Archimedes' Cattle Problem and Pell's Equation

Adrienne Winans

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Abstract

This paper will explain how to solve both parts of Archimedes' cattle problem. The first part is somewhat elementary, but solving the complete problem using Pell's equation requires one to be truly "perfect in this species of wisdom."

1 Introduction

Archimedes' cattle problem is a statement, found in the form of a poem, that was struggled over for about 200 years. While the poem was first brought to light in 1773, found in a German library, the story goes that the problem originated as a challenge. Apparently, Archimedes wrote the poem to frustrate Apollonius, who had criticized his mathematical work. While there is no proof of this story, the poem was originally written in Greek and was extemely hard to solve. The problem was partially solved in 1880, by Amthor, who was able to find the first three digits (776) of the answer and the number of digits (206,545) in the amazingly large integer answers through use of a Pell equation. It was completely solved only with the advent of the computer age, as in 1965 three researchers at the University of Waterloo computed the final answer. All of digits in the final answer were published in 1981, having been found using a CRAY-1 computer.

2 Statement of Archimedes' Cattle Problem

If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colors, one milk white, another a glossy black, a third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all of the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture togther. Now the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd.

If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving seperately the number of well-fed bulls and again the number of females according to each color, thou wouldst not be called unskilled or ignorant of numbers, but yet shalt thou be numbered among the wise.

But come, understand also these conditions regarding the cattle of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a trianguar figure, there being no bulls of other colors in their midst nor none of them lacking.

If thou art able, O stranger, to find out all these things and gather then together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

3 Solution: The First Part

In solving the elementary first part of the problem we only need the ratios of the different types of cattle that is given in the first paragraph of the problem.

Let $\beta =$ the number of black bulls $\quad\zeta =$ the number of black cows

$\Psi =$ the number of white bulls $\quad\varpi =$ the number of white cows

$\Upsilon =$ the number of yellow bulls $\quad\gamma =$ the number of yellow cows

$\Delta =$ the number of dappled bulls $\quad\delta =$ the number of dappled cows

So we have the following relationships between the numbers of the cattle, according to the first paragraph of the problem:

$\Psi = (\frac{1}{2} + \frac{1}{3})\beta + \Upsilon \qquad \varpi = (\frac{1}{3} + \frac{1}{4})(\beta + \zeta)$

$\beta = (\frac{1}{4} + \frac{1}{5})\Delta + \Upsilon \qquad \zeta = (\frac{1}{4} + \frac{1}{5})(\Delta + \delta)$

$\Delta = (\frac{1}{6} + \frac{1}{7})\Psi + \Upsilon \qquad \delta = (\frac{1}{5} + \frac{1}{6})(\Upsilon + \gamma)$

$\gamma = (\frac{1}{6} + \frac{1}{7})(\Psi + \varpi)$

At this point we have seven equations and eight variables, so we put all the equations in terms of one variable and solve it like we would any system of equations.

For example: Take $\Psi = (\frac{1}{2} + \frac{1}{3})\beta + \Upsilon$. So $\Psi = \frac{5}{6}\beta + \Upsilon$. Substituting $\frac{5}{6}\beta + \Upsilon$ for Ψ in every equation with Ψ in it results in:

$\Delta = \frac{65}{252}\beta + \frac{13}{21}\Upsilon$ and $\gamma = \frac{65}{252}\beta + \frac{13}{42}\Upsilon + \frac{13}{42}\varpi$

Then we do the same substitution for $\beta = \frac{9}{20}\Delta + \Upsilon$.

So $\Delta = \frac{65}{252}(\frac{9}{20}\Delta + \Upsilon) + \frac{13}{21}\Upsilon$.

$\Delta = \frac{13}{112}\Delta + \frac{221}{252}\Upsilon$ which is $\frac{99}{112}\Delta = \frac{221}{252}\Upsilon$ and finally we get

[$\Delta = \frac{884}{891}\Upsilon$].

We can plug that back into $\beta = \frac{9}{20}\Delta + \Upsilon$ so that now $\beta = \frac{9}{20}(\frac{884}{891}\Upsilon) + \Upsilon$ so

[$\beta = \frac{716}{495}\Upsilon$].

We put $\beta = \frac{716}{495}\Upsilon$ into $\Psi = \frac{5}{6}\beta + \Upsilon$ and so

[$\Psi = \frac{655}{297}\Upsilon$].

At this point we have Δ, β, and Ψ in terms of Υ. We continue on in this manner, substituting and resubstituting, until we have all of the variables and equations in terms of one of the variables. The first part of the problem has then been solved, for we know the common coefficients for the infinitely many solutions possible for the number of cattle with the first part of the problem. The only way to find the number is to have more limits upon the number of cattle; limits that we will introduce in the second part of the problem. For now the system of equations has been solved so that the number of cattle can be stated as multiples of some integer α where $\alpha \geq 1$.

Our solution to the first part of Archimedes' cattle problem:

$\Psi = 10,366,482\alpha$ $\quad\varpi = 7,206,360\alpha$

$\beta = 7,460,514\alpha$ $\quad\zeta = 4,893,246\alpha$

$\Upsilon = 4,149,387\alpha$ $\quad\gamma = 5,439,213\alpha$

$\Delta = 7,358,060\alpha$ $\quad\delta = 3,515,820\alpha$

The total number of cattle is $50,389,082\alpha$.

4 Solution: The Second Part

Now that we have solved the first part of the problem we can use that information as well as the conditions set forth in the third paragraph of the problem to solve the complete problem.

From the third paragraph we know that the black and white bulls together formed a square. To solve the complete problem, we assume that means their number altogether makes a square number, but if you make the assumption that this means they form a rectangle (as bulls are not perfectly square and are longer than they are wide) the problem is easier. That variation on the cattle problem is considered independent of Archimedes' problem and is instead called Wurm's problem. But solving the complete problem, this condition means that

$$\Psi + \beta = \text{a square number.}$$

The other condition states that the yellow and the dappled bulls form a triangle so we know that

$$\Upsilon + \Delta = \text{a triangular number.}$$

So lets create two positive integers Ω and Ξ so we know that

$$\Upsilon + \Delta = 1 + 2 + 3 + ... + \Xi = \frac{\Xi(\Xi+1)}{2}.$$

From the first part of the problem we know that

$\Psi + \beta = 10,366,482\alpha + 7,460,514\alpha = 17,826,996\alpha = 2 \cdot 2 \cdot 3 \cdot 11 \cdot 29 \cdot 4657\alpha.$

Since this is equal to some square number we can say that

$\alpha = 3 \cdot 11 \cdot 29 \cdot 4657\Omega^2$ where the 2^2 is absorbed by the Ω^2.

So we now have the condition $\alpha = 4,456,749\Omega^2$.

Also from the first part of the problem we know that

$\Upsilon + \Delta = 4,149,387\alpha + 7,358,060\alpha = 11,507,447\alpha = \frac{\Xi(\Xi+1)}{2}.$

Now we can reduce all of the limitations and variables that we have dealt with this entire problem into one very familar equation with two variables.

First we substitute $\alpha = 4,456,749\Omega^2$.

$11,507,447\alpha = 11,507,447(4,456,749\Omega^2) = 102,571,605,819,606\Omega^2 = \frac{\Xi(\Xi+1)}{2}.$

Now we have to solve the equation $\Xi^2 + \Xi - 2 \cdot 102,571,605,819,606\Omega^2 = 0$ for an integer solution. To solve this equation for an integer, which we need,$1 + 4 \cdot 102,571,605,819,606\Omega^2$ must be a square. So we introduce ν, another positive integer.

Now our equation of the hour is $1 + 410,286,423,278,424\Omega^2 = \nu^2$.

To those knowledgable in the area of number theory (which I am not one of), this equation should be familiar as it is a Pell's equation, forms of which can be used to solve all equations. To make the numbers in involved more reasonable than a 15 digit coefficient, we can now factor 410,286,423,278,424 and absorb all the squares into a new positive integer variable, θ. (We pulled this trick earlier going when we created Ω.)

So $1 + 2^3 \cdot 3 \cdot 7 \cdot 11 \cdot 29 \cdot 353 \cdot 4657^2\Omega^2 = 1 + 4,729,494\theta^2$ with θ absorbing 2^2 and 4657^2 along with Ω.

Our last equation is now $\nu^2 - 4,729,494\theta^2 = 1$.

As this equation is Pell's equation, we would now proceed to solve it as we would any Pell equation and then we can feed the resulting values into an explicit equation until we get the exact number of Archimedes' cattle. While I will not do it here, as I was told others are dealing with Pell's equation and continued fractions, the way to solve this problem as it has been solved in the past is to find $\sqrt{4,729,494}$ as a continuing fraction. Then take the last number of an expansion, and the numerator is ν and the denominator is θ. Once we have ν and θ we can find the number of cattle using an explicit formula

$[\frac{25194541}{184119152}\chi^{4658\tau}]$ where τ is a positive integer.

([x] signifies the smallest integer greater than or equal to the x)

We know that $\chi^{4658\tau} = \nu + \theta\sqrt{4,729,494}$, so from there we can generate the actual number of Archimedes' cattle.

5 The Number of the Cattle of the Sun

After solving Archimedes' cattle problem completely the final answer for the smallest number of cattle possible is found to have 202,545 digits. In his article in The American Mathmetical Monthly, Ilan Vardi expresses this immense number in the simple formula,

$$\left[\tfrac{25194541}{184119152}(109931986732829734979866232821433543901088049+ 50549485234315033074477819735540408986340\sqrt{4,729,494})^{4658}\right]$$

([x] signifies the smallest integer greater than or equal to the x)

To give you an idea of the incredible size of this number, here are the first and the last 50 digits of the number of the Cattle of the Sun:

776,027,140,648,681,826,953,023,283,321,388,666,423,232,240,592,33...

...05,994,630,144,292,500,354,883,118,973,723,406,626,719,455,081,800 cattle.

If a person were to sit down and write out the entire number, taking a second to write each digit, it would take 2 days, 8 hours, 15 minutes and 45 seconds! Now that is a lot of bovines.

6 Acknowledgements

I would like to thank Mr. Kobotis for telling me about [1] and [2], where I got the majority of my information.

References

[1] Ilan Vardi: Archimedes' Cattle Problem. The American Mathematical Monthly. April 1998

[2] Archimedes Cattle Problem. Dexel University Department of Mathematics and Computer Science's Website.

[3] Ivan Peterson: Cattle of the Sun. Science News Online. April 18, 1998