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HANKEL DETERMINANTS OF EISENSTEIN SERIES

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Abstract. In this paper we prove Garvan's conjectured formula for the square of the modular discriminant Δ as a 3 by 3 Hankel determinant of classical Eisenstein series E_{2n}. We then obtain similar formulas involving minors of Hankel determinants for $E_{2r}\Delta^m$, for $m = 1, 2, 3$ and $r = 2, 3, 4, 5, 7$, and $E_{14}\Delta^4$. We next use Mathematica to discover, and then the standard structure theory of the ring of modular forms, to derive the general form of our infinite family of formulas extending the classical formula for Δ and Garvan's formula for Δ^2. This general formula expresses the $n \times n$ Hankel determinant $\det(E_{2(i+j)}(q))_{1\le i,j\le n}$ as the product of $\Delta^{n-1}(\tau)$, a homogeneous polynomial in E_4^3 and E_6^2, and if needed, E_4. We also include a simple verification proof of the classical 2 by 2 Hankel determinant formula for Δ. This proof depends upon polynomial properties of elliptic function parameters from Jacobi's Fundamenta Nova. The modular forms approach provides a convenient explanation for the determinant identities in this paper.

1. Introduction

In this paper we prove Garvan's conjectured formula [11] for the square of the modular discriminant Δ as a 3 by 3 Hankel determinant of classical Eisenstein series E_{2n}. We then obtain similar formulas involving minors of Hankel determinants for $E_{2r}\Delta^m$, for $m = 1, 2, 3$ and $r = 2, 3, 4, 5, 7$, and $E_{14}\Delta^4$. We next use Mathematica [37] to discover, and then the modular forms approach of [31, pp. 88–93], as outlined in [5], to derive the general form of our infinite family of formulas extending the classical formula for Δ and Garvan's formula for Δ^2. This general formula expresses the $n \times n$ Hankel determinant $\det(E_{2(i+j)}(q))_{1\le i,j\le n}$ as the product of $\Delta^{n-1}(\tau)$, a homogeneous polynomial in E_4^3 and E_6^2, and if needed, E_4. We also include a simple verification proof of the classical formula for Δ in (1.5) below. This proof depends upon polynomial properties of elliptic function parameters from Jacobi's Fundamenta Nova [16]. The modular forms approach provides a convenient explanation for the determinant identities in this paper.

The modular discriminant Δ is defined in [2, Entry 12, pp. 326] and [28, Eqn. (6.1.11), pp. 196] by means of the following definition.

Definition 1.1. Let $q := \exp(2\pi i\tau)$, where τ is in the upper half-plane $\mathcal{H}$. We then have

$$\Delta(\tau) \equiv \Delta(q) := q\prod_{r=1}^{\infty}(1-q^r)^{24}. \tag{1.1}$$

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The Fourier expansions of the classical Eisenstein series $E_n(\tau)$ as given by [2, pp. 318] and [28, pp. 194–195] are determined by the following definition.

Definition 1.2. Let $q := \exp(2\pi i\tau)$, where τ is in the upper half-plane $\mathcal{H}$, and take $y := \mathrm{Im}(\tau) > 0$. Let $n = 1, 2, 3, \cdots$. We then have

$$E_2(\tau) \equiv E_2(q) := 1 - 24\sum_{r=1}^{\infty} \frac{rq^r}{1-q^r} - \frac{3}{\pi y}, \tag{1.2}$$

and for $n \geq 2$,

$$E_{2n}(\tau) \equiv E_{2n}(q) := 1 - \frac{4n}{B_{2n}} \sum_{r=1}^{\infty} \frac{r^{2n-1}q^r}{1-q^r}, \tag{1.3}$$

with the B_{2n} the Bernoulli numbers defined in [7, pp. 48–49] by

$$\frac{t}{e^t - 1} := \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad \text{for } |t| < 2\pi. \tag{1.4}$$

The fundamental classical formula for the modular discriminant Δ is provided by the following theorem.

Theorem 1.3. *Let $q := \exp(2\pi i\tau)$, where τ is in the upper half-plane $\mathcal{H}$. Let $\Delta(\tau)$ and $E_{2n} \equiv E_{2n}(q)$ be determined by* Definitions 1.1 *and* 1.2, *respectively. Then, for $|q| < 1$,*

$$\Delta(\tau) = \tfrac{1}{1728}(E_4^3 - E_6^2). \tag{1.5}$$

Early elliptic function references for (1.5) are [14], [15, pp. 561], [18, Eqns. (1) and (2), pp. 154], [25], and [26, pp. 27]. (Both Hurwitz and Molin replace q by q^2.) All of these authors contributions refer to earlier background developments in [8, 9, 29]. The chapter notes in [6, pp. 95, pp. 136–137] provide an excellent summary of Dedekind's fundamental [8, Eqn. (3), pp. 281; Eqn. (13), pp. 283; Eqn. (24), pp. 285], Dedekind's comments in [9] on Riemann's work in [29], and the remarks of Fricke [10] and Molin [26, pp. 28] on related methods of Jacobi and Hermite. The classical elliptic function methods for proving (1.5) are discussed in [6, pp. 58–72], [13, pp. 409, pp. 481], and [22, pp. 125–140, pp. 177]. An elementary proof of (1.5) was later found by Ramanujan in [27, Sections 5, 7 and 10]. This and additional work of Ramanujan involving (1.5) is surveyed in [3, pp. 114–140], [4, pp. 43–50], and [34, pp. 1–18]. Two additional elementary proofs of (1.5) are described in [31, pp. 95–96]. Recent references for (1.5) are [2, Entry 12(i), pp. 326], [6, Theorem 7, pp. 71], [28, Eqn. (6.1.14), pp. 197], [30, Eqn. (8), pp. 55; Theorem 8, pp. 70], and [31, Eqn (42), pp. 95]. (The work [19] is a very useful introduction to [30].) Additional applications of (1.5) also appear in [1, 28, 31].

After seeing [23, Theorems 2.1 and 2.2, pp. 15006] and an early version of [24, Theorems 1.5, 1.6, 5.3–5.6] Garvan [11] observed via the well-known relation $E_4^2 = E_8$ that (1.5) immediately becomes the Hankel determinant formula

$$\Delta(\tau) = \tfrac{1}{1728}(E_4E_8 - E_6^2) = \tfrac{1}{1728}\det\begin{vmatrix} E_4 & E_6 \\ E_6 & E_8 \end{vmatrix}. \tag{1.6}$$

He then conjectured the following theorem.

Theorem 1.4 (Garvan). *Let $q := \exp(2\pi i\tau)$, where τ is in the upper half-plane $\mathcal{H}$. Let $\Delta(\tau)$ and $E_{2n} \equiv E_{2n}(q)$ be determined by* Definitions 1.1 *and* 1.2, *respectively. Then, for $|q| < 1$,*

$$\Delta^2(\tau) = -\tfrac{691}{(1728)^2\cdot 250}\det\begin{vmatrix} E_4 & E_6 & E_8 \\ E_6 & E_8 & E_{10} \\ E_8 & E_{10} & E_{12} \end{vmatrix}. \tag{1.7}$$

Proof. Substitute the following three well-known relations from [27, Table I., pp. 141], [28, pp. 195] into the 3 by 3 Hankel determinant in (1.7).

$$E_8 = E_4^2, \qquad E_{10} = E_4E_6, \qquad E_{12} = \tfrac{441}{691}E_4^3 + \tfrac{250}{691}E_6^2. \tag{1.8}$$

Simplifying, factoring, and applying (1.5) then gives the $\Delta^2(\tau)$ on the left-hand-side of (1.7). □

For $\Delta^n(\tau)$ with $n > 2$, formulas analogous to (1.6) and (1.7) generally require a suitable $n+1$ by $n+1$ determinant on the right-hand-side and an additional polynomial factor in E_4^3 and E_6^2 on the left-hand-side. This extra polynomial factor can often be simplified by relations such as (1.8).

We organize the rest of our paper as follows. In Section 2 we first apply recursive methods to obtain our 19 determinental formulas expressing small powers of $\Delta(\tau)$, multiplied by a single Eisenstein series, as a suitable constant times a certain minor of a Hankel determinant of the E_{2r}. These formulas were motivated by Ramanujan's consideration of $E_{2r}\Delta$, for $r = 2, 3, 4, 5, 7$, in [27, Section 16], and the discussion of these $E_{2r}\Delta$ in [33, pp. 302]. The minors here were initially motivated by the n by n minors of the $n+1$ by $n+1$ Hankel determinants as discussed in [17, pp. 244–250]. We next use Mathematica [37] to discover, and then the modular forms approach of [31, pp. 88–93], as outlined in [5], to derive the general form of our infinite family of formulas extending (1.6) and (1.7) that involve $\Delta^{n-1}(\tau)$ and an n by n Hankel determinant of the E_{2r}.

In Section 3 we follow Jacobi's analysis in [16, Section 42] and utilize the Fourier series for the Jacobi elliptic function ns^2 to write down a formula for the Eisenstein series E_{2n}, for $n \geq 2$. We then apply [16, Eqn. (2.), Section 36] to put together a simple verification proof of the classical formula for Δ in (1.5).

Symmetry properties of the coefficients in the Maclaurin series expansion of ns^2 strongly suggest that formulas such as (3.8) and (3.9) in Theorem 3.1 will be useful in a further study of the determinental formulas in Section 2.

2. Additional determinental formulas involving powers of Δ

Our 19 determinental formulas in Theorem 2.3 involving small powers of $\Delta(\tau)$, and the infinite families of identities in Theorem 2.5 are partly motivated by the determinants in the following definition.

Definition 2.1. Let $\{c_\nu\}_{\nu=1}^{\infty}$ be a sequence in $\mathbb{C}^\times$, and let $m, n = 1, 2, 3, \cdots$. We take $H_n^{(1)}$ and $\chi_n^{(m)}$ to be the determinants of $n \times n$ square matrices

$$H_n^{(1)} \equiv H_n^{(1)}(\{c_\nu\}) := \det\begin{pmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \\ c_2 & c_3 & \cdots & c_n & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-2} & c_{2n-1} \end{pmatrix}, \tag{2.1}$$

$$\chi_n^{(m)} \equiv \chi_n^{(m)}(\{c_\nu\}) := \det\begin{pmatrix} c_1 & c_2 & \cdots & c_{n-m} & c_{n-m+2} & \cdots & c_{n+1} \\ c_2 & c_3 & \cdots & c_{n-m+1} & c_{n-m+3} & \cdots & c_{n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-m-1} & c_{2n-m+1} & \cdots & c_{2n} \end{pmatrix}. \tag{2.2}$$

The matrix for $\chi_n^{(m)}$ is obtained from the matrix for $H_{n+1}^{(1)}$ by deleting the $(n-m+1)$-st column and the last row. Others denote $\chi_n^{(1)}$ by χ_n. We also have $H_0^{(1)} = 1$ and $\chi_n^{(m)} = 0$, if $n < m$. Formally, $\chi_n^{(0)} = H_n^{(1)}$.

Applications of the Hankel determinants $H_n^{(1)}$ and determinants $\chi_n^{(m)}$ to continued fractions and orthogonal polynomials are discussed in [17, pp. 244–250]. An excellent survey of the literature on Hankel determinants can be found in Krattenthaler's summary in [20, pp. 20–23 ; pp. 46–48].

Our eventual aim is to study the combinatorial and geometrical implications of the determinants $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$, $\chi_n^{(m)}(\{E_{2(\nu+1)}(q)\})$, and additional minors, where $E_{2(\nu+1)}(q)$, with $\nu \geq 1$, are the Eisenstein series in Definition 1.2. A starting point is Theorems 2.3 and 2.5 below.

Motivated by our proof of Theorem 1.4 we first consider Ramanujan's [2, Entry 14, pp. 332] recursion for the E_{2r} given by the following theorem.

Theorem 2.2 (Ramanujan). *Let $q := \exp(2\pi i\tau)$, where τ is in the upper half-plane $\mathcal{H}$. Let $E_{2n}(\tau)$ be determined by* Definition 1.2. *For convenience, with $n > 1$, define S_{2n} by*

$$S_{2n} := (-1)^{n-1}\frac{B_{2n}}{4n}E_{2n}(\tau). \tag{2.3}$$

If n is an even integer exceeding 4 then

$$\begin{aligned}-\tfrac{(n+2)(n+3)}{2n(n-1)}S_{n+2} = &-20\binom{n-2}{2}S_4S_{n-2}\\ &+\sum_{r=1}^{[(n-2)/4]}{}' \binom{n-2}{2r}\Big\{(n+3-5r)(n-8-5r)\\ &\qquad -5(r-2)(r+3)\Big\}S_{2r+2}S_{n-2r},\end{aligned} \tag{2.4}$$

where the prime on the summation sign indicates that if $(n-2)/4$ is an integer, then the last term of the sum is to be multiplied by $\frac{1}{2}$.

Substituting $n = 6, 8, 10$ into (2.4) yields the relations in (1.8), and setting $n = 12$ in (2.4) leads to the well-known relation

$$E_{14} = E_4^2E_6. \tag{2.5}$$

We next utilize (1.5), Theorem 2.2, and Mathematica [37] to derive our determinental formulas for $E_{2r}\Delta^m$, for $m = 1, 2, 3$ and $r = 2, 3, 4, 5, 7$, and $E_{14}\Delta^4$. In each of the 19 identities below we first used (2.4) to write all the E_{2r} in the determinants on the right-hand-sides as polynomials in E_4 and E_6. Simplifying, factoring, applying (1.5), and then referring to (1.8) and (2.5) as needed yielded the left-hand-side of each identity. We have the following theorem.

Theorem 2.3. *Let $q := \exp(2\pi i\tau)$, where τ is in the upper half-plane $\mathcal{H}$. Let $\Delta(\tau)$ and $E_{2n} \equiv E_{2n}(q)$ be determined by* Definitions 1.1 *and* 1.2, *respectively. Then, for $|q| < 1$,*

$$E_4\Delta(\tau) = -\tfrac{691}{1728\cdot 250}\det\begin{vmatrix} E_4 & E_8\\ E_8 & E_{12}\end{vmatrix}, \tag{2.6}$$

$$E_6\Delta(\tau) = \tfrac{691}{1728\cdot 250}\det\begin{vmatrix} E_4 & E_{12}\\ E_6 & E_{14}\end{vmatrix}, \tag{2.7}$$

$$E_8\Delta(\tau) = \tfrac{3617}{1728\cdot 3\cdot 7^2\cdot 11} \det\begin{vmatrix} E_4 & E_{10} \\ E_{10} & E_{16} \end{vmatrix} \tag{2.8}$$

$$= \tfrac{691}{1728\cdot 441} \det\begin{vmatrix} E_8 & E_{10} \\ E_{10} & E_{12} \end{vmatrix}, \tag{2.9}$$

$$E_{10}\Delta(\tau) = \tfrac{691\cdot 43867}{1728\cdot 2\cdot 3^2\cdot 5^4\cdot 7^2\cdot 13} \det\begin{vmatrix} E_4 & E_{12} \\ E_{10} & E_{18} \end{vmatrix} \tag{2.10}$$

$$= \tfrac{691}{1728\cdot 250} \det\begin{vmatrix} E_8 & E_{12} \\ E_{10} & E_{14} \end{vmatrix}, \tag{2.11}$$

$$E_{14}\Delta(\tau) = \tfrac{691\cdot 593\cdot 131}{1728\cdot 2\cdot 3\cdot 5^3\cdot 7^2\cdot 11\cdot 13} \det\begin{vmatrix} E_4 & E_{14} \\ E_{12} & E_{22} \end{vmatrix} \tag{2.12}$$

$$= -\tfrac{691\cdot 3617}{1728\cdot 2\cdot 3\cdot 5^3\cdot 7^2\cdot 13} \det\begin{vmatrix} E_{10} & E_{14} \\ E_{12} & E_{16} \end{vmatrix}, \tag{2.13}$$

$$E_4\Delta^2(\tau) = \tfrac{691^2}{(1728)^2\cdot (21)^2\cdot 250} \det\begin{vmatrix} E_4 & E_8 & E_{10} \\ E_6 & E_{10} & E_{12} \\ E_8 & E_{12} & E_{14} \end{vmatrix}, \tag{2.14}$$

$$E_6\Delta^2(\tau) = -\tfrac{691^2}{(1728)^2\cdot (250)^2} \det\begin{vmatrix} E_6 & E_8 & E_{10} \\ E_8 & E_{10} & E_{12} \\ E_{10} & E_{12} & E_{14} \end{vmatrix}, \tag{2.15}$$

$$E_8\Delta^2(\tau) = -\tfrac{(691)^2\cdot 3617}{(1728)^2\cdot 2^3\cdot 3\cdot 5^3\cdot 7^2\cdot 467} \det\begin{vmatrix} E_6 & E_8 & E_{12} \\ E_8 & E_{10} & E_{14} \\ E_{10} & E_{12} & E_{16} \end{vmatrix}, \tag{2.16}$$

$$E_{10}\Delta^2(\tau) = \tfrac{(691)^2\cdot 3617}{(1728)^2\cdot 2^2\cdot 3\cdot 5^6\cdot 7^2\cdot 13} \det\begin{vmatrix} E_6 & E_{10} & E_{12} \\ E_8 & E_{12} & E_{14} \\ E_{10} & E_{14} & E_{16} \end{vmatrix}, \tag{2.17}$$

$$E_{14}\Delta^2(\tau) = -\frac{(691)^2\cdot 3617\cdot 43867}{(1728)^2\cdot 2^6\cdot 3\cdot 5^3\cdot 7^2\cdot 97\cdot 7213}\det\begin{vmatrix} E_8 & E_{10} & E_{14} \\ E_{10} & E_{12} & E_{16} \\ E_{12} & E_{14} & E_{18} \end{vmatrix}, \tag{2.18}$$

$$E_{4}\Delta^3(\tau) = -\frac{(691)^3\cdot 3617}{(1728)^3\cdot 2^4\cdot 3\cdot 5^6\cdot 7^2\cdot 467}\det\begin{vmatrix} E_4 & E_6 & E_8 & E_{10} \\ E_6 & E_8 & E_{10} & E_{12} \\ E_8 & E_{10} & E_{12} & E_{14} \\ E_{10} & E_{12} & E_{14} & E_{16} \end{vmatrix}, \tag{2.19}$$

$$E_{6}\Delta^3(\tau) = -\frac{(691)^3\cdot 43867}{(1728)^3\cdot 2^5\cdot 3^2\cdot 5^6\cdot 7^2\cdot 131}\det\begin{vmatrix} E_4 & E_6 & E_8 & E_{12} \\ E_6 & E_8 & E_{10} & E_{14} \\ E_8 & E_{10} & E_{12} & E_{16} \\ E_{10} & E_{12} & E_{14} & E_{18} \end{vmatrix}, \tag{2.20}$$

$$E_{8}\Delta^3(\tau) = \frac{(691)^3\cdot (3617)^2}{(1728)^3\cdot 2^4\cdot 3^2\cdot 5^6\cdot 7^4\cdot 13\cdot 467}\det\begin{vmatrix} E_4 & E_6 & E_{10} & E_{12} \\ E_6 & E_8 & E_{12} & E_{14} \\ E_8 & E_{10} & E_{14} & E_{16} \\ E_{10} & E_{12} & E_{16} & E_{18} \end{vmatrix}, \tag{2.21}$$

$$E_{10}\Delta^3(\tau) = -\frac{(691)^3\cdot 3617\cdot 43867}{(1728)^3\cdot 2^7\cdot 3\cdot 5^6\cdot 7^2\cdot 97\cdot 7213}\det\begin{vmatrix} E_4 & E_8 & E_{10} & E_{12} \\ E_6 & E_{10} & E_{12} & E_{14} \\ E_8 & E_{12} & E_{14} & E_{16} \\ E_{10} & E_{14} & E_{16} & E_{18} \end{vmatrix}, \tag{2.22}$$

$$E_{14}\Delta^3(\tau) = \frac{(691)^3\cdot (3617)^2\cdot 43867\cdot 283\cdot 617}{(1728)^4\cdot 2^3\cdot 3^2\cdot 5^6\cdot 7^2\cdot 31\cdot 3503110621}\det\begin{vmatrix} E_6 & E_8 & E_{10} & E_{14} \\ E_8 & E_{10} & E_{12} & E_{16} \\ E_{10} & E_{12} & E_{14} & E_{18} \\ E_{12} & E_{14} & E_{16} & E_{20} \end{vmatrix}, \tag{2.23}$$

$$E_{14}\Delta^4(\tau)=\tfrac{(691)^4\cdot(3617)^2\cdot 43867\cdot 131\cdot 283\cdot 593\cdot 617}{(1728)^6\cdot 5^9\cdot 7^5\cdot 11\cdot 13\cdot 67\cdot 257\cdot 43721}\det\begin{vmatrix} E_4 & E_6 & E_8 & E_{10} & E_{14}\\ E_6 & E_8 & E_{10} & E_{12} & E_{16}\\ E_8 & E_{10} & E_{12} & E_{14} & E_{18}\\ E_{10} & E_{12} & E_{14} & E_{16} & E_{20}\\ E_{12} & E_{14} & E_{16} & E_{18} & E_{22}\end{vmatrix}. \tag{2.24}$$

The determinants in (1.6), (1.7), (2.14), (2.15), (2.19)–(2.22), (2.24) are of the form $H_2^{(1)}$, $H_3^{(1)}$, $\chi_3^{(2)}$, $\chi_3^{(3)}$, $H_4^{(1)}$, χ_4, $\chi_4^{(2)}$, $\chi_4^{(3)}$, and χ_5, respectively, with entries $c_\nu = E_{2(\nu+1)}(q)$. The rest are certain other minors of $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$, for suitable n.

The determinant evaluations in (2.6)–(2.24) can also be proven, as pointed out in [5], by the methods in the modular forms proof of Theorem 2.5 below. Being able to apply the first part of this proof leads to a characterization of a large class of determinants of Eisenstein series which have evaluations analogous to those in (2.6)–(2.24) and Theorem 2.5. It turns out we only have to consider minors of $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$, for suitable n. We have the following proposition.

Proposition 2.4. *Let A be any $n\times n$ square matrix whose entries are Eisenstein series E_{2r}, with $r\geq 2$. Suppose there are no repeated rows or columns. Recall that the weight of a product $E_{2r_1}E_{2r_2}\cdots E_{2r_n}$ of Eisenstein series is $2(r_1+r_2+\cdots+r_n)$. Then, each term in the $n\times n$ determinant $\det A$ has the same weight if and only if $\det A$ is ± 1 times some $n\times n$ minor of a Hankel determinant $H_m^{(1)}(\{E_{2(\nu+1)}(q)\})\equiv \det(E_{2(i+j)}(q))_{1\leq i,j\leq m}$ of Eisenstein series, with $m\geq n$.*

Proof. First, assume that $\det A$ is some $n\times n$ minor of $\det(E_{2(i+j)}(q))_{1\leq i,j\leq m}$, with $m\geq n$. Let the rows and columns of A be indexed by $\{i_1,\dots i_n\}$ and $\{j_1,\dots j_n\}$, respectively. The weight of the term corresponding to σ in

$$\det A=\sum_{\sigma\in\mathcal{S}_n}\operatorname{sign}(\sigma)\prod_{r=1}^{n}E_{2(i_r+j_{\sigma(r)})}(q), \tag{2.25}$$

is $2((i_1+\cdots i_n)+(j_{\sigma(1)}+\cdots j_{\sigma(n)}))=2((i_1+\cdots i_n)+(j_1+\cdots j_n))$, which is a constant.

Next, for an even more general argument, suppose that each term in the $n\times n$ determinant

$$\det A\equiv\det(E_{p_{i,j}}(q))_{1\leq i,j\leq n}=\sum_{\sigma\in\mathcal{S}_n}\operatorname{sign}(\sigma)\prod_{r=1}^{n}E_{p_{r,\sigma(r)}}(q), \tag{2.26}$$

has the same weight, where the n^2 subscripts $p_{i,j}$ are now arbitrary reals. In this setting we take the weight of the term in (2.26) corresponding to σ to be

$$p_{1,\sigma(1)}+p_{2,\sigma(2)}+\cdots+p_{n,\sigma(n)}. \tag{2.27}$$

We then claim that

$$p_{i,j}-p_{i,j-1}=p_{1,j}-p_{1,j-1},\quad \text{for } i=1,2,\cdots,n \quad\text{and } j=2,3,\cdots,n, \tag{2.28}$$

and

$$p_{i,j}-p_{i-1,j}=p_{i,n}-p_{i-1,n},\quad \text{for } i=2,3,\cdots,n \quad\text{and } j=1,2,\cdots,n. \tag{2.29}$$

To obtain (2.28) and (2.29) first consider the 2×2 submatrix

$$\begin{vmatrix} E_{p_{i-1,j-1}}(q) & E_{p_{i-1,j}}(q) \\ E_{p_{i,j-1}}(q) & E_{p_{i,j}}(q) \end{vmatrix}, \tag{2.30}$$

where $i, j = 2, 3, \cdots, n$. Keeping in mind (2.30), there are at least two terms in (2.26) of the form $B \cdot E_{p_{i-1,j-1}}(q)E_{p_{i,j}}(q)$ and $-B \cdot E_{p_{i,j-1}}(q)E_{p_{i-1,j}}(q)$. Equating the weights of these two terms and simplifying, gives

$$p_{i-1,j-1} + p_{i,j} = p_{i,j-1} + p_{i-1,j}, \quad \text{for } i, j = 2, 3, \cdots, n. \tag{2.31}$$

By rewriting (2.31) in two ways, we have

$$p_{i,j} - p_{i,j-1} = p_{i-1,j} - p_{i-1,j-1}, \quad \text{for } i, j = 2, 3, \cdots, n, \tag{2.32}$$

and

$$p_{i,j} - p_{i-1,j} = p_{i,j-1} - p_{i-1,j-1}, \quad \text{for } i, j = 2, 3, \cdots, n. \tag{2.33}$$

Equation (2.28) is immediate from (2.32) by fixing j and varying i, while (2.29) is immediate from (2.33) by fixing i and varying j.

By permuting the columns and then the rows of the matrix A in (2.26), and factoring out a -1 if necessary, we can assume that the differences in the right-hand-sides of (2.28) and (2.29) are strictly positive.

It is now clear that if we take the n^2 subscripts $p_{i,j}$ to be even integers greater than 2, then $\det A$ is ± 1 times some $n \times n$ minor of a Hankel determinant $\det(E_{2(i+j)}(q))_{1 \leq i,j \leq m}$ of Eisenstein series, with $m \geq n$. $\square$

The simplest application of Proposition 2.4 involves applying the modular forms approach of [31, pp. 88–93], as outlined in [5], to express the $n \times n$ Hankel determinant $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$ as the product of $\Delta^{n-1}(\tau)$, a homogeneous polynomial in E_4^3 and E_6^2, and if needed, E_4. One of our original motivations for studying the determinants $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$ was to note that

$$E_4 = H_1^{(1)}(\{E_{2(\nu+1)}(q)\}) = \det |E_4|, \tag{2.34}$$

and recall the $n = 2, 3, 4$ cases in (1.6), (1.7), and (2.19), respectively. Since E_4 only appears as a factor in the left-hand-sides of (2.34) and (2.19), it was natural to split n up into the classes (mod 3) given by $n = 3r + 1$, $3r + 2$, and $3r + 3$, with $r = 0, 1, 2, \cdots$.

Mathematica [37] computations of $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$ up to $n = 10$, analogous to those of (1.7) and (2.19), first led to the discovery of the general form of the following three infinite families of formulas in (2.35)–(2.37) below. We prove the evaluations in (2.35)–(2.37) by appealing to the standard structure theory of the ring of modular forms in [31, pp. 88–93]. The proof here is a detailed rewriting of the original proof supplied by Borcherds in [5]. We have the following theorem.

Theorem 2.5. *Let* $q := \exp(2\pi i\tau)$, *where* τ *is in the upper half-plane* $\mathcal{H}$. *Let* $\Delta(\tau)$ *and* $E_{2m} \equiv E_{2m}(q)$ *be determined by* Definitions 1.1 *and* 1.2, *respectively. Then, for* $|q| < 1$,

$$E_4\Delta^{3r}(\tau) \cdot P_{3r(r-1)/2}(E_4^3, E_6^2) = d_r H_{3r+1}^{(1)}(\{E_{2(\nu+1)}(q)\}), \quad \textit{for} \quad r = 0, 1, 2, \cdots, \tag{2.35}$$

$$\Delta^{3r+1}(\tau) \cdot Q_{r(3r-1)/2}(E_4^3, E_6^2) = e_r H_{3r+2}^{(1)}(\{E_{2(\nu+1)}(q)\}), \quad \textit{for} \quad r = 0, 1, 2, \cdots, \tag{2.36}$$

$$\Delta^{3r+2}(\tau) \cdot R_{r(3r+1)/2}(E_4^3, E_6^2) = \ f_r H^{(1)}_{3r+3}(\{E_{2(\nu+1)}(q)\}), \quad \textit{for} \quad r = 0, 1, 2, \cdots, \tag{2.37}$$

where d_r, e_r, and f_r are constants depending on r, and $P_n(x, y)$, $Q_n(x, y)$, and $R_n(x, y)$ are homogeneous polynomials in x and y of total degree n (as given above), with integer coefficients, whose monomials are those in $(x - y)^n$.

Proof. Let $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$ denote any of the Hankel determinants in (2.35)–(2.37). Keeping in mind that $E_{2r}(q)$ has weight $2r$, it is immediate from Proposition 2.4 that each term in the Hankel determinant $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$ is an entire modular form of fixed weight $2n(n + 1)$, as is the entire determinant.

Each Eisenstein series $E_{2r}(q)$ in (1.3) is written in [31, Eqn. (34), pp. 92] as a Maclaurin series in q starting with the terms $1 + a_1 q$, with $a_1 \neq 0$. Thus, subtracting the first row from each of the others, factoring q out of each of the resulting $n - 1$ lower rows, and recalling $q := \exp(2\pi i\tau)$, we find that $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$ vanishes to order $n - 1$ at the cusp $\tau = i\infty$.

The Maclaurin series in q for $\Delta(q)$ in (1.1) starts with the term q. The function $\Delta(q) \equiv \Delta(\tau)$ is also a cusp form of weight 12 which vanishes at $\tau = i\infty$. It follows from [31, Theorem 4 (iii), pp. 88] that $\Delta^{n-1}(\tau)$ divides $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$, and that the quotient is a holomorphic modular form of weight $2n(n + 1) - 12(n - 1) = 2(n - 2)(n - 3)$.

By Corollary 2 of [31, pp. 89] the quotient $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})/\Delta^{n-1}(\tau)$ is a linear combination of monomials

$$E_4^{\alpha} E_6^{\beta}, \tag{2.38}$$

with fixed weight

$$4\alpha + 6\beta = 2(n - 2)(n - 3), \tag{2.39}$$

where α and β are nonnegative integers. To obtain the left-hand-sides of (2.35)–(2.37) we next utilize the weight $2(n - 2)(n - 3) \bmod 12$ to simplify (2.38).

First, as in [31, pp. 90], let $4\alpha + 6\beta = 12m$, with m a nonnegative integer. This gives $2\alpha + 3\beta = 6m$, which when divided by 2 and 3, respectively, implies that $\beta/2 = b$ and $\alpha/3 = a$ are nonnegative integers. This gives

$$E_4^{\alpha} E_6^{\beta} = E_4^{3a} E_6^{2b}, \tag{2.40}$$

which in turn leads to (2.36) and (2.37) where $n = 3r + 2$ and $3r + 3$, respectively. The total degree of the polynomials Q and R in (2.36) and (2.37) is immediate from (2.39) by computing

$$(4\alpha + 6\beta)/12 = a + b = (n - 2)(n - 3)/6, \tag{2.41}$$

for $n = 3r + 2$ and $3r + 3$, respectively.

Next, to obtain (2.35), let $4\alpha + 6\beta = 12m + 4$. This gives $2\alpha + 3\beta = 6m + 2$. We must have $\alpha \geq 1$ or else β is not an integer. Dividing by 2 implies that $\beta/2 = b$ is a nonnegative integer. Setting $\beta = 2b$ and solving for α gives $\alpha = 3(m - b) + 1 := 3c + 1$, with c a nonnegative integer. We now have

$$E_4^{\alpha} E_6^{\beta} = E_4 \cdot E_4^{3c} E_6^{2b}, \tag{2.42}$$

which in turn leads to (2.35) when $n = 3r + 1$. In this case we have

$$(4\alpha + 6\beta - 4)/12 = c + b = [(n - 2)(n - 3) - 2]/6 \tag{2.43}$$

which equals $3r(r - 1)/2$. □

It is an interesting open problem to find a concise combinatorial and/or analytical description of the coefficients in the polynomials $P_{3r(r-1)/2}$, $Q_{r(3r-1)/2}$, and $R_{r(3r+1)/2}$.

The $r = 0$ cases of (2.35), (2.36), and (2.37) are (2.34), (1.6), and (1.7), respectively. The first nontrivial case of (2.35) is (2.19). The degree of $P_{3r(r-1)/2}$ in (2.35) is $3\binom{r}{2}$, while the degrees of $Q_{r(3r-1)/2}$ and $R_{r(3r+1)/2}$ in (2.36) and (2.37), respectively, are the pentagonal numbers $r(3r \mp 1)/2$ in [32, sequence **M1336**]. See also [6, pp. 124].

Given (2.14), (2.15), (2.20)–(2.22), (2.24), and the modular forms proof of Theorem 2.5, we can obtain more complicated infinite families of identities analogous to (2.35)–(2.37) for $\chi_n^{(m)}(\{E_{2(\nu+1)}(q)\})$.

Borcherds also observed in [5] that the method of proof of Theorem 2.5 also establishes the determinental identities in Theorem 2.3, up to some constant. The space of modular forms of the appropriate weight happens to have dimension 1, and is thus spanned by an Eisenstein series. The argument through equation (2.39) is the same, with the right-hand-side of (2.39) replaced by the sum of the subscripts of the Eisenstein series in the diagonal entries of the given $n \times n$ matrix, minus $12(n-1)$. Call this expression W_1. We then look at W_1 mod 12, as before. We find that any holomorphic modular form of weight 0,2,4,6,8, or 10 mod 12 is equal to 1, E_{14}, E_4, E_6, E_8, or E_{10} times a homogeneous polynomial in E_4^3 and E_6^2. In the case of Theorem 2.3 we have $W_1 =$ 4,6,8,10,14. These weights correspond to the right spaces of dimension 1 in our list mod 12, the homogeneous polynomial is a constant, and we are done. We have to be careful in the case of $W_1 = 14$. Here, we start with $W_1 = 4\alpha + 6\beta = 12m + 2$, for $m \geq 1$. We end up factoring out $E_4^2 E_6$, with the homogeneous polynomial having total degree $m - 1$. The other cases are simpler, use $m \geq 0$, and the homogeneous polynomials all have total degree m.

The above modular forms proofs of Theorems 2.3 and 2.5, combined with the characterization in Proposition 2.4, lead to only a small number of determinental identities in which the space of modular forms of the appropriate weight W_1 has dimension 1. In particular, as soon as $W_1 \geq 16$, the dimension is ≥ 2, and the homogeneous polynomial in E_4^3 and E_6^2 is no longer a constant. All such identities analogous to those in Theorems 1.3, 1.4, and 2.3 are determined by requiring $W_1 \leq 14$. It is not hard to see that this is not possible if $n \geq 6$. The remaining finite number of possible cases for $n \leq 5$ can be checked directly. The identities in Theorems 1.3, 1.4, and 2.3 cover the basic types that are possible. Moreover, the identities in (1.6) and (1.7) are the only ones involving just a power of Δ, and the identity in (2.24) is unique up to transposition symmetry of the 5×5 determinant. There are no other analogous 5×5 determinental identities whose left-hand-side is of the form $E_{2r}\Delta^4(\tau)$. Thus, (2.24) should be very interesting.

Keeping in mind [35, Eqn. (52.6), pp. 201] it is natural to consider the Hankel determinants $H_n^{(1)}(\{E_{2(\nu+1)}(q)\})$ in which entries $E_{2(\nu+1)}(q)$ are replaced by 0 unless $2(\nu+1)$ satisfies any of a fixed set of congruence conditions. (The "unless" can also be replaced by "whenever"). That is, when all entries in certain of the counter diagonals are 0. For example, the condition $E_{2(\nu+1)}(q) \mapsto 0$ unless $2(\nu+1) \equiv 0 \pmod 6$, leads to interesting determinant evaluations. Similarly, the condition $E_{2(\nu+1)}(q) \mapsto 0$ whenever $2(\nu+1) \equiv 0 \pmod 4$ or $2(\nu+1) \equiv 0 \pmod 6$ leads to reasonable determinants. This is just a small sample of many such possibilities.

3. The Jacobi elliptic function ns^2 and Eisenstein series

In this section we follow Jacobi's analysis in [16, Section 42] and utilize the Fourier series for the Jacobi elliptic function ns^2 to write down a formula for the Eisenstein series E_{2n}, for $n \geq 2$. We then apply [16, Eqn. (2.), Section 36] to put together a simple verification proof of the classical formula for Δ in (1.5).

We require the Jacobi elliptic function parameter

$$z := {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix} \;\middle|\; k^2\right] = 2\boldsymbol{K}(k)/\pi \equiv 2\boldsymbol{K}/\pi, \tag{3.1}$$

with

$$\boldsymbol{K}(k) \equiv \boldsymbol{K} := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \tfrac{\pi}{2}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix} \;\middle|\; k^2\right] \tag{3.2}$$

the complete elliptic integral of the first kind in [21, Eqn. (3.1.3), pp. 51], and k the modulus. We also need the complete elliptic integral of the second kind

$$\boldsymbol{E}(k) \equiv \boldsymbol{E} := \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt = \tfrac{\pi}{2}\ {}_2F_1\left[\begin{matrix}\frac{1}{2}, -\frac{1}{2} \\ 1\end{matrix} \,\middle|\, k^2\right]. \tag{3.3}$$

Finally, we take

$$q := exp(-\pi\boldsymbol{K}(\sqrt{1-k^2})/\boldsymbol{K}(k)) \tag{3.4}$$

The classical Fourier expansion for ns^2, which first appeared in [16, Eqn. (2.), Section 42; Eqn. IV., Section 44], is now given by

$$\mathrm{ns}^2(u,k) = 1 - \frac{\boldsymbol{E}}{\boldsymbol{K}} + \frac{1}{z^2}\csc^2\tfrac{u}{z} - \frac{8}{z^2}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos\tfrac{2nu}{z}. \tag{3.5}$$

More recent references for (3.5) include [12, Eqn. 27, pp. 913] and [36, Ex. 57, pp. 535].

The Fourier expansion in (3.5) may be written as a double sum by first expanding the $\cos\frac{2nu}{z}$ as a Maclaurin series, interchanging summation, and then simplifying. Next, the Laurent series expansion for $\csc^2\frac{u}{z}$ is immediate from differentiating the Laurent series for $\cot\theta$ in [7, Ex. 36, pp. 88]. This analysis yields

$$\begin{aligned}\mathrm{ns}^2(u,k) = \frac{1}{u^2} + 1 - \frac{\boldsymbol{E}}{\boldsymbol{K}} + \sum_{m=1}^{\infty} B_{2m}\frac{(-1)^{m+1}2^{2m-1}}{m\cdot z^{2m}}\frac{u^{2m-2}}{(2m-2)!} \\ - 4\sum_{m=1}^{\infty}\frac{(-1)^{m-1}2^{2m-1}}{z^{2m+2}}\left[\sum_{r=1}^{\infty}\frac{r^{2m-1}q^{2r}}{1-q^{2r}}\right]\frac{u^{2m-2}}{(2m-2)!},\end{aligned} \tag{3.6}$$

with B_{2m} the Bernoulli numbers defined by (1.4).

The Laurent series expansion for $\mathrm{ns}^2(u,k)$ is

$$\mathrm{ns}^2(u,k) = \frac{1}{u^2} + \sum_{m=1}^{\infty}(ns^2)_m(k^2)\frac{u^{2m-2}}{(2m-2)!}, \tag{3.7}$$

where $(ns^2)_m(k^2)$ are polynomials in k^2, with k the modulus.

In what follows we equate the q's in (3.4) and Definition 1.2. That is $2\tau = i\boldsymbol{K}(\sqrt{1-k^2})/\boldsymbol{K}(k)$.

Keeping in mind Definition 1.2, we find that equating coefficients of u^{2m-2}, for $m \geq 2$, in (3.6) and (3.7) leads to a formula for $E_{2m}(q^2)$. Furthermore, applying the Gauß transformation ($q \mapsto \sqrt{q}$, $k \mapsto \frac{2\sqrt{k}}{1+k}$, $\boldsymbol{K} \mapsto (1+k)\boldsymbol{K}$, $z \mapsto (1+k)z$) from [16, Theorem III, Section 37] to the first formula yields a corresponding formula for $E_{2m}(q)$. We have the following theorem.

Theorem 3.1. *Let $z := 2\boldsymbol{K}(k)/\pi \equiv 2\boldsymbol{K}/\pi$, as in (3.1), with k the modulus. Let the Bernoulli numbers B_{2m} be defined by (1.4). Take $(ns^2)_m(k^2)$ to be the elliptic function polynomials of k^2 determined by (3.7). Let q be as in (3.4). Take $E_{2m}(q)$ as in* Definition 1.2*, with $2\tau = i\boldsymbol{K}(\sqrt{1-k^2})/\boldsymbol{K}(k)$. Let $m = 2, 3, 4, \cdots$. We then have*

$$E_{2m}(q^2) \equiv E_{2m}(2\tau) = 1 - z^2 + \frac{(-1)^{m-1}m\cdot z^{2m+2}}{2^{2m-1}\cdot B_{2m}}\cdot(ns^2)_m(k^2), \tag{3.8}$$

$$E_{2m}(q) \equiv E_{2m}(\tau) = 1 - (1+k)^2z^2$$

$$+\frac{(-1)^{m-1}m\cdot z^{2m+2}(1+k)^{2m+2}}{2^{2m-1}\cdot B_{2m}}\cdot(ns^2)_m(4k/(1+k)^2). \qquad (3.9)$$

The $m=2$ and 3 cases of (3.8) are given by

$$E_4(q^2)=z^4(1-k^2+k^4), \qquad (3.10)$$

$$E_6(q^2)=z^6(1+k^2)(1-2k^2)(1-\tfrac{1}{2}k^2). \qquad (3.11)$$

Equations (3.10) and (3.11) appear in [3, Entry 13(i),(ii), pp. 126]. Equation (3.10) is also recorded in [4, Eqn. (12.21), pp. 49], and just under equation (6.) of [16, Section 42]. Jacobi's derivation of his 8 squares formula in Section 42 of [16] only required him to go as far as $E_4(q^2)$.

Our verification proof of (1.5) is a consequence of (3.10), (3.11), and equation (2.) of [16, Section 36] written in our notation as

$$\Delta^{1/4}(2\tau)\equiv\Delta^{1/4}(q^2)=\tfrac{1}{2^2}z^3k(1-k^2)^{1/2}. \qquad (3.12)$$

The fourth power of (3.12) immediately gives

$$\Delta(2\tau)\equiv\Delta(q^2)=\tfrac{1}{2^8}z^{12}(1-k^2)^2k^4. \qquad (3.13)$$

Equation (3.13) also appears in [3, Entry 12(iii), pp. 124] and [4, Theorem 8.3(iii), pp. 29].

Substituting (3.10) and (3.11) into the right-hand-side of (1.5), with q replaced by q^2, simplifying, and obtaining the right-hand-side of (3.13) now completes the proof of (1.5). The z^{12} factored out quickly and reduced the proof to a computation involving polynomials of low degree in k^2.

Our simple verification proof of (1.5) does not seem to have been written down in the literature before.

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