$(1, k)$-Compositions

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Abstract

A $(1, k)$-composition of a positive integer n consists of an ordered sequence of the integers 1 and k whose sum is n. A palindromic $(1, k)$-composition is one for which the sequence is the same from left to right as from right to left. We give recursive equations and generating functions for the total number of such compositions and palindromes, and for the number of 1's, k's, "+"signs and summands in all $(1, k)$-compositions and $(1, k)$-palindromes. We look at patterns in the values for the total number of $(1, k)$-compositions and $(1, k)$-palindromes and derive recusive relations and generating functions for the number of levels, rises and drops in all $(1, k)$-compositions and $(1, k)$-palindromes.

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1. Introduction

A $(1, k)$-*composition* of a positive integer n consists of an ordered sequence of the integers 1 and k whose sum is n. A *palindromic* $(1, k)$-*composition* is one for which the sequence is the same from left to right as from right to left. For the remainder of this paper we will refer to palindromic $(1, k)$-compositions by the short-hand term $(1, k)$-*palindrome*.

Alladi and Hoggatt [1] have considered (1,2)-compositions and (1,2)-palindromes. They count the number of such compositions and palindromes, the number of summands, and the number of times either a 1 or a 2 occurs in all (1,2)-compositions and (1,2)-palindromes, respectively. Furthermore, they count the number of "+"-signs and the number of rises (a summand followed by a larger summand), levels (a summand followed by itself) and drops (a summand followed by a smaller summand) in all such compositions and palindromes, respectively. Hoggatt and Bicknell [3] have looked at more general compositions and palindromes, namely those for which the summands are selected from a finite or countably infinite set S. For example, if only the summands 1 and 2 are allowed in the compositions, then $S = \{1, 2\}$. Hoggatt and Bicknell have derived generating functions for the number of compositions, palindromes, number of summands, "+"-signs, and the number of times a particular summand occurs in all such compositions or palindromes of n. However, since the possible values of the summands come from a very general set, they were not able to develop recurrence relations and generating functions for the number of rises, levels and drops in this setting.

We will focus on a generalization of the (1,2)-compositions, namely, we look at $(1, k)$-compositions and $(1, k)$-palindromes. In Section 2 we establish our notation and derive recurrence relations for the total number of $(1, k)$-compositions and $(1, k)$-palindromes, the number of times either a 1 or a k occurs in all $(1, k)$-compositions and $(1, k)$-palindromes, and the number of summands and "+"-signs in all $(1, k)$-compositions and $(1, k)$-palindromes of n. Furthermore, we state the generating functions for these quantities as a special case of the results in [3]. In Section 3, we investigate and give combinatorial proofs for patterns among the number of $(1, k)$-compositions and $(1, k)$-palindromes for different values of k. In Section 4 we derive recursive formulas and the generating functions for the number of levels, rises and drops in all $(1, k)$-compositions and $(1, k)$-palindromes of n.

2. Notation and basic results

We will use the following notation.

$C_{n,k}, P_{n,k}$	=	the number of $(1, k)$-compositions and $(1, k)$-palindromes of n, respectively, where $C_{0,k} = P_{0,k} = 1$ for all k
$C^{+}_{n,k}$, $P^{+}_{n,k}$	=	the number of "+" signs in all $(1, k)$-compositions and $(1, k)$-palindromes of n, respectively
$C^{\mathrm{s}}_{n,k}$, $P^{\mathrm{s}}_{n,k}$	=	the number of summands in all $(1, k)$-compositions and $(1, k)$-palindromes of n, respectively
$C^{l}_{n,k}$, $P^{l}_{n,k}$	=	the number of l's in all $(1, k)$-compositions and $(1, k)$-palindromes of n, respectively, where $l = 1$ or k
$r_{n,k}$, $l_{n,k}$, $d_{n,k}$	=	the number of rises, levels, and drops in all $(1, k)$-compositions of n, respectively
$\tilde{r}_{n,k}$, $\tilde{l}_{n,k}$, $\tilde{d}_{n,k}$	=	the number of rises, levels, and drops in all $(1, k)$-palindromes of n, respectively
$n \equiv k$, $n \not\equiv k$		denotes n and k having the same and opposite parity, respectively.

Before we derive recurrence relations for the quantities of interest, we will present different ways to create $(1,k)$-compositions and $(1,k)$-palindromes. The first method is a recursive one: for $n > k$, we create the $(1,k)$-compositions of n by either adding a 1 to the right end of the $(1,k)$-compositions of $n-1$, or by adding a k to the right end of the $(1,k)$-compositions of $n-k$. Likewise, for $(1,k)$-palindromes, we add a 1 to both sides of the $(1,k)$-palindromes of $n-2$ or a k to both sides of the $(1,k)$-palindromes of $n-2k$. We will refer to this method as the *recursive creation method*. In addition, we can also enumerate $(1,k)$-palindromes by focusing on the middle summand. Notice that the middle summand must have the same parity as n. Thus, if n is even, either there is no middle summand, and the $(1,k)$-palindrome is created by combining a $(1,k)$-composition of $n/2$ with its reverse, or the middle summand is (an even) k, combined with a $(1,k)$-composition of $(n-k)/2$ on the left and its reverse on the right. If n is odd, then either the middle summand is a 1, combined with a $(1,k)$-composition of $(n-1)/2$ on the left and its reverse on the right, or the middle summand is (an odd) k, combined with $(1,k)$-compositions of $(n-k)/2$. This observation provides for a connection between the number of $(1,k)$-compositions and $(1,k)$-palindromes.

Lemma 1 gives basic results for $(1,k)$-compositions, while Lemma 2 lists basic results for $(1,k)$-palindromes. Recall that the generating function $G_a(x)$ for a sequence $\{a_{n,k}\}_{n=0}^{\infty}$ is given by $G_a(x) = \sum_{n=0}^{\infty} a_{n,k} \cdot x^n$.

Lemma 1 *1.* $C_{n,k} = C_{n-1,k} + C_{n-k,k}$, *with* $C_{n,k} = 1$ *for* $0 \le n < k$. *Alternatively,* $C_{n,k} = \sum_{j=0}^{\lfloor n/k \rfloor} \binom{n-j(k-1)}{j}$ *and* $G_C(x) = \frac{1}{1-x-x^k}$.

2. $C^1_{n,k} = C^1_{n-1,k} + C^1_{n-k,k} + C_{n-1,k}$, *with* $C^1_{n,k} = n$ *for* $0 \le n < k$. *Alternatively,* $C^1_{n,k} = \sum_{j=0}^{\lfloor n/k \rfloor} (n-j\cdot k)\binom{n-j(k-1)}{j}$ *and* $G_{C^1}(x) = \frac{x}{(1-x-x^k)^2}$.

3. $C^k_{n,k} = C^k_{n-1,k} + C^k_{n-k,k} + C_{n-k,k}$, *with* $C^k_{n,k} = 0$ *for* $0 \le n < k$. *Alternatively,* $C^k_{n,k} = \sum_{j=0}^{\lfloor n/k \rfloor} j\binom{n-j(k-1)}{j}$ *and* $G_{C^k}(x) = \frac{x^k}{(1-x-x^k)^2}$.

4. $C^s_{n,k} = C^1_{n,k} + C^k_{n,k}$, *and* $G_{C^s}(x) = \frac{x+x^k}{(1-x-x^k)^2}$.

5. $C^+_{n,k} = C^s_{n,k} - C_{n,k}$ *for* $n \ge 1$ *with* $C^+_{0,k} = 0$, *and* $G_{C^+}(x) = \frac{(x+x^k)^2}{(1-x-x^k)^2}$.

Proof: The recurrence relation for $C_{n,k}$ follows directly from the recursive creation method. Likewise for the recurrence relations of $C^1_{n,k}$ and $C^k_{n,k}$, as we get all the 1's or k's from the $(1,k)$-compositions of $n-1$ and $n-k$, and then one additional 1 or k for each composition to which we add a 1 or k, respectively. The initial conditions for these three quantities follow easily from the fact that the only $(1,k)$-composition of n for $n < k$ consist of n 1's. The alternative formulas for these quantities follow by counting the compositions first according to the number of k's in the $(1,k)$-compositions, and, for $C^1_{n,k}$ and $C^k_{n,k}$, by multiplying these counts by the number of 1's and k's, respectively, then summing according to the number of k's. The recurrence relation for $C^{\mathrm{S}}_{n,k}$ is obvious as each summand has to be either a 1 or a k, and the last recurrence relation follows because in each $(1,k)$-composition, the number of "+" signs is one less than the number of summands. The generating functions follow from Theorem 1.1, Theorem 1.3 and the remarks after Theorem 1.3 of [3], since the function $F(x) = \sum_{a_k \in S} x^{a_k}$ defined in [3] reduces to $F(x) = x + x^k$. □

We now derive the corresponding results for $(1,k)$-palindromes. In this case, the initial conditions depend on the parity of n and k. We will use $n = 2i$ or $n = 2i+1$ and $k = 2j$ or $k = 2j+1$, where i and j are non-negative integers.

Lemma 2 *1.* $P_{n,k} = P_{n-2,k} + P_{n-2k,k}$ *with* $P_{n,k} = 1$ *for* $0 \le n < k$, $P_{n,k} = 1$ *for* $k \le n < 2k$, $n \not\equiv k$, *and* $P_{n,k} = 2$ *for* $k \le n < 2k$, $n \equiv k$. *Alternatively,* $P_{n,k} = C_{i,k}$ *if* $n \not\equiv k$ *and* $P_{n,k} = C_{i-j,k} + C_{i,k}$ *if* $n \equiv k$, *and* $G_P(x) = \frac{1+x+x^k}{1-x^2-x^{2k}}$.

2. $P^1_{n,k} = P^1_{n-2,k} + P^1_{n-2k,k} + 2P_{n-2,k}$, *with* $P^1_{n,k} = n$ *for* $0 \le n < k$, $P^1_{n,k} = n$ *for* $k \le n \le 2k$, $n \not\equiv k$, *and* $P^1_{n,k} = 2n - k$ *for* $k \le n \le 2k$, $n \equiv k$ *with* $G_{P^1}(x) = \frac{x+2x^2+x^3+2x^{2+k}-x^{2k+1}}{(1-x^2-x^{2k})^2}$.

3. $P^k_{n,k} = P^k_{n-2,k} + P^k_{n-2k,k} + 2P_{n-2k,k}$, *with* $P^k_{n,k} = 0$ *for* $0 \le n < k$, $P^k_{n,k} = 0$ *for* $k \le n < 2k$, $n \not\equiv k$, $P^k_{n,k} = 1$ *for* $k \le n < 2k$, $n \equiv k$, $P^k_{2k,k} = 2$ *for* $n \not\equiv k$, *and* $P^k_{2k,k} = 3$ *for* $n \equiv k$ *with* $G_{P^k}(x) = \frac{x^k+2x^{2k}+x^{3k}+2x^{2k+1}-x^{2+k}}{(1-x^2-x^{2k})^2}$.

4. $P^s_{n,k} = P^1_{n,k} + P^k_{n,k}$, *with generating function* $G_{P^S}(x) = \frac{x+2x^2+x^3+x^{k+2}+x^{2k+1}+x^k+2x^{2k}+x^{3k}}{(1-x^2-x^{2k})^2}$.

5. $P^+_{n,k} = P^s_{n,k} - P_{n,k}$ *for* $n \ge 1$ *with* $P^+_{0,k} = 0$, *with* $G_{P^+}(x) = \frac{(x^2+x^{2k})(1+2x+x^2+2x^k+x^{2k})}{(1-x^2-x^{2k})^2}$.

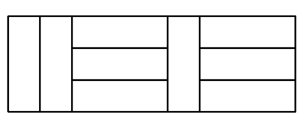
Proof: The first recurrence relation for $P_{n,k}$ follows from the recursive creation method. For the second recurrence relation, based on $(1,k)$-compositions, we need to look at the parity of n and k. If $n = 2i$, then there is either no middle summand, i.e., we get $C_{n/2,k} = C_{i,k}$ palindromes, or, if k

is also even, then we get additional $(1,k)$-palindromes with k as the middle summand, of which there are $C_{(n-k)/2,k} = C_{i-j,k}$. A similar argument can be made for $n = 2i+1$, where the middle summand is either 1 or an odd k, and there are $C_{(n-1)/2,k} = C_{i,k}$ and $C_{(n-k)/2,k} = C_{i-j,k}$ such palindromes, respectively. The recurrence relations for $P^1_{n,k}$ and $P^k_{n,k}$ follow again from the recursive creation method, except that we now get two additional 1's or k's for each palindrome of $n-2$ and $n-2k$, respectively. For all three cases, the initial conditions follow from the fact that for $n < k$, the only $(1,k)$-palindrome is the one consisting of n 1's. If $k \leq n < 2k$, then there is the $(1,k)$-palindrome of n 1's, and, if $n \equiv k$, the additional palindrome which has a k as the middle summand, together with $n-k$ 1's. For $n = 2k$, we get the $(1,k)$-palindrome consisting of two k's in addition to the two types of palindromes for $k \leq n < 2k$. The recurrence relations for $P^{s}_{n,k}$ and $P^{+}_{n,k}$ follow exactly as in the case of $(1,k)$-compositions. Finally, the generating functions follow from Theorem 1.2, Theorem 1.4 and the remarks after Theorem 1.4 in [3]. □

3. Structures in values of $\{C_{n,k}\}_{n=0}^{\infty}$ for different k

We will now look at the sequences $\{C_{n,k}\}_{n=0}^{\infty}$ and $\{P_{n,k}\}_{n=0}^{\infty}$ for different values of k and will give combinatorial proofs of the structures exhibited in Table 1, which contains the values for $C_{n,k}$.

Note that the column for $k = 2$ in Table 1 contains the shifted Fibonacci numbers, and that the columns for $k = 3$ through $k = 9$ appear as sequences A000930, A03269, A003520, and A005708 - A005711 in [5]. There are several examples of objects that are counted by sequence A000930 ($k = 3$), and we show the equivalence of these counts to the number of (1,3)-compositions. The first example indicates that A000930 represents the number of tilings of a 3-by-n rectangle with straight trominoes, i.e., 1-by-3 tiles. It is easy to see how the two counts are related: the trominoes can only be placed vertically or as a block of three horizontal tiles. The first case corresponds to a 1 in the composition, the second to a 3, as indicated in the figure below.



$\longleftrightarrow \; 1+1+3+1+3$

Note that this process can be easily generalized to tilings of a k-by-n rectangle with 1-by-k tiles, thus showing equivalence to the number of (1,k)-compositions.

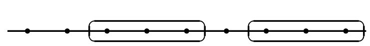
The second example indicates that A000930 represents the number of ordered partitions (= compositions) of $n-1$ consisting of 1's and 2's with no 2's adjacent. Here the correspondence between the counts is not immediately obvious, but can be easily demonstrated with an example. Since the 2's are not adjacent, each 2 is either followed by a 1, or appears as the last summand on the right. To each such composition of $n-1$, add a 1 on the right, making them compositions of n. Now replace every instance of 21 with a 3, which results in compositions of n with 1's and 3's only. This process can be reversed, thus the correspondence is one-to-one. Here is the correspondence for the example given in [5] for $n = 6$.

111111	$\longleftrightarrow$	11111	1131	$\longleftrightarrow$	1121
3111	$\longleftrightarrow$	2111	1113	$\longleftrightarrow$	1112
1311	$\longleftrightarrow$	1211	33	$\longleftrightarrow$	212

Note that this example also points to the obvious generalization: The number of $(1, k)$-compositions of n is equal to the number of compositions of $n-1$ with 1's and $(k-1)$'s, where no $(k-1)$'s are next to each other.

Sequence A000930 is also listed in [5, page 91] as an example of a third order linear recurrence, where $\bar{U}_{-n} = C_{n-1,3}$. The sequence $\bar{U}_n$ is defined by $\bar{U}_n = -\bar{U}_{n-2} + \bar{U}_{n-3}$ with initial conditions $\bar{U}_0 = \bar{U}_1 = 1$, and $\bar{U}_2 = 1$. This corresponds to initial conditions $C_{-3,3} = 1$ and $C_{-2,3} = C_{-1,3} = 0$.

Finally, there is one reference given in [5] which recognizes the sequences A000930, A03269, A003520, and A005708 - A005711 as members of a family with recurrence relation $a(n) = a(n-1) + a(n-k)$. Di Cera and Kong [2] count the number of ways to cover a linear lattice of n sites with molecules that are k sites wide, where there is no overlap of molecules, but gaps are allowed. It is easy to see how this relates to $(1, k)$-compositions — each summand k corresponds to a molecule of size k, and each summand 1 corresponds to an empty site on the lattice, as shown in the figure below:



$\longleftrightarrow \quad 1 + 1 + 3 + 1 + 3$

We will now look at patterns across columns. There are two particularly simple patterns: 1) the upper triangle of 1's; and 2) diagonals of slope -1 consisting of the same integer (from an appropriate starting point onwards). Both of these patterns are the result of the initial conditions, as they cover the cases $0 \leq n < k$ and $k \leq n < 2k$. For $n \geq 2k$, we notice three additional patterns: 1) pairs of repeated values (boxed); 2) diagonal sequences of slope -2 containing consecutive integers (from an appropriate starting point onwards), for example the sequence { 13, 14, 15, 16, 17,....} marked with a $\star$; and 3) sequences on diagonals of slope -3, whose terms have increasing

n \ k	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
2	**2**	1	1	1	1	1	1	1	1
3	**3**	**2**	1	1	1	1	1	1	1
4	5	**3**	**2**	1	1	1	1	1	1
5	8	**4**	**3**	**2**	1	1	1	1	1
6	**13**	6	**4**	**3**	**2**	1	1	1	1
7	**21**	9	**5**	**4**	**3**	**2**	1	1	1
8	34	13 ⋆	7	**5**	**4**	**3**	**2**	1	1
9	55	**19**	10	**6**	**5**	**4**	**3**	**2**	1
10	89	**28**	14⋆	8	**6**	**5**	**4**	**3**	**2**
11	144	**41**	19	11	**7**	**6**	**5**	**4**	**3**
12	233	60	**26**	15⋆	9	**7**	**6**	**5**	**4**
13	377	88	**36**	20	12	**8**	**7**	**6**	**5**
14	610	129	**50**	26	16⋆	10	**8**	**7**	**6**
15	987	189	69	**34**	21	13	**9**	**8**	**7**
16	1597	277	95	**45**	27	17⋆	11	**9**	**8**
17	2584	406	131	**60**	34	22	14	**10**	**9**

Table 1: The number of $(1,k)$-compositions of n

differences, for example the sequence {21, 28, 36, 45, ...} (also in bold). The following theorem makes these patterns more precise.

Theorem 3 *1. For any k, $C_{3k+2,k+1} = C_{3k,k}$.*

2. For $2k \leq n < 3k$, $C_{n+2,k+1} = C_{n,k} + 1$.

3. For $3k \leq n < 4k$, $C_{n+3,k+1} = C_{n,k} + n - 2k + 4$.

Proof: 1. To show the first equality, we give a one-to-one correspondence between the respective compositions. Since $n = 3k$, the $(1,k)$-compositions of $3k$ can have either no k, one k, 2 k's or 3 k's, and the $(1,k+1)$-compositions of $3k+2$ can have either no $k+1$, one $k+1$ or two $(k+1)$'s. There is exactly one composition without any k or $k+1$, respectively, the composition of all 1's. The compositions of $3k$ with exactly two k's are in one-to-one correspondence with those of $3k+2$ containing exactly two $(k+1)$'s, as each k can be replaced by $k+1$. The compositions of $3k$ with exactly one k have $2k+1$ summands, for a total of $2k+1$ such compositions, and there is only one composition of $3k$ that has exactly 3 k's. These are matched by the compositions of $3k+2$ with exactly one $k+1$, of which there are $(3k+2)-(k+1)+1 = 2k+2$. Below is an example illustrating these correspondences for $k=2$ and $n=6$:

111111	$\longleftrightarrow$	11111111		2211	$\longleftrightarrow$	3311
21111	$\longleftrightarrow$	311111		2121	$\longleftrightarrow$	3131
12111	$\longleftrightarrow$	131111		2112	$\longleftrightarrow$	3113
11211	$\longleftrightarrow$	113111		1221	$\longleftrightarrow$	1331
11121	$\longleftrightarrow$	111311		1212	$\longleftrightarrow$	1313
11112	$\longleftrightarrow$	111131		1122	$\longleftrightarrow$	1133
222	$\longleftrightarrow$	111113				

2. For the second equality, we match up the two types of compositions in the same way. Note however, that now there is no $(1,k)$-composition of n with three k's, thus there is no match for one of the (1, $k+1$)-compositions of $n+2$ with exactly one $k+1$.

3. We utilize the explicit formula for counting $(1,k)$-compositions, namely $C_{n,k} = \sum_{j=0}^{\lfloor n/k \rfloor} \binom{n-j(k-1)}{j}$. For $3k \le n < 4k$, $C_{n,k} = \sum_{j=0}^{3} \binom{n-j(k-1)}{j}$, since there can be at most three k's in the $(1,k)$-compositions of n. Likewise, for this range of values for n, there can be at most three $(k+1)$'s in the (1, $k+1$)-compositions of $n+3$, thus $C_{n+3,k+1} = \sum_{j=0}^{3} \binom{n+3-j\cdot k}{j}$. We now just compare the different counts:

$$C_{n,k} = 1 + (n-k+1) + \binom{n-2k+2}{2} + \binom{n-3k+3}{3}$$

and (after simplification)

$$C_{n+3,k+1} = 1 + (n+3-k) + \binom{n+3-2k}{2} + \binom{n+3-3k}{3}.$$

Comparing the respective summands, we see that the first and last ones are identical, the second ones differ by 2, and the third summands are of the form $\binom{m}{2}$ and $\binom{m+1}{2}$, for $m = n-2k+2$. Straightforward computation shows that the difference between these two terms is m, and thus, $C_{n+3,k+1} = C_{n,k} + 2 + n - 2k + 2$, which gives the desired result. □

When looking for patterns in the values for $P_{n,k}$, we need to distinguish between odd and even values of n, as they have different formulas. Thus, the sequence $\{P_{n,k}\}_{n=0}^{\infty}$ is the result of interleaving the two sequences $\{P_{2i+1,k}\}_{i=0}^{\infty}$ and $\{P_{2i,k}\}_{i=0}^{\infty}$. By Lemma 2, part 1, the subsequence for which $n \not\equiv k$ agrees with the sequence for the number of $(1,k)$-compositions.

Furthermore, Lemma 2, part 1 also provides for an easy means to compute the generating function for the subsequence for which $n \equiv k$, since $P_{n,k} = C_{i-j,k} + C_{i,k}$, where $n = 2i$ or $n = 2i+1$ and $k = 2j$ or $k = 2j+1$. Using standard methods for generating functions together with Lemma 1, we get that the generating function for $\hat{P}_{i,k} := P_{2i,k}$ or $\hat{P}_{i,k} := P_{2i+1,k}$ is given by $G_{\hat{P}}(x) = \frac{1+x^j}{(1-x-x^k)}$.

We have tested the sequences for $k = 2, \ldots, 10$ in the Online Encyclopedia of Integer Sequences [5], both using the full sequences and the subsequences for which $n \equiv k$. For $k = 2$, $\{P_{n,2}\}_{n=0}^{\infty}$ consists of two interleaved Fibonacci sequences, and the full sequence is also referenced in [5] as A053602, with a recurrence of the form $a(n) = a(n-1) - (-1)^n a(n-2)$, where $a(n+1) = P_{n,2}$. Thus we get the following two cases: $P_{n,2} = P_{n-1,2} + P_{n-2,2}$ for n even, and $P_{n,2} = P_{n-1,2} - P_{n-2,2}$ for n odd. These recurrences can be explained in terms of the (1,2)-palindromes by using an alternative construction, namely modifying the middle summands rather than the two ends of the palindromes. Note that for even n, the palindrome either has middle summand 2 or an even split; for odd n, the middle summand always is a 1. We can create the (1,2)-palindromes for even n by either increasing the middle summand of a (1,2)-palindrome of $n-1$ (which gives middle summand 2), or by modifying the center of a (1,2)-palindrome of $n-2$, inserting either $1+1$ into those with an even split, or replacing the middle summand of 2 by $2+2$. Thus, $P_{n,2} = P_{n-1,2} + P_{n-2,2}$ for n even. If n is odd, then we get the (1,2)-palindromes of n by inserting a 1 into the center of those (1,2)-palindromes of $n-1$ that have an even split. The number of (1,2)-palindromes of $n-1$ that have a 2 in the center (and thus need to be subtracted) were created by increasing the middle summand of the palindromes of $n-2$ by 1. Thus, $P_{n,2} = P_{n-1,2} - P_{n-2,2}$ for n odd.

For $k \geq 3$, none of the full sequences are listed in [5]. Of the subsequences with $n \equiv k$, only the sequence for $k = 3$ is listed in [5], as A058278, with $P_{2i+1,3} = a(i+2)$.

When looking for patterns across columns, there are several ways to arrange the tables of values. One can look at the complete table of values, which would not show patterns as easily due to the interleaving of the two subsequences that have different formulas. If one looks at the subsequences for odd and even n separately, then there are two choices: 1) making separate tables for sequences in which $n \equiv k$ and $n \not\equiv k$, or 2) making separate tables for the odd and even values of n. We have looked at both choices, and the patterns that arise are similar to the case for $(1,k)$-compositions.

4. Rises, levels and drops in $(1, k)$-compositions

Alladi and Hoggatt have counted the number of rises, levels and drops for (1,2)-compositions and (1,2)-palindromes [1]. We will now look at the general case. Since for each non-palindromic $(1, k)$-composition a corresponding $(1, k)$-composition in reverse order exists, any rise will be matched by a drop and vice versa. In $(1, k)$-palindromes, symmetry provides for the match within the palindrome. Furthermore, each "+"-sign corresponds to either a rise, a level, or a drop, and therefore

$$r_{n,k} = d_{n,k} \text{ and } C^+_{n,k} = r_{n,k} + l_{n,k} + d_{n,k}. \tag{1}$$

Likewise, these formulas hold for $(1, k)$-palindromes. We first give the results for $(1, k)$-compositions.

Theorem 4 *1. For $n > k$, $r_{n,k} = r_{n-1,k} + r_{n-k,k} + C_{n-k-1,k}$, with $r_{n,k} = 0$ for $n \leq k$, and generating function $G_r(x) = \sum_{k=0}^{\infty} r_{n,k} \cdot x^n = \frac{x^{k+1}}{(1-x-x^k)^2}$.*
2. For $n > k$, $l_{n,k} = l_{n-1,k} + l_{n-k,k} + C_{n-2,k} + C_{n-2k,k}$, with $l_{n,k} = n - 1$ for $n \leq k$, and generating function $G_l(x) = \sum_{k=0}^{\infty} l_{n,k} \cdot x^n = \frac{x^2+x^{2k}}{(1-x-x^k)^2}$.

Proof: For $n < k$, the only $(1, k)$-composition of n consists of all 1's, and if $n = k$, there is an additional composition consisting of only k. In either case, no rises occur. If $n > k$, then we look at the creation of the compositions of n from those of $n - 1$ and $n - k$. If a 1 is added, no new rises occur. If a k is added, then additional rises are created if the $(1, k)$-composition of $n - k$ ends in 1. These are exactly the $(1, k)$-compositions of $n - k - 1$, and one new rise is created for each of these, which gives the recursion. To get the generating function, we multiply each term in the recurrence relation by x^n, then sum over $n \geq 0$. (Note that the recurrence relation is also valid for $n \leq k$, since all terms are equal to zero.) Expressing the series in terms of $G_r(x)$ and $G_C(x)$ and using Theorem 1 leads to

$$G_r(x) = \frac{x^{k+1} G_C(x)}{(1 - x - x^k)} = \frac{x^{k+1}}{(1 - x - x^k)^2}.$$

The formula for the levels follows from a similar argument. For $n \leq k$, levels occur only in the $(1, k)$-compositions of all 1's, and there are $n - 1$ of those. When creating $(1, k)$-compositions of n from those of $n - 1$ and $n - k$, additional levels are created when adding either a 1 to a $(1, k)$-composition of $n - 1$ ending in 1, or adding a k to a $(1, k)$-composition of $n - k$ ending in k. There are $C_{n-1-1,k} + C_{n-k-k,k}$ new levels, which gives the recurrence relation. Using Eq. 1, the generating functions is computed as $G_l(x) = G_{C^+}(x) - 2G_r(x) = \frac{x^2+x^{2k}}{(1-x-x^k)^2}$. □

We now derive the corresponding results for $(1, k)$-palindromes. As before, the initial conditions depend on the parity of n and k. Recall that $n \equiv k$ denotes n and k having the same parity.

Theorem 5 *1. For* $n \geq 2(k+1)$, $\tilde{r}_{n,k} = \tilde{r}_{n-2,k} + \tilde{r}_{n-2k,k} + 2P_{n-2(k+1),k}$, *with initial conditions*

$$\tilde{r}_{n,k} = \begin{cases} 0 & \text{for } n \leq k \\ 1 & \text{for } k < n \leq 2k, n \equiv k \\ 0 & \text{for } k < n \leq 2k, n \not\equiv k \\ 2 & \text{for } n = 2k+1, n \equiv k \\ 1 & \text{for } n = 2k+1, n \not\equiv k \end{cases},$$

with $G_{\tilde{r}}(x) = \sum_{k=0}^{\infty} \tilde{r}_{n,k} \cdot x^n = \frac{x^{k+1}(x - x^3 + x^k - x^{3k} + 2x^{k+1} + x^{k+2} + x^{2k+1})}{(1-x^2-x^{2k})^2}$.

Proof: As in the case of $(1,k)$-compositions, there are no rises for $n \leq k$. For $n < k < 2k$, the $(1,k)$-palindromes either consist of all 1's, or can have one occurrence of k, which must be in the center. For this to occur, n and k need to have the same parity, and then there is one rise. If $n = 2k$, we get the additional palindrome $k + k$, which does not have a rise. Finally, for $n = 2k + 1$, we get either all 1's, or the palindrome $k + 1 + k$, and, if n and k have the same parity, the palindrome with a k at the center, combined with all 1's. The recurrence relation is derived similarly to the proof of Theorem 4. When adding a 1 to the $(1,k)$-palindromes of $n-2$, an additional rise occurs on the left side of those $(1,k)$-palindromes which end in k, of which there are $P_{n-2-2k,k}$. Likewise, one additional rise occurs on the right side when adding k on both sides of the $(1,k)$-palindromes of $n - 2k$ which end in 1, of which there are $P_{n-2k-2,k}$.
To compute the generating function, we define $\hat{P}_{n,k}$ as $P_{n,k}$ for $n \geq 0$, and $\hat{P}_{-1,k} = \hat{P}_{-k,k} = 1/2$. Note that $G_{\hat{P}}(x) = G_P(x) + \frac{1}{2}x^{-1} + \frac{1}{2}x^{-k}$. We will show that $\tilde{r}_{n,k} = \tilde{r}_{n-2,k} + \tilde{r}_{n-2k,k} + 2\hat{P}_{n-2(k+1),k}$ for all n. It is clear that this recurrence relation holds for $n > 2k + 1$, and we need to check that it also holds for $n \leq 2k + 1$. The following table gives the values for the different cases:

case	n	parity	$\tilde{r}_{n,k}$	$\tilde{r}_{n-2,k}$	$\tilde{r}_{n-2k,k}$	$\hat{P}_{n-2(k+1),k}$
1	$\leq k$		0	0	0	0
2	$k+1$	(opp)	0	0	0	0
3	$k+2$	(same)	1	0	0	1/2
4	$k+2 < n \leq 2k$	same	1	1	0	0
5	$k+2 < n \leq 2k$	opp	0	0	0	0
6	$2k+1$	same	2	1	0	1/2
7	$2k+1$	opp	1	0	0	1/2

It now becomes clear why we made the definition $\hat{P}_{-1,k} = \hat{P}_{-k,k} = 1/2$. Note also that for cases 4 and 5, $-k < n - 2(k+1) \leq -2$. Multiplying each term in the recurrence relation by x^n, then summing over $n \geq -k$, we get

$$\begin{aligned}\sum_{n\geq -k} \tilde{r}_{n,k} \cdot x^n &= x^2 \sum_{n\geq -k} \tilde{r}_{n-2,k} \cdot x^{n-2} + x^{2k} \sum_{n\geq -k} \tilde{r}_{n-2k,k} \cdot x^{n-2k} \\ &\quad +2 \cdot x^{2(k+1)} \sum_{n\geq -k} \hat{P}_{n-2(k+1),k} \cdot x^{n-2(k+1)}.\end{aligned}$$

Since $\tilde{r}_{n,k} = 0$ for $n \leq 0$,

$$G_{\tilde{r}}(x)(1 - x^2 - x^{2k}) = 2 \cdot x^{2(k+1)} G_{\hat{P}}(x),$$

which after simplification gives the result. □

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