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10781. *Proposed by Leonard Smiley, University of Alaska, Anchorage, AK.* **Prove that**

$$\sum_{i=2}^{n}\binom{n}{i}i^{i-1}(n-i)^{n-i}=\sum_{i=2}^{n}\binom{n}{i-1}(i-1)^{i-1}(n-i)^{n-i},$$

where 0^0 is taken to be 1.

SOLUTION

Dividing through by n, the left side of the proposed identity is

$$f(n)=(n-1)\sum_{k=1}^{n-2}\binom{n-2}{k-1}k^{k-1}(n-k)^{n-k-2}+n^{n-2} \tag{1}$$

while the right side can be expressed as $g(n)=\sum_{k=1}^{n-1}\binom{n-1}{k}(k-1)^{k-1}(n-k)^{n-k-1}$. We will exhibit a class of marked trees (trees with distinguished vertices) counted by $f(n)$, another class counted by $g(n)$, and a "cut-and-paste" bijection between the two classes.

Cayley's classical formula says that the term n^{n-2} in (1) counts the set $\mathcal{T}_n$ of trees on the vertex set $[n]$ $(=\{1,2,\ldots,n\})$. To interpret the other term, let $\mathcal{T}_n'$ $(n\geq 3)$ denote the set of trees on $[n]$ rooted at 1, with a distinguished non-root vertex and at least two children of the root. Then the other term in (1) counts $\mathcal{T}_n'$ by *size k of the subtree X containing the distinguished vertex* as follows

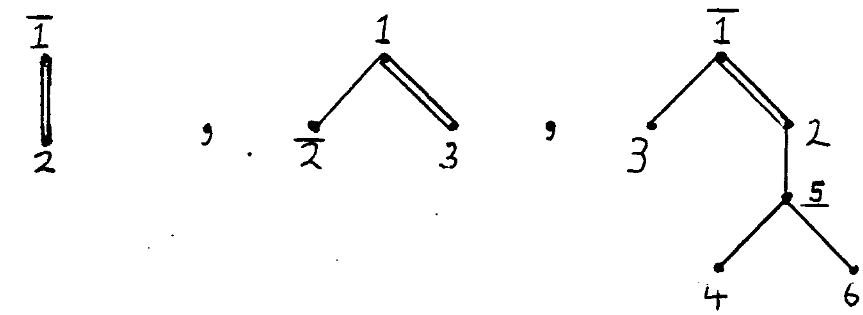
$$\underbrace{(n-1)}_{\text{A}}\sum_{k=1}^{n-2}\underbrace{\binom{n-2}{k-1}}_{\text{B}}\underbrace{k^{k-1}}_{\text{C}}\underbrace{(n-k)^{n-k-2}}_{\text{D}}$$

A: choose a distinguished vertex $i\in[2,n]$
B: choose a $(k-1)$-set $X'\subset[n]\backslash\{1,i\}$
C: choose a rooted tree on $X:=X'\cup\{i\}$; join its root to 1
D: choose a tree on $[n]\backslash X$.

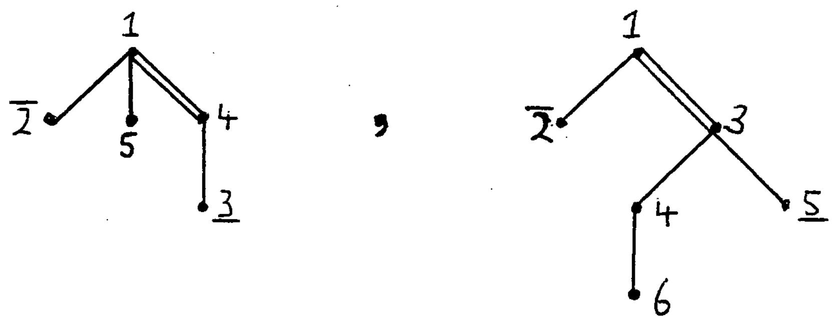
Thus $f(n)=|\mathcal{T}_n|+|\mathcal{T}_n'|$.

To interpret $g(n)$, we say a tree on $[n]$ is *special* when it has the following form: its root 1 has a distinguished child a which itself has at most one child (though it may have many descendants); furthermore if a does have descendants, then (i) all a's descendants are numbered higher than a and (ii) one of a's descendants is distinguished by an underline; finally, one of the vertices outside the subtree headed by a is distinguished by an overline.

Thus, using a double edge to denote the root's distinguished child a,



are special trees while



are not.

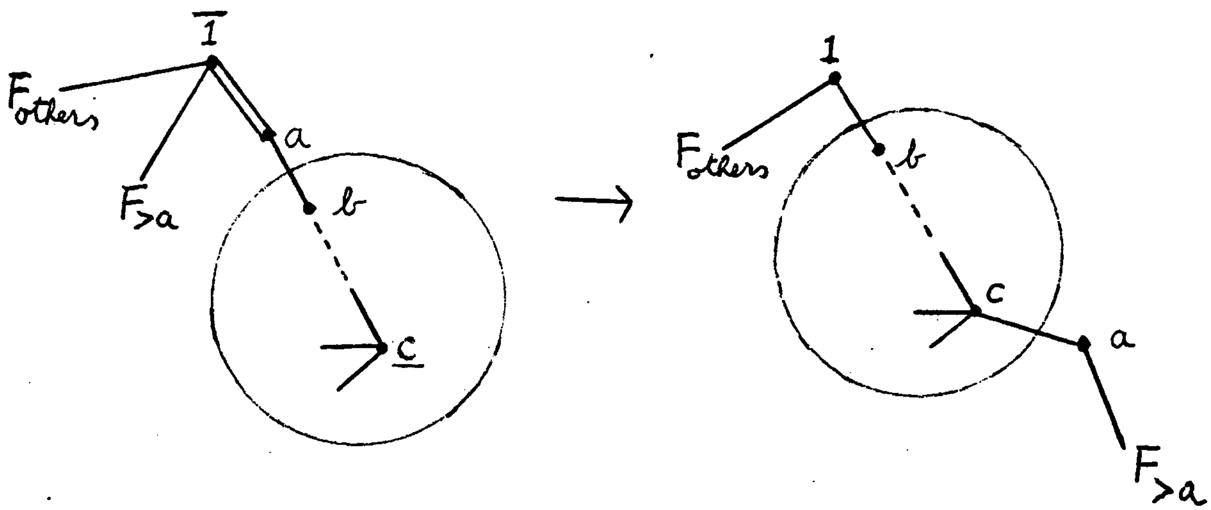
Let $\mathcal{B}_n$ denote the set of special trees on $[n]$. Then $g(n)$ counts $\mathcal{B}_n$ by *size k of the subtree headed by the distinguished child of the root* as follows

$$g(n) = \sum_{k=1}^{n-1} \underbrace{\binom{n-1}{k}}_{\text{A}} \underbrace{(k-1)^{k-1}}_{\text{B}} \underbrace{(n-k)^{n-k-1}}_{\text{C}}$$

A: choose a k-set $X \subset [n]\backslash\{1\}$; let a be the smallest element in X and set $X' = X\backslash\{a\}$
B: choose a doubly-rooted tree on X'; underline the first root, join the second root to a and join a to 1
C: choose a rooted tree on $[n]\backslash X$ and place the overline on its root.

Now partition $\mathcal{B}_n$ as $\mathcal{B}_{n,1} \sqcup \mathcal{B}'_n$ according as the root 1 is the overlined vertex or not. A bijection $\mathcal{B}_{n,1} \to \mathcal{T}_n$ and a bijection $\mathcal{B}'_n \to \mathcal{T}'_n$ are given diagrammatically below, and the proposed identity follows.

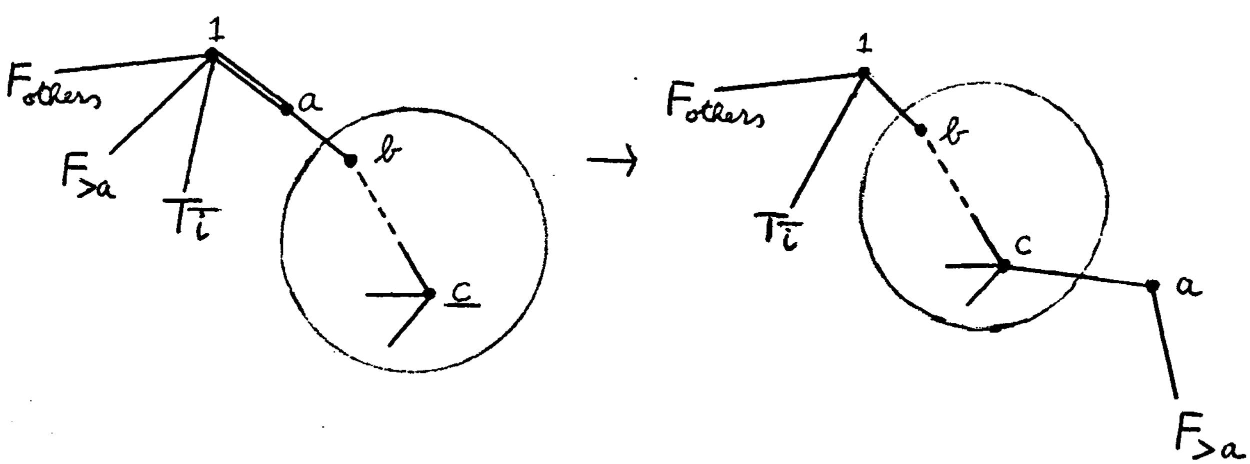
$$\mathcal{B}_{n,1} \longrightarrow \mathcal{T}_n$$



Deleting the root produces a forest of subtrees, and $F_{>a}$ denotes those subtrees consisting entirely of nodes $> a$; F_{others} denotes the remaining subtrees except for the one headed by the distinguished child a. The subtree headed by a's only child is encircled. Of course, any or all of $F_{>a}$, F_{others} and the circle may be empty.

The underline and overline are erased. If the circle is empty, then a is joined directly to the root 1.

$$\mathcal{B}'_n \longrightarrow \mathcal{T}'_n$$



Here $T_{\bar{i}}$ denotes the subtree containing the overlined vertex i. It remains intact, otherwise notation and mapping are as above.

The underline is again erased, and the overlined vertex becomes the distinguished non-root vertex.

The mappings are reversible: once a has been identified, its parent becomes the underlined vertex unless its parent is the root, in which case there is no underlined vertex (and a has no descendents in the preimage). To locate a, pick out the smallest element in each subtree (but ignoring $T_{\bar{i}}$ in the second case). Then a will be the maximum of these minima.

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