Tripods do not pack densely

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Abstract

In 1994, S. K. Stein and S. Szabó posed a problem concerning simple three-dimensional shapes, known as semicrosses, or tripods. By definition, a tripod is formed by a corner and the three adjacent edges of an integer cube. How densely can one fill the space with non-overlapping tripods of a given size? In particular, is it possible to fill a constant fraction of the space as the tripod size tends to infinity? In this paper, we settle the second question in the negative: the fraction of the space that can be filled with tripods of a growing size must be infinitely small.

1 Introduction

In [11, 10], S. K. Stein and S. Szabó posed a problem concerning simple three-dimensional polyominoes, called "semicrosses" in [11], and "tripods" in [10].

A *tripod* of size n is formed by a corner and the three adjacent edges of an integer $n \times n \times n$ cube (see Figure 1). How densely can one fill the space with non-overlapping tripods of size n? In particular, is it possible to keep a constant fraction of the space filled as $n \to \infty$? Despite their simple formulation, these two questions appear to be yet unsolved.

In this paper, we settle the second question in the negative: the density of tripod packing has to approach zero as tripod size tends to infinity. It is easy to prove (see [11]) that this result implies similar results in dimensions higher than three.

Instead of dealing with the problem of packing tripods in space directly, we address an equivalent problem, also introduced in [11, 10]. In this alternative setting, tripods of size n are to be packed without overlap, so that their corner cubes coincide with one of the unit cells of an $n \times n \times n$ cube. We may also assume that all tripods have the same orientation. If we denote by $f(n)$ the maximum number of non-overlapping tripods in such a packing, then the maximum fraction of space that can be filled with arbitrary

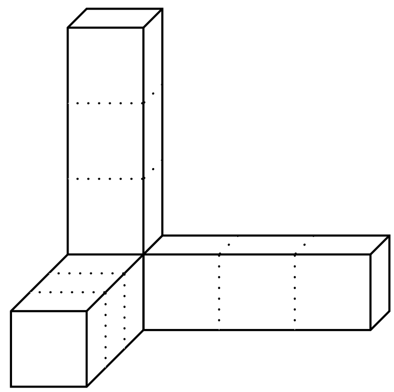

Figure 1: A tripod of size $n = 4$

non-overlapping tripods is proportional to $f(n)/n^2$ (see [11] for a proof). The only known values of function f are $f(1)$ up to $f(5)$: 1, 2, 5, 8, 11. It is easy to see that for all n, $n \leq f(n) \leq n^2$. Stein and Szabó's questions are concerned with the upper bound of this inequality. They can now be restated as follows: what is the asymptotic behaviour of $f(n)/n^2$ as $n \to \infty$? In particular, is $f(n) = o(n^2)$, or is $f(n)$ bounded away from 0?

In this paper we show that, in fact, $f(n) = o(n^2)$, so the fraction of the space that can be filled with tripods of a growing size is infinitely small. Our proof methods are taken from the domain of extremal graph theory. Our main tools are two powerful, widely applicable graph-theoretic results: Szemerédi's Regularity Lemma, and the Blow-up Lemma.

2 Preliminaries

Throughout this paper we use the standard language of graph theory, slightly adapted for our convenience. A graph G is defined by its set of nodes $V(G)$, and its set of edges $E(G)$. All considered graphs are simple and undirected. *The size* of a graph G is the number of its nodes $|V(G)|$; the *edge count* is the number of its edges $|E(G)|$. A graph H is *a subgraph* of G, denoted $H \subseteq G$, iff $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. Subgraph $H \subseteq G$ is *a spanning subgraph* of G, denoted $H \sqsubseteq G$, iff $V(H) = V(G)$.

A complete graph on r nodes is denoted K_r. The term *k-partite graph* is synonymous with "k-coloured". We write *bipartite* for "2-partite", and *tripartite* for "3-partite". Complete bipartite and tripartite graphs are denoted K_{rs} and K_{rst}, where r, s, t are sizes of the colour classes. Graph $K_2 = K_{11}$(short for $K_{1,1}$) is a single edge; we call its complement $\bar{K}_2$ (an empty graph on two nodes) *a nonedge*. Graph $K_3 = K_{111}$ is called *a triangle*; graph K_{121} is called *a diamond* (see Figure 2). We call a k-partite graph *equi-k-partite*, if all its colour classes are of equal size.

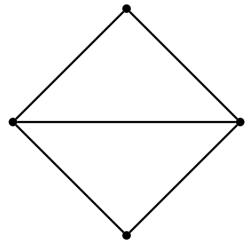

Figure 2: The diamond graph K_{121}

The density of a graph is the ratio of its edge count to the edge count of a complete graph of the same size: if $G \sqsubseteq K_n$, then

$$\mathrm{dens}(G) = |E(G)|/|E(K_n)| = |E(G)|/\binom{n}{2}.$$

The *bipartite density* of a bipartite graph $G \sqsubseteq K_{mn}$ is

$$\mathrm{dens}_2(G) = |E(G)|/|E(K_{mn})| = |E(G)|/(mn).$$

Similarly, the *tripartite density* of a tripartite graph $G \sqsubseteq K_{mnp}$ is

$$\mathrm{dens}_3(G) = |E(G)|/|E(K_{mnp})| = |E(G)|/(mn + np + pm).$$

Let H be an arbitrary graph. Graph G is called *H-covered*, if every edge of G belongs to a subgraph isomorphic to H. Graph G is called *H-free*, if G does not contain any subgraph isomorphic to H. In particular, we will be interested in triangle-covered diamond-free graphs.

Let us now establish an upper bound on the density of an equitripartite diamond-free graph.

Lemma 1. *The tripartite density of an equitripartite diamond-free graph G is at most* $3/4$.

Proof. Denote $|V(G)| = 3n$. By Dirac's generalisation of Turán's theorem (see e.g. [7, p. 300]), we have $|E(G)| \leq (3n)^2/4$. Since $|E(K_{nnn})| = 3n^2$, the theorem follows trivially. ∎

The upper bound of 3/4 given by Lemma 1 is not the best possible. However, that bound will be sufficient to obtain the results of this paper. In fact, any constant upper bound strictly less than 1 would be enough.

3 The Regularity Lemma and the Blow-up Lemma

In most definitions and theorem statements below, we follow [9, 8, 3].

For a graph G, and node sets $X, Y \subseteq V(G)$, $X \cap Y = \emptyset$, we denote by $G(X, Y) \subseteq G$ the bipartite subgraph obtained by removing from G all nodes except those in $X \cup Y$, and all edges adjacent to removed nodes. Let F be a bipartite graph with colour classes A, B. Given some $\epsilon > 0$, graph F is called *ϵ-regular*, if for any $X \subseteq A$ of size $|X| \geq \epsilon \cdot |A|$, and any $Y \subseteq B$ of size $|Y| \geq \epsilon \cdot |B|$, we have

$$|\text{dens}_2(F(X,Y)) - \text{dens}_2(F)| \leq \epsilon.$$

Let G denote an arbitrary graph. We say that G admits *an ϵ-partitioning of order m*, if $V(G)$ can be partitioned into m disjoint subsets of equal size, called *supernodes*, such that for all pairs of supernodes A, B, the bipartite subgraph $G(A,B)$ is ϵ-regular. The ϵ-regular subgraphs $G(A,B)$ will be called *superpairs*.

For different choices of supernodes A, B, the density of the superpair $G(A,B)$ may differ. We will distinguish between superpairs of "low" and "high" density, determined by a carefully chosen threshold. For a fixed d, $0 \leq d \leq 1$, we call a superpair $G(A,B)$ *a superedge*, if $\text{dens}_2(G(A,B)) \geq d$, and *a super-nonedge*, if $\text{dens}_2(G(A,B)) < d$. Now, given a graph G, and its ϵ-partitioning of order m, we can build a high-level representation of G by a graph of size m, which we will call *a d-map* of G. The d-map M contains a node for every supernode of G. Two nodes of M are connected by an edge, if and only if the corresponding supernodes of G are connected by a superedge. Thus, edges and nonedges in M represent, respectively, superedges and super-nonedges of G. For a node pair (edge or nonedge) e in G, we denote by $\mu(e)$ the corresponding pair in the d-map M. We call $\mu : E(G) \to E(M)$ *the mapping function*. The union of all superedges $\mu^{-1}(E(M)) \subseteq E(G)$ will be called *the superedge subgraph of G*. Similarly, the union of all super-nonedges in G will be called *the super-nonedge subgraph of G*.

We rely on the following fact, which is a restricted version of the Blow-up Lemma (see [8]).

Theorem 1 (Blow-up Lemma). *Let $d > \epsilon > 0$. Let G be a graph with an ϵ-partitioning, and let M be its d-map. Let H be a subgraph of M with maximum degree $\Delta > 0$. If $\epsilon \leq (d-\epsilon)^\Delta/(2+\Delta)$, then G contains a subgraph isomorphic to H.*

Proof. See [8]. ■

Since we are interested in diamond-free graphs, we take H to be a diamond. We simplify the condition on d and ϵ, and apply the Blow-up Lemma in the following form: if $\epsilon \leq (d-\epsilon)^3/5$, and G is diamond-free, then its d-map M is also diamond-free.

Our main tool is Szemerédi's Regularity Lemma. Informally, it states that any graph can be transformed into a graph with an ϵ-partitioning by removing a small number of nodes and edges. Its precise statement, slightly adapted from [9], is as follows.

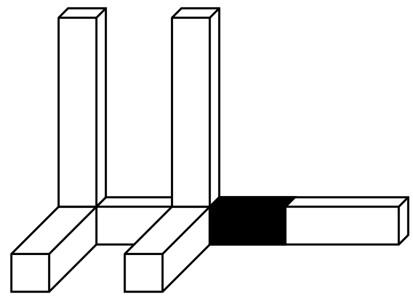

Figure 3: An axial collision

Theorem 2 (Regularity Lemma). *Let G be an arbitrary graph. For every $\epsilon > 0$ there is an $m = m(\epsilon)$ such that for some $G_0 \subseteq G$ with $|E(G) \setminus E(G_0)| \leq \epsilon \cdot |V(G)|^2$, graph G_0 admits an ϵ-partitioning of order at most m.*

Proof. See e.g. [9, 3]. ∎

The given form of the Regularity Lemma is slightly weaker than the standard one. In particular, we allow to remove a "small" number of nodes and edges from the graph G, whereas the standard version only allows to remove a "small" number of nodes (with adjacent edges), and then a "small" number of superpairs. In our context, the difference between two versions is insignificant.

Note that if $|E(G)| = o(|V(G)|^2)$, the statement of the Regularity Lemma becomes trivial. In other words, the Regularity Lemma is only useful for dense graphs.

4 Packing tripods

Consider a packing of tripods of size n in an $n \times n \times n$ cube, of the type described in the Introduction (no overlaps, similar orientation, corner cubes coinciding with $n \times n \times n$ cube cells). A tripod in such a packing is uniquely defined by the coordinates of its corner cube (i, j, k), $0 \leq i, j, k < n$. Moreover, if two of the three coordinates (i, j, k) are fixed, then the packing may contain at most one tripod with such coordinates — otherwise, the two tripods with an equal pair of coordinates would form an *axial collision*, depicted in Figure 3.

We represent a tripod packing by an equitripartite triangle-covered graph $G \sqsubseteq K_{nnn}$ as follows. Three color classes $U = \{u_i\}$, $V = \{v_j\}$, $W = \{w_k\}$, $0 \leq i, j, k < n$, correspond to the three dimensions of the cube. A tripod (i, j, k) is represented by a triangle $\{(u_i, v_j), (v_j, w_k), (w_k, u_i)\}$. To prevent axial collisions, triangles representing different tripods must be edge-disjoint.

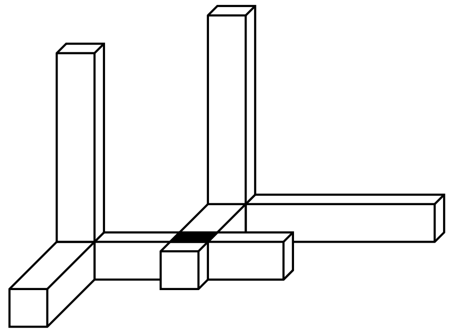

Figure 4: A simple collision

Hence, if m is the number of tripods in the packing, then the representing graph G contains $3m$ edges.

We now prove that the graph G is diamond-free. In general, G might contain a triangle with three edges coming from three different tripods; such a triangle would give rise to three diamonds. To prove that such a situation is impossible, we must consider, apart from axial collisions, also *simple collisions*, depicted in Figure 4.

Lemma 2. *A tripod packing graph is diamond-free.*

Proof. It is sufficient to show that the tripod packing graph does not contain any triangles apart from those representing tripods. Suppose the contrary: there is a triangle $\{(u_i, v_j), (v_j, w_k), (w_k, u_i)\}$, which does not represent any tripod. Then its three edges must come from triangles representing three different tripods; denote these tripods (i, j, k'), (i, j', k), (i', j, k), where $i \neq i'$, $j \neq j'$, $k \neq k'$. Consider the differences $i' - i$, $j' - j$, $k' - k$, all of which are non-zero. At least two of these three differences must have the same sign; without loss of generality assume that $i' - i$, $j' - j$ are of the same sign. Thus, we have either $i' < i$, $j' < j$, or $i' > i$, $j' > j$. In both cases, the tripods (i, j', k), (i', j, k) collide. Hence, our assumption must be false, and the triangle $\{(u_i, v_j), (v_j, w_k), (w_k, u_i)\}$ not representing any tripod cannot exist. Therefore, no triangles in G can share an edge — in other words, G is diamond-free. ■

Thus, tripod packing graphs are equitripartite, triangle-covered and diamond-free. Note that these graph properties are invariant under any permutation of graph nodes within colour classes, whereas the property of a tripod packing being overlap-free is not invariant under permutation of indices within each dimension. Hence, the converse of Lemma 2 does not hold. However, even the loose characterisation of tripod packing graphs by Lemma 2 is sufficient to obtain our results.

The following theorem is a special case of an observation attributed to Szemerédi by Erdős (see [5], [2, p. 48]). Since Szemerédi's proof is apparently unpublished, we give an independent proof of our special case.

Theorem 3. *Consider an equitripartite, triangle-covered, diamond-free graph of size n. The maximum density of such a graph tends to 0 as $n \to \infty$.*

Proof. Suppose the contrary: for some constant $d > 0$, and for an arbitrarily large n (i.e. for some $n \geq n_0$ for any n_0), there is a tripartite, triangle-covered, diamond-free graph G of size n, such that $\text{dens}_3(G) \geq d > 0$. The main idea of the proof is to apply the Regularity Lemma and the Blow-up Lemma to the graph G. This will allow us to "distil" from G a new graph, also triangle-covered and diamond-free, with tripartite density higher than $\text{dens}_3(G)$ by a constant factor λ. Repeating this process, we can raise the density to $\lambda^2 d$, $\lambda^3 d$, etc., until the density becomes higher than 1, which is an obvious contradiction.

Let us now fill in the details of the "distilling" process. We start with a constant γ, $0 < \gamma < 1$; its precise numerical value will be determined later. Select a constant $\epsilon > 0$, such that $\epsilon \leq (\gamma d - \epsilon)^3/5$, as required by the Blow-up Lemma. By the Regularity Lemma, graph G admits an ϵ-regular partitioning, the order of which is constant and independent of the size of G. Denote by M the γd-map of this partitioning, and let $\mu : G \to M$ be the mapping function.

Consider the superedge subgraph $\mu^{-1}(E(M))$. Let $G_\triangle \sqsubseteq \mu^{-1}(E(M)) \sqsubseteq G$ be a spanning subgraph of G, consisting of all triangles *completely contained* in $\mu^{-1}(M)$; in other words, each triangle in $G_\triangle$ is completely contained in some supertriangle of G. We claim that $G_\triangle$ contains a significant fraction of all triangles (and, hence, of all edges) in G. Indeed, the bipartite density of a super-nonedge is by definition at most γd, hence the super-nonedge subgraph has at most $3\gamma d \cdot n^2$ edges. Every triangle not completely contained in $\mu^{-1}(E(M))$ must have at least one edge in the super-nonedge subgraph; since triangles in G are edge-disjoint, the total number of such triangles cannot exceed $3\gamma d \cdot n^2$. By initial assumption, the total number of triangles in G is at least $d \cdot n^2$, therefore the number of triangles in $G_\triangle$ must be at least $(1-3\gamma)d \cdot n^2$. By selecting a sufficiently small γ, we can make the number of triangles in $G_\triangle$ arbitrarily close to $d \cdot n^2$. For the rest the proof, let us fix the constant γ within the range $0 < \gamma < 1/12$, e.g. $\gamma = 1/24$. As a corresponding ϵ we can take e.g. $\epsilon = (\gamma d/2)^3/5 = d^3/(5 \cdot 48^3)$.

It only remains to observe that, since graph G is diamond-free, its γd-map M is diamond-free by the Blow-up Lemma. By Lemma 1, $\text{dens}(M) \leq 3/4$. This means that among all superpairs of G, the fraction of superedges is at most $3/4$. All edges of $G_\triangle$ are contained in superedges of G, therefore the average density of a superedge in $G_\triangle$ is at least $4/3 \cdot \text{dens}(G_\triangle)$. In particular, there must be some superpair in $G_\triangle$ with at least such density. Since $G_\triangle$ consists of edge-disjoint supertriangles, this superpair is contained in a unique supertriple $F \subseteq G_\triangle$, with

$$\mathrm{dens}_3(F) \geq 4/3 \cdot \mathrm{dens}_3(G_\triangle) \geq 4/3 \cdot (1 - 3\gamma)d = 4/3 \cdot (1 - 3 \cdot 1/24)d = 7/6 \cdot d.$$

In our previous notation, we have $\lambda = 7/6 > 1$.

We define the supertriple F to be our new "distilled" equitripartite triangle-covered diamond-free graph. Graph size has only been reduced by a constant factor, equal to the size of the ϵ-partitioning. By taking the original graph G large enough, the "distilled" graph F can be made arbitrarily large. Its density $\mathrm{dens}_3(F) \geq \lambda d = 7/6 \cdot d > d$. By repeating the whole process, we can increase the graph density to $(7/6)^2 \cdot d$, $(7/6)^3 \cdot d, \ldots,$ and eventually to values higher than 1, which contradicts the definition of density (in fact, values higher than 3/4 will already contradict Lemma 1). Hence, the initial assumption of existence of arbitrarily large equitripartite triangle-covered diamond-free graphs with constant positive density must be false. Negating this assumption, we obtain our theorem. ■

The solution of Stein and Szabó's problem is now an easy corollary of Lemma 2 and Theorem 3.

Corollary 1. *Consider a tripod packing of size n. The maximum density of such a packing tends to 0 as $n \to \infty$.*

5 Conclusion

We have proved that the density of a tripod packing must be infinitely small as its size tends to infinity. Since the Regularity Lemma only works for dense graphs, the question of determining the precise asymptotic growth of a maximum tripod packing size remains open.

Nevertheless, we can obtain an upper bound on this growth. Let d be the maximum density of tripod packing of size n. In our proof of Theorem 3, it is established that the maximum density of tripod packing of size $m(d^3/(5 \cdot 48^3)) \cdot n$ is at most $6/7 \cdot d$, where $m(\cdot)$ is the function defined by the Regularity Lemma. In [1, 4] it is shown that for tripartite graphs, $m(t) \leq 4^{t^{-5}}$. These two bounds together yield a desired upper bound on d as a function of n, which turns out to be a rather slow-growing function. By applying the "descending" technique from [11], we can also obtain an upper bound on the size of a maximal r-pod, which can tile (without any gaps) an r-dimensional space. The resulting bound is a fast-growing function of r. In [11] it is conjectured that this bound can be reduced to $r - 2$ for any $r \geq 4$. The conjecture remains open.

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