5}, 24FE^{5}, 256F^{5}, 257E^{5}, 25BB^{5}, 25DE^{5}, 25EB^{5}, 25EE^{5}, 287F^{5}, 28F7^{5}, 28FD^{5}, 297B^{5}, 299F^{5}, 29B7^{5}, 2AAF^{5}, 2ABB^{5}, 2C7D^{5}, 2CAF^{5}, 2CDE^{5}, 2D57^{5}, 2D6E^{5}, 2EAB^{5}, 30BF^{5}, 30EF^{5}, 30FB^{5}, 315F^{5}, 3177^{5}, 31AF^{5}, 31BE^{5}, 31FA^{5}, 347D^{5}, 34AF^{5}, 34DE^{5}, 3557^{5}, 356E^{5}, 36AB^{5}, 387D^{5}, 38AF^{5}, 38DE^{5}, 3957^{5}, 396E^{5}, 3AAB^{5}, 3C3B^{5}, 3C3E^{5}, 3C4F^{5}, 3C67^{5}, 3C6B^{5}, 3C73^{5}, 3C76^{5}, 3C7A^{5}, 3C97^{5}, 3C9B^{5}, 3C9D^{5}, 3CB3^{5}, 3CB5^{5}, 3CB6^{5}, 3CCD^{5}, 3CD9^{5}, 3CE6^{5}, 3CE9^{5}, 3D5A^{5}, 3D9A^{5}, 3D55^{1}, 3EAA^{1}; remove block 1FF
$C(17,8,7) \leq 3040$ $[F_{8,7}]$ (1,2,4,3,6,7,5)(9,10,12,11,14,15,13), (0,1)(2,3)(4,5)(6,7)(8,9)(10,11)(12,13)(14,15)	36F^{56}, 3F5^{56}, 773^{56}, 7AD^{56}, 7AE^{56}, F36^{56}, F53^{56}, F65^{56}, 1717^{56}, 172B^{56}, 173A^{56}, 174D^{56}, 175A^{56}, 1765^{56}, 178E^{56}, 1799^{56}, 17B1^{56}, 17B2^{56}, 17C3^{56}, 17D4^{56}, 17E8^{56}, 1F51^{56}, 1F61^{56}, 1F8A^{56}, 1FA4^{56}, 3F11^{56}, 3F14^{56}, 3F48^{56}, 10357^{56}, 1035E^{56}, 10376^{56}, 103D6^{56}, 1070F^{56}, 10735^{56}, 10759^{56}, 1076C^{56}, 10793^{56}, 107A6^{56}, 107CA^{56}, 107F0^{56}, 10F07^{56}, 10F70^{56}, 1171C^{56}, 11738^{56}, 11746^{56}, 11762^{56}, 11785^{56}, 117A1^{56}, 11F12^{56}, 11F24^{56}, 11F41^{56}, 11F88^{56}, 33F^{28}, 3CF^{28}, 3FC^{28}, 3F03^{28}, 1007F^{8}, 17F00^{8}
$C(19,5,4) \leq 846$ $[\mathbb{Z}_{18}]$ (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17)	4F^{18}, 75^{18}, CB^{18}, 11B^{18}, 159^{18}, 1A3^{18}, 253^{18}, 295^{18}, 299^{18}, 325^{18}, 3C1^{18}, 417^{18}, 439^{18}, 543^{18}, 589^{18}, 661^{18}, 893^{18}, 8C5^{18}, 915^{18}, A23^{18}, B09^{18}, C0D^{18}, CA1^{18}, E11^{18}, 1069^{18}, 10D1^{18}, 1131^{18}, 1283^{18}, 1451^{18}, 1485^{18}, 1889^{18}, 2185^{18}, 2229^{18}, 2309^{18}, 2491^{18}, 4002D^{18}, 40063^{18}, 400A9^{18}, 40107^{18}, 40191^{18}, 40425^{18}, 404C1^{18}, 40851^{18}, 40921^{18}, 41045^{18}, 40603^{9}, 40A05^{9}, 41209^{9}, 42211^{9}

TABLE III (cont.).

Bound [Group] Generators	Base Blocks
$C(20,5,4) \leq 1083$ [$\mathbb{Z}_{19}$] (0,1,2,3,4,5,6,7,8,9,10,11, 12,13,14,15,16,17,18)	2F^{19}, 73^{19}, E9^{19}, 147^{19}, 21B^{19}, 383^{19}, 459^{19}, 48D^{19}, 4C3^{19}, 4D1^{19}, 50B^{19}, 525^{19}, 645^{19}, 6A1^{19}, 711^{19}, 895^{19}, 931^{19}, 961^{19}, A23^{19}, A51^{19}, B05^{19}, B09^{19}, C13^{19}, C29^{19}, D81^{19}, 1055^{19}, 1087^{19}, 1113^{19}, 1189^{19}, 1229^{19}, 1291^{19}, 1449^{19}, 18A1^{19}, 1941^{19}, 1C21^{19}, 2087^{19}, 20B1^{19}, 2115^{19}, 220D^{19}, 2243^{19}, 2321^{19}, 4249^{19}, 4423^{19}, 80099^{19}, 8010B^{19}, 80151^{19}, 801A1^{19}, 80215^{19}, 80229^{19}, 80407^{19}, 80431^{19}, 80825^{19}, 808C1^{19}, 80903^{19}, 81061^{19}, 81105^{19}, 81241^{19}
$C(22,7,6) \leq 13244$ [$F_{11,10}$] (1,2,4,8,5,10,9,7,3,6)(12, 13,15,19,16,21,20,18,14, 17), (0,1,2,3,4,5,6,7,8,9, 10)(11,12,13,14,15,16,17, 18,19,20,21)	8E7^{110}, 8FC^{110}, 93E^{110}, 979^{110}, 182F^{110}, 189B^{110}, 18BA^{110}, 18F1^{110}, 190F^{110}, 1A36^{110}, 1A6A^{110}, 1AC6^{110}, 1B07^{110}, 1B94^{110}, 1C5C^{110}, 1D15^{110}, 381D^{110}, 384B^{110}, 3872^{110}, 38A3^{110}, 391A^{110}, 392A^{110}, 3961^{110}, 3991^{110}, 39C8^{110}, 3A51^{110}, 3A58^{110}, 3A8A^{110}, 3B0C^{110}, 3C98^{110}, 5827^{110}, 5874^{110}, 58AC^{110}, 58C3^{110}, 58D8^{110}, 5926^{110}, 5931^{110}, 5952^{110}, 59B0^{110}, 5A0E^{110}, 5A2C^{110}, 5A61^{110}, 5A85^{110}, 5AA2^{110}, 5B18^{110}, 5B44^{110}, 5B88^{110}, 5C0D^{110}, 5C32^{110}, 5C4A^{110}, 5C51^{110}, 5C98^{110}, 5CA1^{110}, 5CC4^{110}, 5D50^{110}, 5D82^{110}, 5E30^{110}, 5E42^{110}, 5F01^{110}, 5F08^{110}, 7829^{110}, 784C^{110}, 7885^{110}, 7914^{110}, 7A12^{110}, B80E^{110}, B813^{110}, B82C^{110}, B834^{110}, B889^{110}, B922^{110}, BA44^{110}, BA90^{110}, BC09^{110}, BD40^{110}, BD80^{110}, D813^{110}, D845^{110}, D868^{110}, D960^{110}, DA22^{110}, F830^{110}, F8C0^{110}, 13807^{110}, 13889^{110}, 138B0^{110}, 13922^{110}, 13A03^{110}, 13A14^{110}, 13A28^{110}, 13A42^{110}, 13C41^{110}, 13D08^{110}, 17821^{110}, 17940^{110}, 17A80^{110}, 17C10^{110}, 1B841^{110}, 1B860^{110}, 1B884^{110}, 1B910^{110}, 1BA01^{110}, 1BC02^{110}, 1F804^{110}, 27811^{110}, 27822^{110}, 27888^{110}, 27A02^{110}, 27D00^{110}, 2E803^{110}, 2E882^{110}, 2E908^{110}, 4B850^{110}, 4F880^{110}, 7F^{55}, AF4^{55}, 1AE8^{55}, 1C47^{55}, 38C5^{55}, 3925^{55}, 3AD0^{55}, 3B30^{55}, D984^{55}, 2E814^{55}, 2EC80^{55}, 3F800^{55}, A3B^{22}, 5B820^{22}

TABLE III (cont.).

Bound [Group] Generators	Base Blocks
$C(23,8,7) \leq 38962$ $[F_{23,22}]$ (1,5,2,10,4,20,8,17,16,11,9,22,18,21,13,19,3,15,6,7,12,14), (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22)	27F^{506}, 57D^{506}, 5BB^{506}, 5CF^{506}, 5D7^{506}, 96F^{506}, 9D7^{506}, B5D^{506}, CE7^{506}, DA7^{506}, E4F^{506}, 11BB^{506}, 11D7^{506}, 11EB^{506}, 12B7^{506}, 135B^{506}, 1375^{506}, 13CD^{506}, 1477^{506}, 168F^{506}, 16E3^{506}, 1747^{506}, 17A3^{506}, 191F^{506}, 192F^{506}, 1A67^{506}, 1A6B^{506}, 1B17^{506}, 1B87^{506}, 215F^{506}, 21BD^{506}, 22D7^{506}, 232F^{506}, 23AB^{506}, 23CB^{506}, 24AF^{506}, 24F9^{506}, 2537^{506}, 25E3^{506}, 261F^{506}, 2753^{506}, 2787^{506}, 287B^{506}, 28DD^{506}, 29D3^{506}, 2AE3^{506}, 2C57^{506}, 2D47^{506}, 2E2B^{506}, 34C7^{506}, 416F^{506}, 419F^{506}, 41B7^{506}, 433D^{506}, 43E5^{506}, 44CF^{506}, 46D5^{506}, 46F1^{506}, 49CB^{506}, 49E9^{506}, 4AAB^{506}, 4B33^{506}, 4F15^{506}, 4F43^{506}, 5317^{506}, 5395^{506}, 54D5^{506}, 564D^{506}, 5953^{506}, 865D^{506}, 8DC3^{506}, 9553^{506}, 9D23^{506}, A497^{506}, 102EB^{506}, FF^{253}, 37B^{253}, 1CA7^{253}, 352B^{253}
$C(24,7,6) \leq 23276$ $[F_{23,22}]$ See $C(23,8,7)$	19F^{506}, 277^{506}, 2CF^{506}, 34F^{506}, 4BB^{506}, 4F5^{506}, 6C7^{506}, 83F^{506}, 8E7^{506}, 9B5^{506}, B17^{506}, C97^{506}, 10B7^{506}, 11E9^{506}, 123D^{506}, 130F^{506}, 1563^{506}, 1587^{506}, 1617^{506}, 1663^{506}, 1917^{506}, 1A87^{506}, 20BB^{506}, 2179^{506}, 2437^{506}, 2457^{506}, 2695^{506}, 26A3^{506}, 322B^{506}, 4297^{506}, 80006F^{506}, 80021F^{506}, 800533^{506}, 800547^{506}, 800927^{506}, 800939^{506}, 800953^{506}, 800C87^{506}, 80108F^{506}, 801171^{506}, 8020E9^{506}, 17D^{253}, 1D7^{253}, 727^{253}, 1C47^{253}, 80016D^{253}, 8002B5^{253}, 8002CD^{253}, 80058D^{253}, 80068B^{253}, 8008F1^{253}

TABLE III (cont.).

Bound [Group] Generators	Base Blocks
$C(25,8,7) \leq 78012$ $[L_2(11)]$ (0,10,9,8,7,6,5,4,3,2,1)(12,22,21,20,19,18,17,16,15,14,13), (0,11)(1,10)(2,5)(3,7)(4,8)(6,9)(12,23)(13,22)(14,17)(15,19)(16,20)(18,21)	$305F^{660}$, $30BD^{660}$, 3137^{660}, $316B^{660}$, $317C^{660}$, $31BA^{660}$, $31D6^{660}$, $31D9^{660}$, $331D^{660}$, $332E^{660}$, $33B4^{660}$, $347A^{660}$, $34DC^{660}$, $F03A^{660}$, $F05C^{660}$, $F071^{660}$, $F096^{660}$, $F0A9^{660}$, $F0C5^{660}$, $F115^{660}$, $F12C^{660}$, $F149^{660}$, $F162^{660}$, $F18A^{660}$, $F1A4^{660}$, $F1B0^{660}$, $F1C2^{660}$, $F21C^{660}$, $F289^{660}$, $F291^{660}$, $F2D0^{660}$, $F2E0^{660}$, $F312^{660}$, $F321^{660}$, $F416^{660}$, $F483^{660}$, $F48C^{660}$, $F4A8^{660}$, $F4C1^{660}$, $F550^{660}$, $F682^{660}$, $F8A2^{660}$, $F981^{660}$, $FA90^{660}$, $1F00D^{660}$, 27039^{660}, 27056^{660}, $2708B^{660}$, $2709C^{660}$, $270C5^{660}$, $270D1^{660}$, 27158^{660}, $271B0^{660}$, 27213^{660}, 27292^{660}, 27415^{660}, $2741A^{660}$, 27609^{660}, 27628^{660}, 27650^{660}, $3F042^{660}$, $3F104^{660}$, $3F220^{660}$, $5F022^{660}$, $6F012^{660}$, $7700A^{660}$, $7B006^{660}$, $7D005^{660}$, $7D00C^{660}$, $7D018^{660}$, $7D082^{660}$, $7D102^{660}$, $7D202^{660}$, $100007F^{660}$, $100107B^{660}$, $100311D^{660}$, 1003169^{660}, $10031D2^{660}$, 1003334^{660}, 1003478^{660}, $10034D4^{660}$, $100700F^{660}$, 1007017^{660}, 1007027^{660}, $100703C^{660}$, 1007059^{660}, $100706A^{660}$, $10070E4^{660}$, $100714A^{660}$, 1007191^{660}, 1007258^{660}, 1007289^{660}, 1007344^{660}, 1007530^{660}, 1007921^{660}, 1007930^{660}, $100F098^{660}$, $100F0A1^{660}$, $100F211^{660}$, $100F230^{660}$, $100F282^{660}$, $100F380^{660}$, $100F482^{660}$, $100F490^{660}$, $101F028^{660}$, 1027007^{660}, 1027049^{660}, $10270D0^{660}$, 1027411^{660}, $102F006^{660}$, $102F024^{660}$, $102F050^{660}$, $102F101^{660}$, $102F208^{660}$, $102F600^{660}$, $102F880^{660}$, $107F000^{660}$, $364E^{132}$, $5F240^{132}$, $13F008^{132}$, $100013F^{132}$, $100117C^{132}$, $101F240^{132}$

TABLE III (cont.).

Bound [Group] Generators	Base Blocks
$C(26,8,7) \leq 105000$ $[F_{25,24}]$ (1,5,23,22,17,24,2,10,16, 19,9,18,4,20,7,8,13,6,3,15, 14,11,21,12), (0,1,2,3,4)(5, 6,7,8,9)(10,11,12,13,14) (15,16,17,18,19)(20,21,22, 23,24)	$5EE^{600}$, $6BB^{600}$, $6DE^{600}$, $6ED^{600}$, $79B^{600}$, $7B6^{600}$, $7F1^{600}$, $C3F^{600}$, $CF5^{600}$, $D5E^{600}$, $D79^{600}$, $DC7^{600}$, DCD^{600}, $DF2^{600}$, $ED6^{600}$, FAA^{600}, $14F9^{600}$, 1576^{600}, $15BC^{600}$, $15D3^{600}$, $165B^{600}$, 1697^{600}, $170F^{600}$, $178E^{600}$, $1C6D^{600}$, $1C8F^{600}$, $1D59^{600}$, $1D8B^{600}$, $1F15^{600}$, $2C57^{600}$, $2D3A^{600}$, $849F^{600}$, $84D7^{600}$, $84FA^{600}$, $853D^{600}$, $854F^{600}$, $85A7^{600}$, $85B3^{600}$, $85D5^{600}$, $85D9^{600}$, $85EA^{600}$, $85F4^{600}$, 8667^{600}, $86F8^{600}$, 8735^{600}, 8772^{600}, 8787^{600}, $87A9^{600}$, $87D2^{600}$, $88CF^{600}$, $88F3^{600}$, $88FC^{600}$, $89BC^{600}$, $8A1F^{600}$, $8A79^{600}$, $8AAE^{600}$, $8B2B^{600}$, $8B66^{600}$, $8B96^{600}$, $8BC9^{600}$, $8C5D^{600}$, $8D78^{600}$, $8D8B^{600}$, $8DE4^{600}$, $8E2E^{600}$, $8E87^{600}$, $8F45^{600}$, $8F4A^{600}$, $8FD0^{600}$, 9077^{600}, $90EB^{600}$, $91EC^{600}$, $923B^{600}$, $928F^{600}$, $92CD^{600}$, $935C^{600}$, $945E^{600}$, $95B1^{600}$, $95C6^{600}$, 9655^{600}, 9672^{600}, $96AA^{600}$, $96E1^{600}$, 9725^{600}, $98F1^{600}$, 9917^{600}, $994D^{600}$, $99A3^{600}$, $9A2D^{600}$, $9A36^{600}$, $9A4E^{600}$, $9A53^{600}$, $9AE8^{600}$, $9B1C^{600}$, $9B2A^{600}$, $9C53^{600}$, $9D0E^{600}$, $9DA2^{600}$, $9F48^{600}$, $A1DA^{600}$, $A36A^{600}$, $A3D4^{600}$, $A475^{600}$, $A5E1^{600}$, $A63A^{600}$, $A6E2^{600}$, $A770^{600}$, $A91D^{600}$, $AB61^{600}$, $B0D6^{600}$, $B1B4^{600}$, $B263^{600}$, $B4D1^{600}$, $BAC2^{600}$, $200047B^{600}$, $20004BE^{600}$, $20004CF^{600}$, 2000557^{600}, $200057C^{600}$, 2000597^{600}, 2000637^{600}, $200063E^{600}$, $200069D^{600}$, $20006D3^{600}$, $200070F^{600}$, 2000759^{600}, $200075A^{600}$, $20007A5^{600}$, $2000C7A^{600}$, $2000CAD^{600}$, $2000D1B^{600}$, $2000D2B^{600}$, $2000D8B^{600}$, $2000EA9^{600}$, 2001437^{600}, $200145E^{600}$, $200146E^{600}$, $20014F2^{600}$, $2001C36^{600}$, $2001D0D^{600}$, $200843E^{600}$, 2008476^{600}, $20084F1^{600}$, $20085B8^{600}$, $200864E^{600}$, $200866A^{600}$, 2008699^{600}, $20086D4^{600}$, 2008726^{600}, 2008758^{600}, $200885B^{600}$, 2008876^{600}, $200889E^{600}$, $200894E^{600}$, $20089C6^{600}$, $2008ACA^{600}$, $2008AE1^{600}$, $2008B58^{600}$, $2008C5A^{600}$, 2009196^{600}, $20092B8^{600}$, $1F7^{300}$, $EE3^{300}$, $15EA^{300}$, $17C9^{300}$, $1C73^{300}$, $1E3C^{300}$, $1E59^{300}$, $1E93^{300}$, $2E8B^{300}$, $9E62^{300}$, $ADA8^{300}$, $AEA1^{300}$, $20007E2^{300}$, $20007E8^{300}$, $20007F0^{300}$, $2000CE6^{300}$, $2000D63^{300}$, $2000DB1^{300}$, $2000DD8^{300}$, $2000E71^{300}$, $2000EB8^{300}$, $2000ECC^{300}$, $2000F46^{300}$, $20014E5^{300}$, $20015D4^{300}$, 2001669^{300}, $200172A^{300}$, 2001792^{300}

TABLE III (cont.).

Bound [Group] Generators	Base Blocks
$C(27,5,4) \leq 3848$ [$\mathbb{Z}_{26}$] (0,1,2,3,4,5,6,7,8,9,10,11, 12,13,14,15,16,17,18,19, 20,21,22,23,24,25)	6D^{26}, 127^{26}, 14D^{26}, 199^{26}, 21D^{26}, 361^{26}, 471^{26}, 4C5^{26}, 613^{26}, 649^{26}, 781^{26}, 88D^{26}, 8B1^{26}, 913^{26}, A07^{26}, D09^{26}, 1151^{26}, 12A1^{26}, 1429^{26}, 1823^{26}, 2059^{26}, 20C3^{26}, 2161^{26}, 2213^{26}, 2305^{26}, 2415^{26}, 2845^{26}, 2A81^{26}, 2E01^{26}, 300B^{26}, 3025^{26}, 3085^{26}, 3501^{26}, 4129^{26}, 4245^{26}, 4289^{26}, 4303^{26}, 4419^{26}, 44A1^{26}, 4605^{26}, 4815^{26}, 4843^{26}, 4A21^{26}, 4D01^{26}, 500D^{26}, 5013^{26}, 50C1^{26}, 5601^{26}, 6023^{26}, 6105^{26}, 7201^{26}, 802D^{26}, 804B^{26}, 80E1^{26}, 8291^{26}, 8423^{26}, 8807^{26}, 8A41^{26}, 9105^{26}, 9411^{26}, 9809^{26}, A0A1^{26}, A409^{26}, A901^{26}, B041^{26}, C083^{26}, C811^{26}, E009^{26}, 100A3^{26}, 10149^{26}, 10185^{26}, 10243^{26}, 10491^{26}, 10819^{26}, 11031^{26}, 11209^{26}, 11881^{26}, 12111^{26}, 14211^{26}, 14441^{26}, 18045^{26}, 18501^{26}, 18821^{26}, 1A201^{26}, 1C101^{26}, 20443^{26}, 20521^{26}, 20681^{26}, 20861^{26}, 20885^{26}, 20A09^{26}, 21015^{26}, 21083^{26}, 21241^{26}, 22205^{26}, 22481^{26}, 24061^{26}, 24181^{26}, 28301^{26}, 30409^{26}, 31101^{26}, 40449^{26}, 40621^{26}, 4080B^{26}, 40C05^{26}, 41019^{26}, 41121^{26}, 42209^{26}, 42811^{26}, 48109^{26}, 49081^{26}, 50205^{26}, 51041^{26}, 81811^{26}, 82091^{26}, 82109^{26}, 82221^{26}, 84085^{26}, 88241^{26}, A0841^{26}, 4000039^{26}, 4000063^{26}, 40000A5^{26}, 4000183^{26}, 400020B^{26}, 4000425^{26}, 4000541^{26}, 4000981^{26}, 4000C03^{26}, 4001045^{26}, 4001089^{26}, 4001A01^{26}, 4004091^{26}, 4004481^{26}, 4004821^{26}, 4005101^{26}, 4008111^{26}, 4009201^{26}, 400C041^{26}, 4010121^{26}, 4010281^{26}, 4010403^{26}, 4010805^{26}, 4020221^{26}, 4040841^{26}, 4006003^{13}, 400A005^{13}, 4012009^{13}, 4022011^{13}, 4042021^{13}, 4082041^{13}
$C(27,6,5) \leq 15210$ [$F_{27,26}$] (1,3,9,5,15,23,13,17,20,4, 12,14,11,2,6,18,7,21,16,26, 22,10,8,24,25,19), (0,1,2) (3,4,5)(6,7,8)(9,10,11)(12, 13,14)(15,16,17)(18,19,20) (21,22,23)(24,25,26)	77^{702}, 24F^{702}, 366^{702}, 39A^{702}, 6A9^{702}, 715^{702}, E52^{702}, 125C^{702}, 126A^{702}, 12AA^{702}, 12B4^{702}, 1319^{702}, 142E^{702}, 14B2^{702}, 14CA^{702}, 1543^{702}, 1570^{702}, 158C^{702}, 188B^{702}, 161A^{351}, 1683^{351}, 16D0^{351}, 1C1C^{351}, 3384^{351}, DB^{117}

TABLE III (cont.).

Bound [Group] Generators	Base Blocks
$C(27,7,6) \leq 50895$ $[F_{27,26}]$ See $C(27,6,5)$	$23F^{702}$, $3AB^{702}$, $3F8^{702}$, 675^{702}, $6BC^{702}$, $6D9^{702}$, $6DA^{702}$, 787^{702}, $7A9^{702}$, $7B2^{702}$, $E1B^{702}$, $E5C^{702}$, $EE8^{702}$, $12AB^{702}$, $12B5^{702}$, $134E^{702}$, 1399^{702}, $13A6^{702}$, $149B^{702}$, $151D^{702}$, 1571^{702}, $15C3^{702}$, $161E^{702}$, 1627^{702}, $163A^{702}$, $16C5^{702}$, $16E2^{702}$, $178C^{702}$, $17D0^{702}$, $19C5^{702}$, $1A55^{702}$, $1AC9^{702}$, $1B32^{702}$, $1BA1^{702}$, $1C59^{702}$, $1CCA^{702}$, $1E61^{702}$, $1E62^{702}$, $1F09^{702}$, $1F24^{702}$, 3256^{702}, 3265^{702}, $330B^{702}$, 3389^{702}, $344E^{702}$, 3463^{702}, $349C^{702}$, $34AA^{702}$, $34CC^{702}$, 3591^{702}, $389A^{702}$, 3925^{702}, 3949^{702}, $72A2^{702}$, $730C^{702}$, 7321^{702}, 9266^{702}, 9472^{702}, $950B^{702}$, $9B0A^{702}$, $AC62^{702}$, $40B2A^{702}$, $40E51^{702}$, DF^{351}, EF^{351}, $4024F^{351}$, $402A7^{351}$, 40317^{351}, 40359^{351}, $403B1^{351}$, 40457^{351}, $4048F^{351}$, $404DC^{351}$, 40574^{351}, $405AC^{351}$, 40867^{351}, $408F2^{351}$, $4090F^{351}$, $4099A^{351}$, $41A43^{351}$, 43341^{351}, 44685^{351}

TABLE III (cont.).

Bound [Group] Generators	Base Blocks
$C(27,9,8) \leq 334854$ [AΓL$_1$(27)] (0,1,2,3,4,5,6,7,8)(9,10,11,12,13,14,15,16,17)(18,19,20,21,22,23,24,25,26), (0,26,18,13,8,19)(1,16,20,15,6,5)(2,10,14,23,3,11)(4,7,25,12,9,21)(22,24)	3BF^{2106}, 7DE^{2106}, 7E7^{2106}, 7ED^{2106}, B5F^{2106}, B7D^{2106}, BEB^{2106}, EF9^{2106}, F76^{2106}, FAE^{2106}, 13EB^{2106}, 16CF^{2106}, 16DD^{2106}, 16F6^{2106}, 177A^{2106}, 17D3^{2106}, 1AD7^{2106}, 1AED^{2106}, 1AF3^{2106}, 1B3E^{2106}, 1B4F^{2106}, 1BAD^{2106}, 1BF8^{2106}, 1EAB^{2106}, 1F1B^{2106}, 1F3C^{2106}, 1F63^{2106}, 1F69^{2106}, 1FC5^{2106}, 22FB^{2106}, 26FA^{2106}, 2776^{2106}, 279D^{2106}, 27E3^{2106}, 27EC^{2106}, 2A6F^{2106}, 2AF5^{2106}, 2BD3^{2106}, 2BD9^{2106}, 2E73^{2106}, 2E7C^{2106}, 2ECD^{2106}, 2EE3^{2106}, 2F39^{2106}, 2F6A^{2106}, 325F^{2106}, 329F^{2106}, 32E7^{2106}, 32FC^{2106}, 36B6^{2106}, 36B9^{2106}, 36E9^{2106}, 372D^{2106}, 379A^{2106}, 3A3E^{2106}, 3A6D^{2106}, 3ACE^{2106}, 3B1E^{2106}, 3B35^{2106}, 3B72^{2106}, 3B9C^{2106}, 3E27^{2106}, 3E2B^{2106}, 3EB4^{2106}, 3ED2^{2106}, 3EE4^{2106}, 3F0E^{2106}, 3F25^{2106}, 3F54^{2106}, 3F64^{2106}, 3F83^{2106}, 46F9^{2106}, 476E^{2106}, 4A77^{2106}, 4BB3^{2106}, 4E4F^{2106}, 4ED5^{2106}, 4F5A^{2106}, 52DB^{2106}, 52EE^{2106}, 5397^{2106}, 53DC^{2106}, 56D3^{2106}, 571E^{2106}, 57E4^{2106}, 5A7C^{2106}, 5B65^{2106}, 5E35^{2106}, 5E56^{2106}, 5E6A^{2106}, 5E93^{2106}, 5F26^{2106}, 5F43^{2106}, 5FD0^{2106}, 6679^{2106}, 6778^{2106}, 67B2^{2106}, 6AD3^{2106}, 6AD6^{2106}, 6B69^{2106}, 6B74^{2106}, 6ECA^{2106}, 6ED1^{2106}, 6F38^{2106}, 725E^{2106}, 72DA^{2106}, 73AC^{2106}, 73E2^{2106}, 762E^{2106}, 764B^{2106}, 7732^{2106}, 7745^{2106}, 774A^{2106}, 7754^{2106}, 7783^{2106}, 7A27^{2106}, 7A55^{2106}, 7A59^{2106}, 7A95^{2106}, 7E62^{2106}, 7E92^{2106}, 7F88^{2106}, 926F^{2106}, 96DA^{2106}, 9747^{2106}, 9755^{2106}, 97B4^{2106}, 9E39^{2106}, 9ED2^{2106}, 9F98^{2106}, 9FA2^{2106}, AE95^{2106}, AEA5^{2106}, AED8^{2106}, B723^{2106}, B725^{2106}, BAE1^{2106}, BF12^{2106}, CECC^{2106}, CF07^{2106}, CF91^{2106}, EEC1^{2106}, F790^{2106}, FA0E^{2106}, 1564E^{2106}, 156B2^{2106}, 16E8C^{2106}, 16F81^{2106}, 40E57^{2106}, 412F9^{2106}, 4171E^{2106}, 41735^{2106}, 4196D^{2106}, 41D27^{2106}, 4235E^{2106}, 458DA^{2106}, 469E4^{2106}, 4873C^{2106}, 494F2^{2106}

TABLE III (cont.).

Bound [Group] Generators	Base Blocks
$C(28,5,4) \leq 4550$ $[\mathbb{Z}_{26}]$ See $C(27,5,4)$	F1^{26}, 18B^{26}, 293^{26}, 2A9^{26}, 44B^{26}, 495^{26}, 529^{26}, 5C1^{26}, 839^{26}, 88D^{26}, 8C9^{26}, 9A1^{26}, B09^{26}, C23^{26}, 1065^{26}, 10A3^{26}, 1207^{26}, 1229^{26}, 1681^{26}, 180B^{26}, 1905^{26}, 1911^{26}, 1C09^{26}, 202B^{26}, 2053^{26}, 2261^{26}, 2311^{26}, 24A1^{26}, 2505^{26}, 2603^{26}, 2883^{26}, 3031^{26}, 3103^{26}, 3205^{26}, 4033^{26}, 4099^{26}, 4145^{26}, 4251^{26}, 4305^{26}, 4825^{26}, 5031^{26}, 5043^{26}, 5085^{26}, 5109^{26}, 6015^{26}, 6181^{26}, 6209^{26}, 6A01^{26}, 7401^{26}, 801B^{26}, 8035^{26}, 8151^{26}, 8249^{26}, 8291^{26}, 8621^{26}, 8807^{26}, 8D01^{26}, 9181^{26}, 9403^{26}, A141^{26}, B009^{26}, C601^{26}, E021^{26}, 10185^{26}, 1020B^{26}, 10407^{26}, 10489^{26}, 10861^{26}, 10E01^{26}, 11013^{26}, 11205^{26}, 11501^{26}, 11841^{26}, 120C1^{26}, 12411^{26}, 140A1^{26}, 14441^{26}, 14803^{26}, 18083^{26}, 18141^{26}, 2010D^{26}, 20243^{26}, 20321^{26}, 2040D^{26}, 20503^{26}, 20851^{26}, 20981^{26}, 21091^{26}, 22045^{26}, 22809^{26}, 24049^{26}, 240C1^{26}, 24411^{26}, 280A1^{26}, 28205^{26}, 28441^{26}, 29041^{26}, 34101^{26}, 40429^{26}, 40445^{26}, 40C11^{26}, 41211^{26}, 42049^{26}, 42281^{26}, 42411^{26}, 44103^{26}, 44881^{26}, 48121^{26}, 50221^{26}, 60821^{26}, 81061^{26}, 84111^{26}, 84221^{26}, 84409^{26}, 84841^{26}, 88821^{26}, 4000027^{26}, 4000069^{26}, 4000143^{26}, 4000215^{26}, 4000219^{26}, 40002C1^{26}, 4000451^{26}, 4000609^{26}, 4000A03^{26}, 4001121^{26}, 4001411^{26}, 4001881^{26}, 4004405^{26}, 4004811^{26}, 4008301^{26}, 4008841^{26}, 4009005^{26}, 400C081^{26}, 4010421^{26}, 4010901^{26}, 4020111^{26}, 4020403^{26}, 4021009^{26}, 4040809^{26}, 4041041^{26}, 8000047^{26}, 8000059^{26}, 8000131^{26}, 800020D^{26}, 8000303^{26}, 8000A81^{26}, 8001089^{26}, 8001141^{26}, 8001405^{26}, 8001821^{26}, 8002025^{26}, 8002089^{26}, 8002841^{26}, 8004281^{26}, 8004421^{26}, 8004803^{26}, 8005201^{26}, 8006041^{26}, 8008061^{26}, 8008481^{26}, 8009011^{26}, 800C101^{26}, 8010241^{26}, 8020203^{26}, 8020441^{26}, C000091^{26}, C000105^{26}, C000805^{26}, C000C01^{26}, C002101^{26}, C004011^{26}, 4006003^{13}, 400A005^{13}, 4012009^{13}, 4042021^{13}, 4082041^{13}, 8022011^{13}
$C(28,6,5) \leq 18369$ $[F_{27,26}]$ See $C(27,6,5)$	77^{702}, 695^{702}, 70D^{702}, 732^{702}, 746^{702}, 758^{702}, E49^{702}, E54^{702}, E62^{702}, 121E^{702}, 122B^{702}, 1436^{702}, 14E8^{702}, 1570^{702}, 1788^{702}, 18D1^{702}, 9254^{702}, 9262^{702}, A862^{702}, 800022D^{702}, 80002D1^{702}, 8000394^{702}, 8000709^{702}, 80012E0^{702}, 1C2A^{351}, 800004F^{351}, 800005D^{351}, 800006B^{351}, DB^{117}

TABLE III (cont.).

Bound [Group] Generators	Base Blocks
$C(28,7,6) \leq 65286$ $[F_{27,26}]$ See $C(27,6,5)$	$2EE^{702}$, $2F5^{702}$, $3B6^{702}$, $62F^{702}$, $66D^{702}$, 673^{702}, $69E^{702}$, $72D^{702}$, 753^{702}, 759^{702}, $7C6^{702}$, $7E1^{702}$, $E55^{702}$, $EC6^{702}$, $EE8^{702}$, 1237^{702}, $12C7^{702}$, $12E9^{702}$, 1395^{702}, $13B1^{702}$, $145B^{702}$, 1547^{702}, $15B2^{702}$, $161B^{702}$, 1669^{702}, $16CA^{702}$, 1762^{702}, $17D0^{702}$, 1879^{702}, $18F4^{702}$, $192D^{702}$, $196A^{702}$, $19A5^{702}$, $19C6^{702}$, $1A55^{702}$, $1A99^{702}$, $1B16^{702}$, $1B23^{702}$, $1C96^{702}$, $1CE4^{702}$, $1D49^{702}$, $1D64^{702}$, $1EA1^{702}$, $1F0C^{702}$, $32A5^{702}$, 3386^{702}, 3389^{702}, $348E^{702}$, $34E8^{702}$, 3519^{702}, 3551^{702}, $384B^{702}$, $38A3^{702}$, 7249^{702}, 7252^{702}, $72A2^{702}$, $95A4^{702}$, $A866^{702}$, $4046B^{702}$, $406A5^{702}$, $408B3^{702}$, $40B51^{702}$, $40E61^{702}$, $4144D^{702}$, $41F04^{702}$, $80000DD^{702}$, $800023D^{702}$, 8000257^{702}, $80002F8^{702}$, 8000327^{702}, $80003E2^{702}$, $80006A9^{702}$, $80006C3^{702}$, $80006E4^{702}$, 8000713^{702}, 8000758^{702}, $8000E4C^{702}$, 8001271^{702}, $800131A^{702}$, $80013C4^{702}$, 8001472^{702}, 8001561^{702}, 8001585^{702}, $800168C^{702}$, $800188B^{702}$, $8001C92^{702}$, $7F^{351}$, 40317^{351}, $4048F^{351}$, $404DC^{351}$, 40574^{351}, 40867^{351}, $411E2^{351}$, $41A43^{351}$, 8001629^{351}, 8001634^{351}, $80017A0^{351}$, $8001A86^{351}$, $8001B60^{351}$, $8001C2A^{351}$

TABLE III (cont.).

Bound [Group] Generators	Base Blocks
$C(28,9,8) \leq 470340$ $[\mathrm{A\Gamma L}_1(27)]$ See $C(27,9,8)$	3BF^{2106}, 7F6^{2106}, B77^{2106}, BEE^{2106}, EDD^{2106}, EEB^{2106}, EF9^{2106}, F2F^{2106}, F5E^{2106}, F97^{2106}, F9B^{2106}, FE3^{2106}, 127F^{2106}, 12FD^{2106}, 13F3^{2106}, 1776^{2106}, 17AB^{2106}, 17D9^{2106}, 1AB7^{2106}, 1ADB^{2106}, 1AF6^{2106}, 1B2F^{2106}, 1B6E^{2106}, 1BB9^{2106}, 1BF8^{2106}, 1E6B^{2106}, 1EE6^{2106}, 1F55^{2106}, 1F69^{2106}, 1F8D^{2106}, 1FAC^{2106}, 23EE^{2106}, 26BD^{2106}, 27BC^{2106}, 27D3^{2106}, 2A7D^{2106}, 2AFC^{2106}, 2B6B^{2106}, 2E5B^{2106}, 2E67^{2106}, 2EDC^{2106}, 2F17^{2106}, 2F39^{2106}, 2F4D^{2106}, 2F74^{2106}, 2F9A^{2106}, 2FCC^{2106}, 32E7^{2106}, 3337^{2106}, 333E^{2106}, 362F^{2106}, 36AD^{2106}, 3A7C^{2106}, 3ACE^{2106}, 3AD5^{2106}, 3BE2^{2106}, 3E71^{2106}, 3E9C^{2106}, 3EA5^{2106}, 3F0B^{2106}, 3F32^{2106}, 3F46^{2106}, 3FE0^{2106}, 4767^{2106}, 4AD7^{2106}, 4B37^{2106}, 4B7C^{2106}, 4E6D^{2106}, 4E73^{2106}, 4F71^{2106}, 4F93^{2106}, 4FB4^{2106}, 4FC6^{2106}, 4FC9^{2106}, 52CF^{2106}, 539D^{2106}, 5679^{2106}, 5753^{2106}, 57CC^{2106}, 57D4^{2106}, 5A4F^{2106}, 5A57^{2106}, 5A6E^{2106}, 5A7C^{2106}, 5B65^{2106}, 5BCC^{2106}, 5EAA^{2106}, 5ED1^{2106}, 5F0E^{2106}, 5F62^{2106}, 67B2^{2106}, 67E8^{2106}, 6A7A^{2106}, 6AF1^{2106}, 6AF2^{2106}, 6B71^{2106}, 6B72^{2106}, 6EE4^{2106}, 6F4C^{2106}, 723D^{2106}, 725E^{2106}, 7396^{2106}, 73D8^{2106}, 761E^{2106}, 76A6^{2106}, 770D^{2106}, 77A8^{2106}, 7A2E^{2106}, 7A59^{2106}, 7AB4^{2106}, 7B1C^{2106}, 7B94^{2106}, 7E49^{2106}, 7E8A^{2106}, 7EC4^{2106}, 7EC8^{2106}, 925F^{2106}, 96DA^{2106}, 9A9E^{2106}, 9B9A^{2106}, 9BE4^{2106}, AED4^{2106}, AF85^{2106}, AFA1^{2106}, B6A3^{2106}, B725^{2106}, BAA6^{2106}, BB07^{2106}, BE34^{2106}, BF11^{2106}, CE4E^{2106}, CE78^{2106}, CF1A^{2106}, CF26^{2106}, CF54^{2106}, CFA2^{2106}, D738^{2106}, D78A^{2106}, D792^{2106}, EEC1^{2106}, F7C0^{2106}, FAE0^{2106}, 156B2^{2106}, 407C7^{2106}, 40B6D^{2106}, 40E57^{2106}, 411CF^{2106}, 4129F^{2106}, 41735^{2106}, 41CF2^{2106}, 458CB^{2106}, 458EA^{2106}, 800037E^{2106}, 800039F^{2106}, 80003DB^{2106}, 800066F^{2106}, 80006BB^{2106}, 8000757^{2106}, 8000779^{2106}, 80007B5^{2106}, 80007DC^{2106}, 8000A3F^{2106}, 8000AF5^{2106}, 8000B4F^{2106}, 8000E75^{2106}, 8000EA7^{2106}, 8000EBC^{2106}, 8000EE9^{2106}, 8000F2D^{2106}, 8000F53^{2106}, 8000FF0^{2106}, 80012BB^{2106}, 800139B^{2106}, 80016B9^{2106}, 80016D6^{2106}, 800171D^{2106}, 800174B^{2106}, 800176C^{2106}, 80017E1^{2106}, 80017E4^{2106}, 8001A5D^{2106}, 8001A6E^{2106}, 8001ACB^{2106}, 8001AF8^{2106}, 8001B74^{2106}, 8001E93^{2106}, 8001EC9^{2106}, 8001F07^{2106}, 8001F1A^{2106}, 8001F23^{2106}, 8001F25^{2106}, 8001FC1^{2106}, 80023AD^{2106}, 800269E^{2106}, 8002763^{2106}, 8002799^{2106}, 8002AD3^{2106}, 8002BB4^{2106}, 8002E33^{2106}, 8002E6C^{2106}, 8002F32^{2106}, 8002F45^{2106}, 8002F58^{2106}, 8003669^{2106}, 8003770^{2106}, 80037A4^{2106}, 8003A65^{2106}, 8003A72^{2106}, 8003EA2^{2106}, 8003F28^{2106}, 8003F41^{2106}, 8003F42^{2106}, 80046CB^{2106}, 800474D^{2106}, 8004F34^{2106}, 800565A^{2106}, 8005A69^{2106}, 8005A93^{2106}, 8005F12^{2106}, 800664E^{2106}, 8006AF0^{2106}, 800BA34^{2106}, 800CFC0^{2106}, 8005BC8^{702}