GENERALISED PATTERN AVOIDANCE

ANDERS CLAESSON

Abstract. Recently, Babson and Steingrímsson have introduced generalised permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. We consider pattern avoidance for such patterns, and give a complete solution for the number of permutations avoiding any single pattern of length three with exactly one adjacent pair of letters. We also give some results for the number of permutations avoiding two different patterns. Relations are exhibited to several well studied combinatorial structures, such as set partitions, Dyck paths, Motzkin paths, and involutions. Furthermore, a new class of set partitions, called monotone partitions, is defined and shown to be in one-to-one correspondence with non-overlapping partitions.

1. Introduction

In the last decade a wealth of articles has been written on the subject of pattern avoidance, also known as the study of "restricted permutations" and "permutations with forbidden subsequences". Classically, a pattern is a permutation $\sigma \in \mathcal{S}_k$, and a permutation $\pi \in \mathcal{S}_n$ avoids σ if there is no subsequence in π whose letters are in the same relative order as the letters of σ. For example, $\pi \in \mathcal{S}_n$ avoids 132 if there is no $1 \leq i \leq j \leq k \leq n$ such that $\pi(i) \leq \pi(k) \leq \pi(j)$. In [6] Knuth established that for all $\sigma \in \mathcal{S}_3$, the number of permutations in $\mathcal{S}_n$ avoiding σ equals the nth Catalan number, $C_n = \frac{1}{1+n}\binom{2n}{n}$. One may also consider permutations that are required to avoid several patterns. In [7] Simion and Schmidt gave a complete solution for permutations avoiding any set of patterns of length three. Even patterns of length greater than three have been considered. For instance, West showed in [10] that permutations avoiding both 3142 and 2413 are enumerated by the Shröder numbers, $S_n = \sum_{i=0}^{n}\binom{2n-i}{i}C_{n-i}$.

In [1] Babson and Steingrímsson introduced generalised permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. The motivation for Babson and Steingrímsson in introducing these patterns was the study of Mahonian statistics, and they showed that essentially all Mahonian permutation statistics in the literature can be written as linear combinations of such patterns. An example of a generalised pattern is $(a\text{–}cb)$. An $(a\text{–}cb)$-subword of a permutation $\pi = a_1a_2\cdots a_n$ is a subword $a_ia_ja_{j+1}$, $(i < j)$, such that $a_i < a_{j+1} < a_j$. More generally, a pattern p is a word over the alphabet $a < b < c < d \cdots$ where two adjacent letters may or may not be separated by a dash. The absence of a dash between two adjacent letters in a p indicates that the corresponding letters in a p-subword of a permutation must be adjacent. Also,

Date: July 24, 2006.

the ordering of the letters in the p-subword must match the ordering of the letters in the pattern. This definition, as well as any other definition in the introduction, will be stated rigorously in Section 2. All classical patterns are generalised patterns where each pair of adjacent letters is separated by a dash. For example, the generalised pattern equivalent to 132 is $(a\text{–}c\text{–}b)$.

We extend the notion of pattern avoidance by defining that a permutation avoids a (generalised) pattern p if it does not contain any p-subwords. We show that this is a fruitful extension, by establishing connections to other well known combinatorial structures, not previously shown to be related to pattern avoidance. The main results are given below.

P	$\lvert\mathcal{S}_n(P)\rvert$	*Description*
$a\text{–}bc$	B_n	Partitions of $[n]$
$a\text{–}cb$	B_n	Partitions of $[n]$
$b\text{–}ac$	C_n	Dyck paths of length $2n$
$a\text{–}bc$, $ab\text{–}c$	B_n^*	Non-overlapping partitions of $[n]$
$a\text{–}bc$, $a\text{–}cb$	I_n	Involutions in $\mathcal{S}_n$
$a\text{–}bc$, $ac\text{–}b$	M_n	Motzkin paths of length n

Here $\mathcal{S}_n(P) = \{\pi \in \mathcal{S}_n : \pi \text{ avoids } p \text{ for all } p \in P\}$, and $[n] = \{1, 2, \dots, n\}$. When proving that $\lvert\mathcal{S}_n(a\text{–}bc, ab\text{–}c)\rvert = B_n^*$ (the nth Bessel number), we first prove that there is a one-to-one correspondence between $\{a\text{–}bc, ab\text{–}c\}$-avoiding permutations and *monotone partitions*. A partition is monotone if its non-singleton blocks can be written in increasing order of their least element and increasing order of their greatest element, simultaneously. This new class of partitions is then shown to be in one-to-one correspondence with non-overlapping partitions.

2. Preliminaries

By an *alphabet* X we mean a non-empty set. An element of X is called a *letter*. A *word* over X is a finite sequence of letters from X. We consider also the *empty word*, that is, the word with no letters; it is denoted by ϵ. Let $x = x_1x_2\cdots x_n$ be a word over X. We call $|x| := n$ the *length* of x. A *subword* of x is a word $v = x_{i_1}x_{i_2}\cdots x_{i_k}$, where $1 \le i_1 < i_2 < \cdots < i_k \le n$. A *segment* of x is a word $v = x_ix_{i+1}\cdots x_{i+k}$. If X and Y are two linearly ordered alphabets, then two words $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ over X and Y, respectively, are said to be *order equivalent* if $x_i < x_j$ precisely when $y_i < y_j$.

Let $X = A \cup \{\text{–}\}$ where A is a linearly ordered alphabet. For each word x let $\bar{x}$ be the word obtained from x by deleting all dashes in x. A word p over X is called a *pattern* if it contains no two consecutive dashes and $\bar{p}$ has no repeated letters. By slight abuse of terminology we refer to the *length of a pattern* p as the length of $\bar{p}$. Two patterns p and q of equal length are said to be *dash equivalent* if the ith letter in p is a dash precisely when the ith letter in q is a dash. If p and q are dash and order equivalent, then p and q are *equivalent*. In what follows a pattern will usually be taken to be over the alphabet $\{a, b, c, d, \dots\} \cup \{\text{–}\}$ where $\{a, b, c, d, \dots\}$ is ordered so that $a < b < c < d < \cdots$.

Let $[n] := \{1, 2, \dots, n\}$ (so $[0] = \emptyset$). A *permutation* of $[n]$ is bijection from $[n]$ to $[n]$. Let $\mathcal{S}_n$ be the set of permutations of $[n]$. We shall usually think of a permutation π as the word $\pi(1)\pi(2)\cdots\pi(n)$ over the alphabet $[n]$. In particular, $\mathcal{S}_0 = \{\epsilon\}$, since there is only one bijection from $\emptyset$ to $\emptyset$, the empty map. We say that a subword σ of π is a *p-subword* if by replacing (possibly empty) segments of π with dashes we can obtain a pattern q equivalent to p such that $\bar{q} = \sigma$. However, all patterns that we will consider will have a dash at the beginning and one at the end. For convenience, we therefore leave them out. For example, $(a\text{–}bc)$ is a pattern, and the permutation 491273865 contains three $(a\text{–}bc)$-subwords, namely 127, 138, and 238. A permutation is said to be *p-avoiding* if it does not contain any p-subwords. Define $\mathcal{S}_n(p)$ to be the set of p-avoiding permutations in $\mathcal{S}_n$ and, more generally, $\mathcal{S}_n(A) = \bigcap_{p\in A} \mathcal{S}_n(p)$.

We may think of a pattern p as a permutation statistic, that is, define $p\,\pi$ as the number of p-subwords in π, thus regarding p as a function from $\mathcal{S}_n$ to $\mathbb{N}$. For example, $(a\text{–}bc)\,491273865 = 3$. In particular, π is p-avoiding if and only if $p\,\pi = 0$. We say that two permutation statistics stat and stat$'$ are *equidistributed* over $A \subseteq \mathcal{S}_n$, if

$$\sum_{\pi\in A} x^{\operatorname{stat}\pi} = \sum_{\pi\in A} x^{\operatorname{stat}'\pi}.$$

In particular, this definition applies to patterns.

Let $\pi = a_1a_2\cdots a_n \in \mathcal{S}_n$. An i such that $a_i > a_{i+1}$ is called a *descent* in π. We denote by $\operatorname{des}\pi$ the number of descents in π. Observe that des can be defined as the pattern (ba), that is, $\operatorname{des}\pi = (ba)\pi$. A *left-to-right minimum* of π is an element a_i such that $a_i < a_j$ for every $j < i$. The number of left-to-right minima is a permutation statistic. Analogously we also define *left-to-right maximum*, *right-to-left minimum*, and *right-to-left maximum*.

In this paper we will relate permutations avoiding a given set of patterns to other better known combinatorial structures. Here follows a brief description of these structures.

Set partitions. A *partition* of a set S is a family, $\pi = \{A_1, A_2, \dots, A_k\}$, of pairwise disjoint non-empty subsets of S such that $S = \cup_i A_i$. We call A_i a *block* of π. The total number of partitions of $[n]$ is called a *Bell number* and is denoted B_n. For reference, the first few Bell numbers are

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597.$$

Let $S(n, k)$ be the number of partitions of $[n]$ into k blocks; these numbers are called the *Stirling numbers of the second kind*.

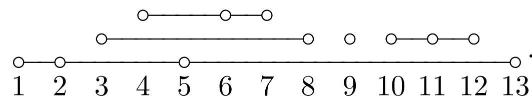
Non-overlapping partitions. Two blocks A and B of a partition π *overlap* if

$$\min A < \min B < \max A < \max B.$$

A partition is *non-overlapping* if no pairs of blocks overlap. Thus

$$\pi = \{\{1, 2, 5, 13\}, \{3, 8\}, \{4, 6, 7\}, \{9\}, \{10, 11, 12\}\}$$

is non-overlapping. A pictorial representation of π is

$\pi =$ 

Let B_n^* be the number of non-overlapping partitions of $[n]$; this number is called the nth *Bessel number* [4, p. 423]. The first few Bessel numbers are

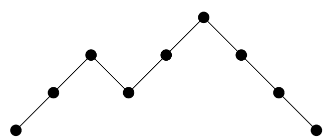
$$1, 1, 2, 5, 14, 43, 143, 509, 1922, 7651, 31965, 139685, 636712.$$

We denote by $S^*(n, k)$ the number of non-overlapping partitions of $[n]$ into k blocks.

Involutions. An *involution* is a permutation which is its own inverse. We denote by I_n the number of involutions in $\mathcal{S}_n$. The sequence $\{I_n\}_0^\infty$ starts with

$$1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152.$$

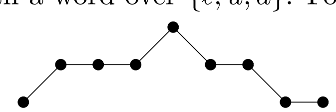
Dyck paths. A *Dyck path* of length $2n$ is a lattice path from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$ that never goes below the x-axis. Letting u and d represent the steps $(1, 1)$ and $(1, -1)$ respectively, we code such a path with a word over $\{u, d\}$. For example, the path



is coded by *uuduuddd*. The nth *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the number of Dyck paths of length $2n$. The sequence of Catalan numbers starts with

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012.$$

Motzkin paths. A *Motzkin path* of length n is a lattice path from $(0, 0)$ to $(n, 0)$ with steps $(1, 0)$, $(1, 1)$, and $(1, -1)$ that never goes below the x-axis. Letting ℓ, u, and d represent the steps $(1, 0)$, $(1, 1)$, and $(1, -1)$ respectively, we code such a path with a word over $\{\ell, u, d\}$. For example, the path



is coded by $u\ell\ell ud\ell d\ell$. The nth *Motzkin number* M_n is the number of Motzkin paths of length n. The first few of the Motzkin numbers are

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511.$$

3. Three classes of patterns

Let $\pi = a_1 a_2 \cdots a_n \in \mathcal{S}_n$. Define the *reverse* of π as $\pi^r := a_n \cdots a_2 a_1$, and define the *complement* of π by $\pi^c(i) = n + 1 - \pi(i)$, where $i \in [n]$.

Proposition 1. *With respect to being equidistributed, the twelve pattern statistics of length three with one dash fall into the following three classes.*

(i) *a–bc, c–ba, ab–c, cb–a.*
(ii) *a–cb, c–ab, ba–c, bc–a.*
(iii) *b–ac, b–ca, ac–b, ca–b.*

Proof. The bijections $\pi \mapsto \pi^r$, $\pi \mapsto \pi^c$, and $\pi \mapsto (\pi^r)^c$ give the equidistribution part of the result. Calculations show that these three distributions differ pairwise on $\mathcal{S}_4$. ☐

4. Permutations avoiding a pattern of class one or two

Proposition 2. *Partitions of $[n]$ are in one-to-one correspondence with $(a\text{–}bc)$-avoiding permutations in $\mathcal{S}_n$. Hence $|\mathcal{S}_n(a\text{–}bc)| = B_n$.*

First proof. Recall that the Bell numbers satisfy $B_0 = 1$, and

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

We show that $|\mathcal{S}_n(a\text{–}bc)|$ satisfy the same recursion. Clearly, $\mathcal{S}_0(a\text{–}bc) = \{\epsilon\}$. For $n > 0$, let $M = \{2, 3, \ldots, n+1\}$, and let S be a k element subset of M. For each $(a\text{–}bc)$-avoiding permutation σ of S we construct a unique $(a\text{–}bc)$-avoiding permutation π of $[n+1]$. Let τ be the word obtained by writing the elements of $M \setminus S$ in decreasing order. Define $\pi := \sigma 1 \tau$.

Conversely, if $\pi = \sigma 1 \tau$ is a given $(a\text{–}bc)$-avoiding permutation of $[n+1]$, where $|\sigma| = k$, then the letters of τ are in decreasing order, and σ is an $(a\text{–}bc)$-avoiding permutation of the k element set $\{2, 3, \ldots, n+1\} \setminus \{i : i \text{ is a letter in } \tau\}$. ☐

Second proof. Given a partition π of $[n]$, we introduce a standard representation of π by requiring that:

(a) Each block is written with its least element first, and the rest of the elements of that block are written in decreasing order.
(b) The blocks are written in decreasing order of their least element, and with dashes separating the blocks.

Define $\widehat{\pi}$ to be the permutation we obtain from π by writing it in standard form and erasing the dashes. We now argue that $\widehat{\pi} := a_1 a_2 \cdots a_n$ avoids $(a\text{–}bc)$. If $a_i < a_{i+1}$, then a_i and a_{i+1} are the first and the second element of some block. By the construction of $\widehat{\pi}$, a_i is a left-to-right minimum, hence there is no $j \in [i-1]$ such that $a_j < a_i$.

Conversely, π can be recovered uniquely from $\widehat{\pi}$ by inserting a dash in $\widehat{\pi}$ preceding each left-to-right minimum, apart from the first letter in $\widehat{\pi}$. Thus $\pi \mapsto \widehat{\pi}$ gives the desired bijection. ☐

Example 3. As an illustration of the map defined in the above proof, let

$$\pi = \{\{1,3,5\}, \{2,6,9\}, \{4,7\}, \{8\}\}.$$

Its standard form is 8–47–296–153. Thus $\widehat{\pi} = 847296153$.

Porism 4. *Let $L(\pi)$ be the number of left-to-right minima of π. Then*

$$\sum_{\pi \in \mathcal{S}_n(a-bc)} x^{L(\pi)} = \sum_{k \geq 0} S(n,k) x^k.$$

Proof. This result follows readily from the second proof of Proposition 2. We here give a different proof, which is based on the fact that the Stirling numbers of the second kind satisfy

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Let $T(n,k)$ be the number of permutations in $\mathcal{S}_n(a\text{–}bc)$ with k left-to-right minima. We show that the $T(n,k)$ satisfy the same recursion as the $S(n,k)$.

Let π be an $(a\text{–}bc)$-avoiding permutation of $[n-1]$. To insert n in π, preserving $(a\text{–}bc)$-avoidance, we can put n in front of π or we can insert n immediately after each left-to-right minimum. Putting n in front of π creates a new left-to-right minimum, while inserting n immediately after a left-to-right minimum does not. □

Proposition 5. *Partitions of* $[n]$ *are in one-to-one correspondence with* $(a\text{–}cb)$*-avoiding permutations in* $\mathcal{S}_n$*. Hence* $|\mathcal{S}_n(a\text{–}cb)| = B_n$*.*

Proof. Let π be a partition of $[n]$. We introduce a standard representation of π by requiring that:

(a) The elements of a block are written in increasing order.
(b) The blocks are written in decreasing order of their least element, and with dashes separating the blocks.

Notice that this standard representation is different from the one given in the second proof of Proposition 2. Define $\widehat{\pi}$ to be the permutation we obtain from π by writing it in standard form and erasing the dashes. It easy to see that $\widehat{\pi}$ avoids $(a\text{–}cb)$. Conversely, π can be recovered uniquely from $\widehat{\pi}$ by inserting a dash in between each descent in $\widehat{\pi}$. □

Example 6. As an illustration of the map defined in the above proof, let

$$\pi = \{\{1,3,5\},\{2,6,9\},\{4,7\},\{8\}\}.$$

Its standard form is 8–47–269–135. Thus $\widehat{\pi} = 847269135$.

Porism 7.

$$\sum_{\pi\in\mathcal{S}_n(a-cb)} x^{1+\operatorname{des}\pi} = \sum_{k\geq 0} S(n,k)x^k.$$

Proof. From the proof of Proposition 5 we see that π has $k+1$ blocks precisely when $\widehat{\pi}$ has k descents. □

Proposition 8. *Involutions in* $\mathcal{S}_n$ *are in one-to-one correspondence with permutations in* $\mathcal{S}_n$ *that avoid* $(a\text{–}bc)$ *and* $(a\text{–}cb)$*. Hence*

$$|\mathcal{S}_n(a\text{–}bc, a\text{–}cb)| = I_n.$$

Proof. We give a combinatorial proof using a bijection that is essentially identical to the one given in the second proof of Proposition 2.

Let $\pi \in \mathcal{S}_n$ be an involution. Recall that π is an involution if and only if each cycle of π is of length one or two. We now introduce a standard form for writing π in cycle notation by requiring that:

(a) Each cycle is written with its least element first.

(b) The cycles are written in decreasing order of their least element.

Define $\widehat{\pi}$ to be the permutation obtained from π by writing it in standard form and erasing the parentheses separating the cycles.

Observe that $\widehat{\pi}$ avoids $(a\text{–}bc)$: Assume that $a_i < a_{i+1}$, that is $(a_i\, a_{i+1})$ is a cycle in π, then a_i is a left-to-right minimum in π. This is guaranteed by the construction of $\widehat{\pi}$. Thus there is no $j < i$ such that $a_j < a_i$.

The permutation $\widehat{\pi}$ also avoids $(a\text{–}cb)$: Assume that $a_i > a_{i+1}$, then a_{i+1} must be the smallest element of some cycle. Then a_i is a left-to-right minimum in π.

Conversely, if $\widehat{\pi} := a_1 \dots a_n$ is an $\{a\text{–}bc, a\text{–}cb\}$-avoiding permutation then the involution π is given by: $(a_i\, a_{i+1})$ is a cycle in π if and only if $a_i < a_{i+1}$. □

Example 9. The involution $\pi = 826543719$ written in standard form is

$$(9)(7)(4\,5)(3\,6)(2)(1\,8),$$

and hence $\widehat{\pi} = 974536218$.

Porism 10.

$$\mathcal{S}_n(a\text{–}bc, a\text{–}cb) = \mathcal{S}_n(a\text{–}bc, acb) = \mathcal{S}_n(abc, a\text{–}cb) = \mathcal{S}_n(abc, acb).$$

Proof. Kitaev [5] observed that the dashes in the patterns $(a\text{–}bc)$ and $(a\text{–}cb)$ are immaterial for the proof of Proposition 8. The result may, however, also be proved directly. For an example of such a proof see the proof of Lemma 21. □

Porism 11. *The number of permutations in $\mathcal{S}_{n+k}(a\text{–}bc, a\text{–}cb)$ with $n-1$ descents equals the number of involutions in $\mathcal{S}_{n+k}$ with $n-k$ fixed points.*

Proof. Under the bijection $\pi \mapsto \widehat{\pi}$ in the proof of Proposition 8, a cycle of length two in π corresponds to an occurrence of (ab) in $\widehat{\pi}$. Hence, if π has $n-2k$ fixed points, then $\widehat{\pi}$ has $n-k-1$ descents. Substituting $n+k$ for n we get the desired result. □

To take the analysis of descents in $\{a\text{–}bc, a\text{–}cb\}$-avoiding permutations further, we introduce the polynomial

$$A_n(x) = \sum_{\pi \in \mathcal{S}_n(a-bc, a-cb)} x^{1+\operatorname{des}\pi},$$

and call it the nth *Eulerian polynomial for $\{a\text{–}bc, a\text{–}cb\}$-avoiding permutations.* Direct enumeration shows that the sequence $\{A_n(x)\}$ starts with

$$\begin{aligned}
A_0(x) &= 1\\
A_1(x) &= \quad + x\\
A_2(x) &= x + x^2\\
A_3(x) &= 3x^2 + x^3\\
A_4(x) &= 3x^2 + 6x^3 + x^4\\
A_5(x) &= 15x^3 + 10x^4 + x^5\\
A_6(x) &= 15x^3 + 45x^4 + 15x^5 + x^6\\
A_7(x) &= 105x^4 + 105x^5 + 21x^6 + x^7.
\end{aligned}$$

We will relate these polynomials to the so called Bessel polynomials. The nth *Bessel polynomial* $y_n(x)$ is defined by

$$y_n(x) = \sum_{k=0}^{n} \binom{n+k}{k} \binom{n}{k} \frac{k!}{2^k} x^k. \tag{1}$$

The first six of the Bessel polynomials are

$$\begin{aligned}
y_0(x) &= 1 \\
y_1(x) &= 1 + x \\
y_2(x) &= 1 + 3x + 3x^2 \\
y_3(x) &= 1 + 6x + 15x^2 + 15x^3 \\
y_4(x) &= 1 + 10x + 45x^2 + 105x^3 + 105x^4 \\
y_5(x) &= 1 + 15x + 105x^2 + 420x^3 + 945x^4 + 945x^5.
\end{aligned}$$

These polynomials satisfy the second order differential equation

$$x^2 \frac{d^2y}{dx^2} + 2(x+1)\frac{dy}{dx} = n(n+1)y.$$

Moreover, the Bessel polynomials satisfy the recurrence relation

$$y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x). \tag{2}$$

Proposition 12. *Let $y_n(x)$ be the nth Bessel polynomial, and let $A_n(x)$ be the nth Eulerian polynomial for $\{a{-}bc, a{-}cb\}$-avoiding permutations. Then*

(i) $\sum_n y_n(x)(xt)^n$ *generates* $\{A_n(t)\}$*, that is*

$$\sum_{n\geq 0} A_n(t)x^n = \sum_{n\geq 0} y_n(x)(xt)^n.$$

(ii) $A_0(x) = 1$, $A_1(x) = x$, *and for* $n \geq 2$, *we have*

$$A_{n+2}(x) = x(1 + x + 2x\frac{d}{dx})A_n(x).$$

(iii) $A_n(x)$ *is explicitly given by*

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{k} \frac{k!}{2^k} x^{n-k}.$$

Proof. Let I_n^k denote the number of involutions in $\mathcal{S}_n$ with k fixed points. Then Porism 11 is equivalently stated as

$$A_n(x) = \sum_{k\geq 0} I_n^{2k-n} x^k. \tag{3}$$

In [3] Dulucq and Favreau showed that the Bessel polynomials are given by

$$y_n(x) = \sum_{k\geq 0} I_{n+k}^{n-k} x^k. \tag{4}$$

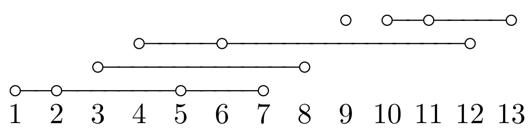
To prove (i), multiply Equation (4) by $(xt)^n$ and sum over n.

$$\begin{aligned}
\sum_{n\geq 0} y_n(x)(xt)^n &= \sum_{n\geq 0}\sum_{k\geq 0} I_{n+k}^{n-k} t^n x^{n+k} \\
&= \sum_{k\geq 0}\sum_{n\geq 0} I_{k}^{2n-k} t^n x^{k} && \text{By substituting } n-k \text{ for } k. \\
&= \sum_{k\geq 0} A_k(t) x^k && \text{By Equation (3).}
\end{aligned}$$

We now multiply Equation (2) by $(xt)^n$ and sum over n. Tedious but straightforward calculations then yield (ii) from (i). Finally, we obtain (iii) from Equation (1) by identifying coefficients in (i). $\square$

Definition 13. Let π be an arbitrary partition whose non-singleton blocks $\{A_1, \dots, A_k\}$ are ordered so that for all $i \in [k-1]$, $\min A_i > \min A_{i+1}$. If $\max A_i > \max A_{i+1}$ for all $i \in [k-1]$, then we call π a *monotone partition*. The set of monotone partitions of $[n]$ is denoted by $\mathcal{M}_n$.

Example 14. The partition

$\pi =$ 

is monotone.

Proposition 15. *Monotone partitions of* $[n]$ *are in one-to-one correspondence with permutations in* $\mathcal{S}_n$ *that avoid* $(a\text{–}bc)$ *and* $(ab\text{–}c)$. *Hence*

$$|\mathcal{S}_n(a\text{–}bc, ab\text{–}c)| = |\mathcal{M}_n|.$$

Proof. Given π in $\mathcal{M}_n$, let $A_1\text{–}A_2\text{–}\cdots\text{–}A_k$ be the result of writing π in the standard form given in the second proof of Proposition 2, and let $\widehat{\pi} = A_1A_2\cdots A_k$. By the construction of $\widehat{\pi}$ the fist letter in each A_i is a left-to-right minimum. Furthermore, since π is monotone the second letter in each non-singleton A_i is a right-to-left maximum. Therefore, if xy is an (ab)-subword of $\widehat{\pi}$, then x is left-to-right minimum and y is a right-to-left maximum. Thus $\widehat{\pi}$ avoids both $(a\text{–}bc)$ and $(ab\text{–}c)$.

Conversely, given $\widehat{\pi}$ in $\mathcal{S}_n(a\text{–}bc, ab\text{–}c)$, let $A_1\text{–}A_2\text{–}\cdots\text{–}A_k$ be the result of inserting a dash in $\widehat{\pi}$ preceding each left-to-right minimum, apart from the first letter in $\widehat{\pi}$. Since $\widehat{\pi}$ is $(ab\text{–}c)$-avoiding, the second letter in each non-singleton A_i is a right-to-left maximum. The second letter in A_i is the maximal element of A_i when A_i is viewed as a set. Thus $\pi = \{A_1, A_2, \dots, A_k\}$ is monotone. $\square$

What is left for us to show is that there is a one-to-one correspondence between monotone partitions and non-overlapping partitions. The proof we give is strongly influenced by the work of Flajolet [4].

Proposition 16. *Monotone partitions of* $[n]$ *are in one-to-one correspondence with non-overlapping partitions of* $[n]$. *Hence* $|\mathcal{M}_n| = B_n^*$.

Proof. If k is the minimal element of a non-singleton block, then call k the *first* element of that block. Similarly, If k is the maximal element of a non-singleton block, then call k the *last* element of that block. An element of a non-singleton that is not a first or last element is called an *intermediate* element. Let us introduce an ordering of the blocks of a partition. A block A is smaller than a block B if $\min A < \min B$.

We define a map Φ that to each non-overlapping partition π of $[n]$ gives a unique monotone partition $\Phi(\pi)$ of $[n]$. Let the integer k range from 1 to n.

(a) If k is the first element of a block of π, then open a new block in $\Phi(\pi)$ by letting k be its first element. (A block B is open if $\max B < k$.)
(b) If k is the last element of a block of π, then close the smallest open block of $\Phi(\pi)$ by letting k be its last element.
(c) If k is an intermediate element of some block B of π, and B is the i:th largest open block of π, then let k belong to the ith largest open block of $\Phi(\pi)$.
(d) If $\{k\}$ is a singleton block of π, then let $\{k\}$ be a singleton block of $\Phi(\pi)$.

Observe that $\Phi(\pi)$ is monotone. Indeed, it is only in (b) that we close a block of $\Phi(\pi)$, and we always close the smallest open block of $\Phi(\pi)$.

Conversely, we give a map Ψ that to each monotone partition π be a of $[n]$ gives a unique non-overlapping partition $\Psi(\pi)$ of $[n]$. Define Ψ the same way as Φ is defined, except for case (c), where we instead of closing the smallest open block close the largest open block.

It is easy to see that Φ and Ψ are each others inverses and hence they are bijections. $\square$

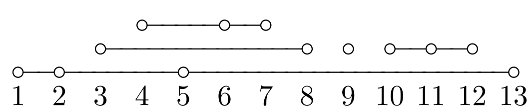
Corollary 17. *The NOPs (non-overlapping partitions) of* $[n]$ *are in one-to-one correspondence with permutations in* $\mathcal{S}_n$ *that avoid* $(a\text{–}bc)$ *and* $(ab\text{–}c)$. *Hence*

$$|\mathcal{S}_n(a\text{–}bc, ab\text{–}c)| = B_n^*.$$

Proof. Follows immediately from Proposition 15 together with Proposition 16. $\square$

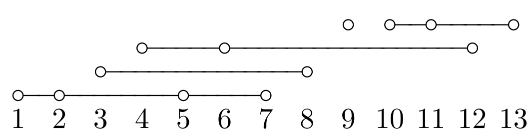
Example 18. By the proof of Proposition 16, the non-overlapping partition

$\pi =$



corresponds to the monotone partition

$\Phi(\pi) =$



that according to the proof of Proposition 15 corresponds to the $\{a\text{–}bc, ab\text{–}c\}$-avoiding permutation

$$\widehat{\Phi(\pi)} = 10\ 13\ 11\ 9\ 4\ 12\ 6\ 3\ 8\ 1\ 7\ 5\ 2.$$

Porism 19. *Let $L(\pi)$ be the number of left-to-right minima of π. Then*

$$\sum_{\pi\in\mathcal{S}_n(a\text{–}bc,ab\text{–}c)} x^{L(\pi)} = \sum_{k\geq 0} S^*(n,k)x^k.$$

Proof. Under the bijection $\pi \mapsto \widehat{\pi}$ in the proof of Proposition 15, the number of blocks in π determines the number of left-to-right minima of $\widehat{\pi}$, and vice versa. The number of blocks is not changed by the bijection $\Phi_1 \circ \Psi_2$ in the proof of Proposition 16. □

5. Permutations avoiding a pattern of class three

In [6] Knuth observed that there is a one-to-one correspondence between (b–a–c)-avoiding permutations and Dyck paths. For completeness and future reference we give this result as a lemma, and prove it using one of the least known bijections. First we need a definition. For each word $x = x_1x_2\cdots x_n$ without repeated letters, we define the *projection* of x onto S_n, which we denote proj(x), by

$$\mathrm{proj}(x) = a_1a_2\cdots a_n\,, \quad \text{where} \;\; a_i = |\{j \in [n] : x_i \geq x_j\}|.$$

Equivalently, proj(x) is the permutation in $\mathcal{S}_n$ which is order equivalent to x. For example, proj(265) = 132.

Lemma 20. $|\mathcal{S}_n(b\text{–}a\text{–}c)| = C_n$.

Proof. Let $\pi = a_1a_2\cdots a_n$ be a permutation of $[n]$ such that $a_k = 1$. Then π is (b–a–c)-avoiding if and only if $\pi = \sigma 1\tau$, where $\sigma := a_1\cdots a_{k-1}$ is a (b–a–c)-avoiding permutation of $\{n, n{-}1, \ldots, n{-}k{+}1\}$, and $\tau := a_{k+1}\cdots a_n$ is a (b–a–c)-avoiding permutation of $\{2, 3, \ldots, k\}$.

We define recursively a mapping Φ from $\mathcal{S}_n(b\text{–}a\text{–}c)$ onto the set of Dyck paths of length $2n$. If π is the empty word, then so is the Dyck path determined by π, that is, $\Phi(\epsilon) = \epsilon$. If $\pi \neq \epsilon$, then we can use the factorisation $\pi = \sigma 1\tau$ from above, and define $\Phi(\pi) = u\,(\Phi\circ\mathrm{proj})(\sigma)\,d\,(\Phi\circ\mathrm{proj})(\tau)$. It is easy to see that Φ may be inverted, and hence is a bijection. □

Lemma 21. *A permutation avoids (b–ac) if and only if it avoids (b–a–c).*

Proof. The sufficiency part of the proposition is trivial. The necessity part is not difficult either. Assume that π contains a (b–a–c)-subword. Then there exist

$$A, B, C, n_1, n_2, \ldots, n_r \in [n], \text{ where } A < B < C,$$

such that BAC is a subword of π, and $An_1\cdots n_rC$ is a segment of π. If $n_1 > B$, then BAn_1 form a (b–ac)-subword in π. Assume that $n_1 < B$. Indeed, to avoid forming a (b–ac)-subword we will have to assume that $n_i < B$ for all $i \in [r]$, but then Bn_rC is a (b–ac)-subword. Accordingly we conclude that there exists at least one (b–ac)-subword in π. □

Proposition 22. *Dyck paths of length $2n$ are in one-to-one correspondence with (b–ac)-avoiding permutations in $\mathcal{S}_n$. Hence*

$$|\mathcal{S}_n(b\text{–}ac)| = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Follows immediately from Lemma 20 and Lemma 21. □

Proposition 23. *Let $L(\pi)$ be the number of left-to-right minima of π. Then*

$$\sum_{\pi \in \mathcal{S}_n(b\text{–}ac)} x^{L(\pi)} = \sum_{k \geq 0} \frac{k}{2n-k}\binom{2n-k}{n} x^k.$$

Proof. A *return step* in a Dyck path δ is a d such that $\delta = \alpha u \beta d \gamma$, for some Dyck paths α, β, and γ. A useful observation is that every non-empty Dyck path δ can be uniquely decomposed as $\delta = u\alpha d\beta$, where α and β are Dyck paths. This is the so-called *first return decomposition* of δ. Let $R(\delta)$ denote the number of return steps in δ.

In [2] Deutsch showed that the distribution of R over all Dyck paths of length $2n$ is the distribution we claim that L has over $\mathcal{S}_n(b\text{–}ac)$.

Let γ be a Dyck path of length $2n$, and let $\gamma = u\alpha d\beta$ be its first return decomposition. Then $R(\gamma) = 1 + R(\beta)$. Let $\pi \in \mathcal{S}_n(b\text{–}ac)$, and let $\pi = \sigma 1 \tau$ be the decomposition given in the proof of Lemma 20. Then $L(\pi) = 1 + L(\sigma)$. The result now follows by induction. □

In addition, it is easy to deduce that left-to-right minima, left-to-right maxima, right-to-left minima, and right-to-left maxima all share the same distribution over $\mathcal{S}_n(b\text{–}ac)$.

Proposition 24. *Motzkin paths of length n are in one-to-one correspondence with permutations in $\mathcal{S}_n$ that avoid $(a\text{–}bc)$ and $(ac\text{–}b)$. Hence*

$$|\mathcal{S}_n(a\text{–}bc, ac\text{–}b)| = M_n.$$

Proof. We mimic the proof of Lemma 20. Let $\pi \in \mathcal{S}_n(a\text{–}bc, ac\text{–}b)$. Since π avoids $(ac\text{–}b)$ it also avoids $(a\text{–}c\text{–}b)$ by Lemma 21 via $\pi \mapsto (\pi^c)^r$. Thus we may write $\pi = \sigma n \tau$, where $\pi(k) = n$, τ is an $\{a\text{–}bc, ac\text{–}b\}$-avoiding permutation of $\{n-1, n-2, \dots, n-k+1\}$, and τ is an $\{a\text{–}bc, ac\text{–}b\}$-avoiding permutation of $[n-k]$. If $\sigma \neq \epsilon$ then $\sigma = \sigma' r$ where $r = n-k+1$, or else an $(a\text{–}bc)$-subword would be formed with n as the 'c' in $(a\text{–}bc)$. Define a map Φ from $\mathcal{S}_n(a\text{–}bc, ac\text{–}b)$ to the set of Motzkin paths by $\Phi(\epsilon) = \epsilon$ and

$$\Phi(\pi) = \begin{cases} \ell\,(\Phi \circ \mathrm{proj})(\sigma) & \text{if } \pi = n\sigma, \\ u\,(\Phi \circ \mathrm{proj})(\sigma)\, d\, \Phi(\tau) & \text{if } \pi = \sigma r n \tau \text{ and } r = n-k+1. \end{cases}$$

Its routine to find the inverse of Φ. □

Example 25. Let us find the Motzkin path associated to the $\{a\text{–}bc, ac\text{–}b\}$-avoiding permutation 76453281.

$$\begin{aligned} \Phi(764532\mathbf{8}1) &= u\Phi(\mathbf{5}4231)d\Phi(\mathbf{1}) \\ &= u\ell\Phi(\mathbf{4}231)d\ell \\ &= u\ell\ell\Phi(\mathbf{23}1)d\ell \\ &= u\ell\ell ud\Phi(\mathbf{1})d\ell \\ &= u\ell\ell ud\ell d\ell \end{aligned}$$

Acknowledgement

I am greatly indebted to my advisor Einar Steingrímsson, who put his trust in me and gave me the opportunity to study mathematics on a post-graduate level. This work has benefited from his knowledge, enthusiasm and generosity.

References

[1] E. Babson and E. Steingrímsson. Generalized permutation patterns and a classification of the Mahonian statistics. *Séminaire Lotharingien de Combinatoire*, B44b:18pp, 2000.

[2] E. Deutsch. Dyck path enumeration. *Discrete Math.*, 204(1-3):167–202, 1999.

[3] S. Dulucq and L. Favreau. Un modèle combinatoire pour les polynômes de Bessel. In *Séminaire Lotharingien de Combinatoire (Salzburg, 1990)*, pages 83–100. Univ. Louis Pasteur, Strasbourg, 1991.

[4] P. Flajolet and R. Schott. Non–overlapping partitions, continued fractions, Bessel functions and a divergent series. *European Journal of Combinatorics*, 11:421–432, 1990.

[5] S. Kitaev. Private communication, 2000.

[6] D. E. Knuth. *The art of computer programming*, volume 1. Addison-Wesley, 1973.

[7] R. Simion and F. W. Schmidt. Restricted permutations. *European Journal of Combinatorics*, 6:383–406, 1985.

[8] N. J. A. Sloane and S. Plouffe. *The Encyclopedia of Integer Sequences*. Academic Press, 1995. http://www.research.att.com/~njas/sequences/.

[9] R. P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, 1997.

[10] J. West. Permutation trees and the Catalan and Schröder numbers. *Discrete Mathematics*, 146:247–262, 1995.

Matematik, Chalmers tekniska högskola och Göteborgs universitet, S-412 96 Göteborg, Sweden

E-mail address: `claesson@math.chalmers.se`